Sharp norm comparison of the maxima of a sequence and its predictable projection

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Abstract

Let $f = (f_n)$ be an adapted sequence of integrable Banach-space valued random variables and $g = (g_n)$ denote its predictable projection. We prove that, for $1 \leq p \leq \infty$,

$$\big\| \sup_n ||g_n|| \big\|_p \leq \left(1 + \frac{(p-1)^{p-1}}{p^p}\right) \big\| \sup_n ||f_n|| \big\|_p$$

and the constant $1 + \frac{(p-1)^{p-1}}{p^p}$ is the best possible.

Key words: predictable projection, norm inequality

1. Introduction

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a fixed probability space, filtered by a nondecreasing sequence $(\mathcal{F}_n)$ of sub-σ-algebras of $\mathcal{F}$. Let $f = (f_n)_{n \geq 0}$ be an adapted sequence of integrable real-valued random variables and let $g = (g_n)$ stand for the predictable projection of the sequence f, that is, $g_0 = f_0$ and $g_n = \mathbb{E}(f_n|\mathcal{F}_{n-1})$, for $n = 1, 2, \ldots$. For $1 \leq p, q \leq \infty$ we define

$$||f||_{p,q} = ||f||_{L^p(\ell^q)} = \left[\mathbb{E}\left(\sum_{k=1}^{\infty} |f_n|^q\right)^{p/q}\right]^{1/p},$$

with the usual convention if p or q is infinite. The problem of comparing the norms of f and g was first studied by Stein (1970), who showed that for $1 < p < \infty$ and $1 \leq q \leq \infty$ there is a universal $C_{p,q} < \infty$ (not depending on f, g or the probability space) such that

$$||g||_{p,q} \leq C_{p,q}||f||_{p,q}. \tag{1}$$

In fact, Stein established the inequality for $q = 2$ only, but the proof works for other values of q as well. It is also worth to mention that the result is true for the sequences f which are not necessarily adapted.

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A number of authors studied various extensions of the inequality (1). Johnson et al. (1979) extended it by replacing L^p by a rearrangement invariant space X with Boyd indices satisfying $0 < \beta_X \leq \alpha_X < 1$. Bourgain (1983) showed the estimate for $p = 1$ and $q = 2$ with $C_{1,2} = 3$ and Lépingle (1978) decreased this constant to 2. Delbaen & Schachermayer (1994) needed the version of (1) with $p = 2$ and $q = \infty$, and in Delbaen & Schachermayer (1995) they proved the estimate for $1 \leq p \leq q \leq \infty$ with $C_{p,q} = 2$ (in fact the proof yields $C_{p,q} = 2^{1/p}$). The constant $2^{1/p}$, for $p \in \{1, \infty\}$ and $q = \infty$, turns out to be the best possible. The case $q = 1$ was studied by Burkholder (1973) and Garsia (1973); finally, Wang (1991) proved that the best constant $C_{p,1}$ equals p. Let us finish the short overview by noting that the inequality (1) can be investigated using decoupled conditionally independent tangent sequences; for details, see the book by Kwapień & Woyczyński (1992).

The contribution of this paper is to provide the optimal values of the constants $C_{p,\infty}$ for $p \in (1, \infty)$. Let $f_n^* = \max_{0 \leq k \leq n} ||f_k||$ and $f^* = \sup_n ||f_n||$. Here is our main result.

Theorem 1.1. *For any $1 < p < \infty$ we have*

$$||g^*||_p \leq \left(1 + \frac{(p-1)^{p-1}}{p^p}\right) ||f^*||_p. \tag{2}$$

The inequality is sharp.

The inequality generalizes to the sequences f of strongly integrable Banach space valued variables; see Remark 3.2 below.

2. Technical lemmas

Throughout the paper, the number p is fixed and belongs to the interval $(1, \infty)$. Let

$$C_p = 1 + \frac{(p-1)^{p-1}}{p^p}, \quad A_p = C_p \cdot \frac{p-1}{p} \cdot \left[1 - \left(\frac{p-1}{p}\right)^p\right]^{1/(p-1)}.$$

Lemma 2.1. *We have $C_p^p - 1 \geq A_p^{p-1}$.*

Proof. One can easily verify that the inequality is equivalent to $J\big((\frac{p-1}{p})^p\big) \geq 1$, where

$$J(x) = \left(1 + \frac{x}{p-1}\right)^{p-1} \left(1 - x + \frac{px^2}{p-1}\right), \qquad x \geq 0.$$

Since $J(0) = 1$, we will be done if we show that J is nondecreasing. It suffices to note that

$$J'(x) = \left(1 + \frac{x}{p-1}\right)^{p-2} \left[\frac{px}{p-1} + \frac{px^2}{p-1} + \frac{2px^2}{(p-1)^2}\right] > 0$$

for $x > 0$.

Lemma 2.2. *We have*

$$-(p+1)C_p + \left(\frac{p}{p-1}\right)^{p-1} C_p^2 + p \geq 0. \tag{3}$$

Proof. The left hand side equals

$$p(1-C_p)-C_p+\left(\frac{p}{p-1}\right)^{p-1}C_p^2=-\left(\frac{p-1}{p}\right)^{p-1}+C_p\left[\left(\frac{p}{p-1}\right)^{p-1}+\frac{1}{p}-1\right].$$

Now, if $p\geq 2$, then the estimate is a consequence of

$$\left(\frac{p-1}{p}\right)^{p-1}\leq 1,\ C_p\geq 1 \text{ and } \left(\frac{p}{p-1}\right)^{p-1}\geq 2.$$

If $p<2$, then, by Bernoulli's inequality,

$$\left(\frac{p-1}{p}\right)^{p-1}\leq 1-\frac{p-1}{p}=\frac{1}{p}\leq C_p\left[\left(\frac{p}{p-1}\right)^{p-1}+\frac{1}{p}-1\right].$$

Lemma 2.3. *(i) We have* $A_p<C_p\cdot\frac{p-1}{p}<1$.
(ii) For any $q\in[0,1]$, *we have* $q(1-q)^{p-1}<A_p^{p-1}$.

Proof. (i) The first inequality is trivial. To show the second one, note that

$$C_p\cdot\frac{p-1}{p}\leq\left(1+\frac{1}{p}\right)\cdot\frac{p-1}{p}=1-\frac{1}{p^2}.$$

(ii) The function $q\mapsto q(1-q)^{p-1}$, $q\in[0,1]$, attains its maximum at $q=1/p$ and hence it suffices to show that $\frac{(p-1)^{p-1}}{p^p}<A_p^{p-1}$, or, equivalently, $1<pC_p^{p-1}\left[1-\left(\frac{p-1}{p}\right)^p\right]$. However, we have $1-\left(\frac{p-1}{p}\right)^p>1-\left(\frac{p-1}{p}\right)=\frac{1}{p}$, which yields the desired estimate.

Lemma 2.4. *For any* $s\in[A_p^{p-1},1]$ *we have*

$$(C_p^p-s)\left(s^{p/(p-1)}-s+A_p^{p-1}\right)^{p-1}\geq A_p^{p(p-1)}. \tag{4}$$

Proof. Denote the left hand side of (4) by $F(s)$. We have that

$$F'(s)=\left(s^{p/(p-1)}-s+A_p^{p-1}\right)^{p-2}f(s),$$

where $f:[0,\infty)\to\mathbb{R}$ is given by

$$f(s)=-(p+1)s^{p/(p-1)}+ps+pC_p^p s^{1/(p-1)}-(p-1)C_p^p-A_p^{p-1}.$$

Some information about F and f: by Lemma 2.3 (i), we have $(C_p\frac{p-1}{p})^{p-1}\in[A_p^{p-1},1]$,

$$F\left(\left(C_p\frac{p-1}{p}\right)^{p-1}\right)-A_p^{p(p-1)}=0 \qquad\text{and}\qquad f\left(\left(C_p\frac{p-1}{p}\right)^{p-1}\right)=0. \tag{5}$$

In addition,

$$f(0)<0 \qquad\text{and}\qquad \lim_{s\to\infty}f(s)=-\infty. \tag{6}$$

Furthermore, $f'(s)=\frac{p}{p-1}s^{(2-p)/(p-1)}(-(p+1)s+C_p^p)+p$, so

$$f'(0+)=\lim_{s\downarrow 0}f'(s)\in[0,\infty] \quad \text{and} \quad f'\left(\left(C_p\frac{p-1}{p}\right)^{p-1}\right)\geq 0, \tag{7}$$

the latter being equivalent to (3). Finally, we have

$$f''(s)=\frac{p}{(p-1)^2}s^{-2+1/(p-1)}\left[-(p+1)s+(2-p)C_p^p\right]. \tag{8}$$

Now let us put (5) – (8) together; it is evident that there exists $s>(C_p\frac{p-1}{p})^{p-1}$ such that $f>0$ on $((C_p\frac{p-1}{p})^{p-1},s)$ and $f\leq 0$ on the compliment of this interval. Since F' and f have the same sign on $(A_p^{p-1},1)$, the estimate (4) will be established once we have shown that $F(1)\geq 0$. However, we have proved this inequality in Lemma 2.1.

Lemma 2.5. *For any $q\in[0,1]$ and any $w\geq 1$ we have*

$$q\left\{\left[(q+(1-q)w)^p-C_p^p\right]\vee(-A_p^{p-1})\right\}-(1-q)w^pA_p^{p-1}\leq -A_p^{p-1}. \tag{9}$$

Proof. It is easy to see that the inequality holds if $q=0$ or $q=1$, so we may assume that q lies in the interior of $[0,1]$. Since $q(-A_p^{p-1})-(1-q)w^pA_p^{p-1}=A_p^{p-1}(-q-(1-q)w^p)\leq -A_p^{p-1}$, it suffices to show that $q\left[(q+(1-q)w)^p-C_p^p\right]-(1-q)w^pA_p^{p-1}\leq -A_p^{p-1}$. Substitute $x=(1-q)w^p\geq 1-q$. The inequality takes form

$$q\left[(q+(1-q)^{1-1/p}x^{1/p})^p-C_p^p\right]\leq A_p^{p-1}(x-1). \tag{10}$$

For a fixed q, the function $G_q:[0,\infty)\to\mathbb{R}$, given by

$$G_q(s)=q\left[(q+(1-q)^{1-1/p}s^{1/p})^p-C_p^p\right]-A_p^{p-1}(s-1),$$

is concave and, by Lemma 2.3 (ii), tends to $-\infty$ as $s\to\infty$. Furthermore, it is increasing for $s<s_0$ and decreasing for $s>s_0$, where $s_0=s_0(q)$ satisfies

$$q^{1/(p-1)}\left[\frac{q(1-q)^{1/p}}{s_0^{1/p}}+(1-q)\right]=A_p \tag{11}$$

(such s_0 exists, as $A_p>q^{1/(p-1)}(1-q)$ in view of Lemma 2.3 (ii)). Now if $q<A_p^{p-1}$, then, by (11), we have $s_0(q)<1-q$, and so $G_q(x)\leq G_q(1-q)=q(1+A_p^{p-1}-C_p^p)\leq 0$, by Lemma 2.1. Suppose then, that $q\geq A_p^{p-1}$ and let $s=A_p^{p-1}/q\in[A_p^{p-1},1]$. By (11),

$$\begin{aligned}\frac{G_q(x)}{q}\leq\frac{G_q(s_0)}{q}&=\frac{s_0}{1-q}\left(\frac{q(1-q)^{1/p}}{s_0}+1-q\right)^p-ss_0-(C_p^p-s)\\ =\frac{s_0}{1-q}\left[\frac{A_p^p}{q^{p/(p-1)}}-s+qs\right]-(C_p^p-s)&=\frac{s_0}{1-q}(s^{p/(p-1)}-s+A_p^{p-1})-(C_p^p-s)\\ &=\frac{A_p^{p(p-1)}}{(s^{p/(p-1)}-s+A_p^{p-1})^p}\cdot(s^{p/(p-1)}-s+A_p^{p-1})-(C_p^p-s),\end{aligned}$$

which is nonpositive by (4). The proof is complete.

3. The proof of the main theorem

Let $U_p : [0,\infty) \times [0,\infty) \to \mathbb{R}$ be given by $U_p(x,y) = \big(y^p - C_p^p x^p\big) \vee (-A_p^{p-1}x^p)$.

Lemma 3.1. *For any x, $y \geq 0$ and any nonnegative integrable variable X we have*

$$\mathbb{E}U_p(x \vee X, y \vee \mathbb{E}X) \leq U_p(x,y). \tag{12}$$

Proof. We start from some reductions. First, we may assume $x = 1$. Secondly, observe that it suffices to show the estimate for simple variables X, that is, taking only a finite number of values. Moreover, we may assume that X takes at most two values. To see this, note that there exists a finite sequence of pairwise disjoint events A_1, A_2, ..., A_n of positive probability such that for any i, X takes at most two values on A_i and $\mathbb{E}(XI_{A_i}|A_i) = \mathbb{E}X$. The existence can be easily shown by induction on the number of values taken by X. Applying (12) to the variable XI_{A_i}, conditionally on A_i, we obtain

$$\mathbb{E}\left[U_p(1 \vee XI_{A_i}, y \vee \mathbb{E}(XI_{A_i}|A_i))\big|A_i\right] \leq U_p(1,y),$$

or $\mathbb{E}(U_p(1 \vee X, y \vee \mathbb{E}X)I_{A_i}) \leq \mathbb{P}(A_i)U_p(1,y)$. Now if we sum these inequalities for $i = 1, 2, \ldots, n$, we obtain (12). The next reduction is that we may assume $\mathbb{P}(X \geq 1) = 1$, replacing X by $1 \vee X$, if necessary. Furthermore, we may restrict ourselves to the variables X satisfying $\mathbb{E}X \geq y$, since if it is not the case, we have

$$\begin{aligned}\mathbb{E}U_p(1 \vee X, y \vee \mathbb{E}X) = &\ \mathbb{E}U_p(X,y) = \mathbb{E}\big[(y^p - C_p^p X^p) \vee (-A_p^{p-1}X^p)\big] \\ \leq &\ \mathbb{E}\big[(y^p - C_p^p) \vee (-A_p^{p-1})\big] = U_p(1,y).\end{aligned}$$

As $U_p(1,y) \geq U_p(1,1) = -A_p^{p-1}$ in view of Lemma 2.1, the proof will be complete if we show that

$$\mathbb{E}U_p(X, \mathbb{E}X) \leq -A_p^{p-1} \tag{13}$$

and note that we may drop the assumption $\mathbb{E}X \geq y$.

Summarizing, we have reduced the problem of proving (12) to the problem of showing (13) for any nonnegative variable X taking at most two values not smaller than 1.

Now, if X is constant, say, $X = a \geq 1$, then (13) is evident; indeed,

$$\mathbb{E}U_p(X, \mathbb{E}X) = U_p(a,a) = a^p U_p(1,1) \leq U_p(1,1).$$

If X takes two values: 1 and $w > 1$, and we denote $\mathbb{P}(X = 1)$ by q, then the inequality (13) is precisely the estimate (9). Finally, if X takes two values: $a > 1$ and $w > a$, then

$$\mathbb{E}U_p(X, \mathbb{E}X) = a^p \mathbb{E}U_p\left(X/a, \mathbb{E}X/a\right).$$

Observe that X/a takes two values, one of which equals 1; hence we may use the previous case and write $\mathbb{E}U_p(X, \mathbb{E}X) \leq a^p U_p(1,1) < U_p(1,1)$. This completes the proof.

Proof of the inequality (2). Clearly, it suffices to show the estimate for the sequences f, for which the right hand side of (2) is finite. Furthermore, we may restrict ourselves to the case of finite sequences f, that is, such that there exists N for which we have $0 = f_N = f_{N+1} = f_{N+2} = \ldots$. Since $U_p(x,y) \geq y^p - C_p^p x^p$, we will be done once we have shown that $\mathbb{E}U_p(f_n^*, g_n^*) \leq 0$ for any nonnegative integer n. We will prove a bit more, namely, that the process $(U_p(f_n^*, g_n^*))$ is a supermartingale with respect to $(\mathcal{F}_n)$. To this end, apply the inequality (12) conditionally on $\mathcal{F}_{n-1}$, $n \geq 1$, to get

$$\mathbb{E}\left[U_p(f_n^*, g_n^*)|\mathcal{F}_{n-1}\right] = \mathbb{E}\left[U_p(f_{n-1}^* \vee f_n, g_{n-1}^* \vee \mathbb{E}(f_n|\mathcal{F}_{n-1}))|\mathcal{F}_{n-1}\right] \leq U_p(f_{n-1}^*, g_{n-1}^*).$$

Therefore $\mathbb{E}U_p(f_n^*, g_n^*) \leq \mathbb{E}U_p(f_0^*, g_0^*) = \mathbb{E}U_p(f_0, g_0) \leq 0$, as $U_p(x, x) \leq 0$ for any $x \geq 0$. The estimate follows.

Remark 3.1. *As we have* $U_p(x, y) = y^p \vee \left[(C_p^p - A_p^{p-1})x^p\right] - C_p^p x^p \geq (x \vee y)^p - C_p^p x^p$, *we see that a stronger estimate is valid: for any sequence* f *as in Theorem 1.1,*

$$||f^* \vee g^*||_p \leq C_p ||f^*||_p, \;\; 1 < p < \infty.$$

Since C_p *is optimal in (2) (as proved below), the inequality above is also sharp.*

Remark 3.2. *The inequality (2) extends to the case of strongly integrable sequences* f *taking values in a certain Banach space* $(B, ||\cdot||_B)$. *Indeed, in such a case consider real valued* $\overline{f} = (||f_1||_B, ||f_2||_B, \ldots)$ *and observe that* $||f||_{p,\infty} = ||\overline{f}||_{p,\infty}$, $||g||_{p,\infty} \leq ||\overline{g}||_{p,\infty}$.

Sharpness of (2). We will construct appropriate examples on the interval $[0, 1]$ equipped with its Borel subsets and Lebesgue's measure. To this end, fix $p \in (1, \infty)$ and let $q = 1 - \left(\frac{p-1}{p}\right)^p$, $\quad w = \frac{p}{p-1} > 1$. Define the sequence (f_n) by $f_0 = I_{[0,1]}$ and

$$f_n = w^n I_{[0,(1-q)^n]} + w^{n-1} I_{((1-q)^n,(1-q)^{n-1}]}, \;\; n = 1, 2, \ldots,$$

and let $(\mathcal{F}_n)$ be a filtration generated by the sequence (f_n). Then $g_0 = I_{[0,1]}$ and $g_n = w^{n-1}(w(1-q)+q)I_{[0,(1-q)^{n-1}]} = C_p w^{n-1} I_{[0,(1-q)^{n-1}]}$, $n = 1, 2, \ldots$. Therefore,

$$f_n^* = w^n I_{[0,(1-q)^n]} + \sum_{k=1}^n w^{k-1} I_{((1-q)^k,(1-q)^{k-1}]},$$

$$g_n^* = C_p w^{n-1} I_{[0,(1-q)^n]} + C_p \sum_{k=1}^n w^{k-1} I_{((1-q)^k,(1-q)^{k-1}]}.$$

Since $(1-q)w^p = 1$, it can be easily verified that we have $||f_n^*||_p^p = 1 + qn$ and $||g_n^*||_p^p = C_p^p \left[\left(\frac{p-1}{p}\right)^p + qn\right]$. Letting $n \to \infty$ we see that the constant C_p in (2) can not be replaced by a smaller one.

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