SHARP $L^1(\ell^q)$ ESTIMATE FOR A SEQUENCE AND ITS PREDICTABLE PROJECTION

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ABSTRACT. Let $f = (f_n)_{n\geq 0}$ be a sequence of integrable Banach-space valued random variables and $g = (g_n)_{n\geq 0}$ denote its predictable projection. We prove that, for $1 \leq q < \infty$,

$$\mathbb{E}\left(\sum_{n=0}^{\infty}|g_n|^q\right)^{1/q} \leq 2^{(q-1)/q}\mathbb{E}\left(\sum_{n=0}^{\infty}|f_n|^q\right)^{1/q}$$

and that the constant $2^{(q-1)/q}$ is the best possible. The proof rests on the construction of a certain special function enjoying appropriate majorization and concavity.

1. Introduction

Suppose that $(\Omega, \mathcal{F}, \mathbb{P})$ is a fixed probability space, equipped with a filtration $(\mathcal{F}_n)_{n\geq 0}$, a nondecreasing sequence of sub-σ-algebras of $\mathcal{F}$. Let $f = (f_n)_{n\geq 0}$ be an adapted sequence of integrable real-valued random variables and let $g = (g_n)_{n\geq 0}$ stand for the predictable projection of the sequence f, that is, $g_0 = f_0$ and $g_n = \mathbb{E}(f_n|\mathcal{F}_{n-1})$, for $n = 1, 2, \ldots$. For $1 \leq p,\ q \leq \infty$, we introduce the mixed norms

$$||f||_{p,q} = ||f||_{L^p(\ell^q)} = \left[\mathbb{E}\left(\sum_{n=1}^{\infty}|f_n|^q\right)^{p/q}\right]^{1/p},$$

with the usual convention if p or q is infinite. The problem of comparing the (p, q)-norms of f and g was first studied by Stein [10]. He showed that for $1 < p < \infty$ and $1 \leq q \leq \infty$ there is an absolute constant $C_{p,q} < \infty$ (not depending on f, g, the probability space or filtration) such that

$$||g||_{p,q} \leq C_{p,q}||f||_{p,q}. \tag{1.1}$$

Actually, Stein focused on the case $q = 2$ only, but his proof works for other values of q as well. It is also worth to mention here that the result is true for the sequences f which are not necessarily adapted.

The inequality (1.1) has been studied by many authors. Johnson, Maurey, Schechtman and Tzafriri [6] established an extension with L^p replaced by a rearrangement invariant space X with Boyd indices satisfying $0 < \beta_X \leq \alpha_X < 1$. Then Bourgain [1] showed the estimate for $p = 1$ and $q = 2$ with $C_{1,2} = 3$; later Lépingle and Yor (see [7]) managed to decrease this constant to 2. See also the recent note by Qiu [9] for the noncommutative counterpart of this result. Delbaen and Schachermayer needed in [3] the version of (1.1) with $p = 2$ and $q = \infty$, and

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in [4] they proved the estimate for $1 \leq p \leq q \leq \infty$ with $C_{p,q} = 2$ (in fact the proof yields $C_{p,q} = 2^{1/p}$). The constant $2^{1/p}$, for $p \in \{1, \infty\}$ and $q = \infty$, turns out to be the best possible. The author identified in [8] the value of $C_{p,\infty}$ for $1 < p < \infty$: it is equal to $1 + (p-1)^{p-1}/p^p$. The case $q = 1$ is strictly related to estimates for martingale conditional square function. It was studied by Burkholder [2], Garsia in [5] and Wang [11].

The contribution of this paper is to provide the optimal values of the constants $C_{1,q}$ for $1 \leq q < \infty$. Here is our main result.

Theorem 1.1. *For any* $1 \leq q < \infty$ *we have*

$$||g||_{1,q} \leq 2^{(q-1)/q} \, ||f||_{1,q} \,. \tag{1.2}$$

The inequality is sharp.

Of course, this result remains valid if we allow the sequence f to take values in a separable Banach space $\mathbb{B}$ (with an appropriate modification of $||f||_{p,q}$: we need to replace $|f_n|$ with $||f_n||_{\mathbb{B}}$, the norm of f_n in $\mathbb{B}$). This follows at once from the fact that the passage from $(f_n)_{n\geq 0}$ to $(||f_n||_{\mathbb{B}})_{n\geq 0}$ does not change the (p,q)-norm of the sequence and does not decrease the (p,q)-norm of the projection.

The proof of the inequality (1.2) will rest on the existence of a certain special function. Our approach is novel and can be regarded as a version of the so-called Bellman function method (or Burkholder's method), a powerful technique which has been successfully applied in numerous problems in probability and analysis. We do not know whether this approach can be used to identify the optimal constants $C_{p,q}$ in the full range of exponents.

2. Proof of Theorem 1.1

2.1. **A special function and its properties.** Let $1 \leq q < \infty$ be fixed. Introduce a function $u : [0, \infty) \times [0, \infty) \to \mathbb{R}$, given by the formula

$$u(x,y) = \begin{cases} -(x^q - y^q)^{1/q} & \text{if } x \geq 2^{1/q} y, \\ y - 2^{(q-1)/q} x & \text{if } x < 2^{1/q} y. \end{cases}$$

It is easy to see that first-order partial derivatives of u exist and are continuous on $(0,\infty) \times (0,\infty)$. The next three lemmas are devoted to certain pointwise estimates for u.

Lemma 2.1. *For any* x, $y \geq 0$ *we have the majorization*

$$u(x,y) \geq y - 2^{(q-1)/q} x. \tag{2.1}$$

Proof. Clearly, it is enough to show the claim for $x \geq 2^{1/q} y$. By homogeneity, we may assume that $y = 1$; then the bound can be rewritten in the equivalent form

$$F(x) := 1 - 2^{(q-1)/q} x + (x^q - 1)^{1/q} \leq 0$$

for $x \geq 2^{1/q}$. However, one easily checks that $F(2^{1/q}) = F'(2^{1/q}+) = 0$ (where $F'(2^{1/q}+)$ denotes the right-hand derivative of F at the point $2^{1/q}$). In addition, a direct differentiation shows that F is concave on $[2^{1/q}, \infty)$. This proves the claim. □

Lemma 2.2. *Let* x, $y \geq 0$. *Then the function* $\psi(s) = u((x^q + s)^{1/q}, (y^q + s)^{1/q})$ *is nonincreasing on* $[0, \infty)$.

Proof. We will consider two cases.

The case $x \leq 2^{1/q}y$. Then for each s we have

$$(x^q + s)^{1/q} \leq 2^{1/q}(y^q + s)^{1/q} \tag{2.2}$$

(this is evident: it suffices to rise both sides to the power q and do some trivial manipulations) and hence $\psi(s) = (y^q + s)^{1/q} - 2^{(q-1)/q}(x^q + s)^{1/q}$. So,

$$\psi'(s) = q^{-1}\left[(y^q + s)^{1/q-1} - 2^{(q-1)/q}(x^q + s)^{1/q-1}\right] \leq 0,$$

where the latter bound is again due to (2.2).

The case $x > 2^{1/q}y$. For such x and y, we have $(x^q + s)^{1/q} > 2^{1/q}(y^q + s)^{1/q}$ when $s < x^q - 2y^q$ and $(x^q+s)^{1/q} \leq 2^{1/q}(y^q+s)^{1/q}$ for $s \geq x^q - 2y^q$. Consequently,

$$\psi(s) = \begin{cases} -(x^q - y^q)^{1/q} & \text{if } s < x^q - 2y^q, \\ (y^q + s)^{1/q} - 2^{(q-1)/q}(x^q + s)^{1/q} & \text{if } s \geq x^q - 2y^q. \end{cases}$$

So, ψ is constant on $[0, x^q - 2y^q]$ and nonincreasing on $[x^q - 2y^q, \infty)$, where the latter fact is proved word-by-word as in the preceding case. □

The main property of u is described by the following concavity-type condition.

Lemma 2.3. *For any* x, $y \geq 0$ *and any simple random variable* $d \geq 0$*, we have*

$$\mathbb{E}u\left((x^q + d^q)^{1/q}, (y^q + (\mathbb{E}d)^q)^{1/q}\right) \leq u(x, y). \tag{2.3}$$

Proof. Consider the C^1 function

$$\xi(s) = u\left((x^q + s^q)^{1/q}, (y^q + (\mathbb{E}d)^q)^{1/q}\right), \qquad s \geq 0,$$

and let ζ stand for the smallest concave majorant of ξ. Clearly, the left-hand side of (2.3) can be bounded from above by $\zeta(\mathbb{E}d)$, and in what follows, we will find the explicit formulas for ζ, depending on the relations between x, y and $\mathbb{E}d$. We will need the following elementary fact: if $A \geq 0$, then the function $s \mapsto (s^q + A)^{1/q}$ is convex on $[0, \infty)$; if $A < 0$, then the function $s \mapsto (s^q + A)^{1/q}$ is concave on $[(-A)^{1/q}, \infty)$.

For the sake of convenience and clarity, we consider two cases separately.

The case $x^q \geq y^q + (\mathbb{E}d)^q$. For such x, y and d, we have

$$\xi(s) = \begin{cases} -(x^q - y^q - (\mathbb{E}d)^q + s^q)^{1/q} & \text{if } s^q > 2y^q + 2(\mathbb{E}d)^q - x^q, \\ (y^q + (\mathbb{E}d)^q)^{1/q} - 2^{(q-1)/q}(x^q + s^q)^{1/q} & \text{if } s^q \leq 2y^q + 2(\mathbb{E}d)^q - x^q. \end{cases}$$

Using the above fact concerning the convexity/concavity of the function $s \mapsto (s^q + A)^{1/q}$, we see that the function ξ is concave and hence $\zeta = \xi$. Consequently, the left-hand side of (2.3) does not exceed $u\left((x^q + (\mathbb{E}d)^q)^{1/q}, (y^q + (\mathbb{E}d)^q)^{1/q}\right)$, which, by Lemma 2.2, is not bigger than $u(x, y)$.

The case $x^q < y^q + (\mathbb{E}d)^q$. For these x, y and d, the function ξ has the same explicit formula as above, but its behavior is slightly different. Namely, it is concave on $[0, (2y^q + 2(\mathbb{E}d)^q - x^q)^{1/q}]$ and convex on $[(2y^q + 2(\mathbb{E}d)^q - x^q)^{1/q}, \infty)$. To find the least concave majorant ζ, observe that $\lim_{s\to\infty} \xi'(s) = -1$; furthermore, a little calculation shows that $x \leq (2y^q + 2(\mathbb{E}d)^q - x^q)^{1/q}$ and $\xi'(x) = -1$. Consequently, ζ is given by

$$\zeta(s) = \begin{cases} (y^q + (\mathbb{E}d)^q)^{1/q} - 2^{(q-1)/q}(x^q + s^q)^{1/q} & \text{if } s \leq x, \\ (y^q + (\mathbb{E}d)^q)^{1/q} - x - s & \text{if } s > x. \end{cases}$$

Therefore, if $\mathbb{E}d \leq x$, then the left-hand side of (2.3) does not exceed

$$(y^q + (\mathbb{E}d)^q)^{1/q} - 2^{(q-1)/q}(x^q + (\mathbb{E}d)^q)^{1/q} = u\left((x^q + (\mathbb{E}d)^q)^{1/q}, (y^q + (\mathbb{E}d)^q)^{1/q}\right),$$

which is not larger than $u(x, y)$ by means of Lemma 2.2. On the other hand, if $\mathbb{E}d > x$, then

$$\mathbb{E}u\left((x^q + d^q)^{1/q}, (y^q + (\mathbb{E}d)^q)^{1/q}\right) \leq (y^q + (\mathbb{E}d)^q)^{1/q} - x - \mathbb{E}d.$$

However, the right-hand side increases if we decrease $\mathbb{E}d$ to x; consequently,

$$(y^q + (\mathbb{E}d)^q)^{1/q} - x - \mathbb{E}d \leq (y^q + x^q)^{1/q} - 2x = u((x^q + x^q)^{1/q}, (y^q + x^q)^{1/q}),$$

which is not bigger than $u(x, y)$ by Lemma 2.2. This completes the proof. □

Remark 2.1. *The lemma above implies that for any $s \geq 0$ we have*

$$u(s, s) \leq u(0, 0) = 0. \tag{2.4}$$

To see this, it suffices to apply (2.3) *with $x = y = 0$ and $d \equiv s$.*

2.2. **Proof of Theorem 1.1.** We split the reasoning into two parts.

Proof of (1.2). For any sequence $a = (a_n)_{n \geq 0}$ and any $q \geq 1$, define the operator

$$S_n(a) = S_n^{(q)}(a) = \left(\sum_{k=0}^{n} |a_k|^q\right)^{1/q}.$$

Fix f, g as in the statement; with no loss of generality, we may and do assume that f is nonnegative, passing from $(f_n)_{n \geq 0}$ to $(|f_n|)_{n \geq 0}$, if necessary. The crucial property is that for such f and g, the sequence $\big(u(S_n(f), S_n(g))\big)_{n \geq 0}$ is a supermartingale. To see this, pick an arbitrary $n \geq 0$ and write

$$u(S_{n+1}(f), S_{n+1}(g)) = u\left((S_n^q(f) + f_{n+1}^q)^{1/q}, (S_n^q(g) + g_{n+1}^q)^{1/q}\right).$$

It suffices to apply the inequality (2.3), conditionally with respect to $\mathcal{F}_n$, with $x = S_n(f)$, $y = S_n(g)$ and $d = f_{n+1}$, to establish the aforementioned supermartingale property. Hence, by the majorization (2.1) and then (2.4), we may write

$$\mathbb{E}S_n(g) - 2^{(q-1)/q}\mathbb{E}S_n(f) \leq \mathbb{E}u(S_n(f), S_n(g)) \leq \mathbb{E}u(S_0(f), S_0(g)) = \mathbb{E}u(f_0, f_0) \leq 0.$$

Thus, the claim follows by letting $n \to \infty$ and applying Lebesgue's monotone convergence theorem. □

Sharpness. Assume that the underlying probability space is the interval $[0, 1)$ equipped with its Borel subsets and Lebesgue measure. We will construct a family of appropriate examples. Let r be an arbitrary number belonging to $(0, 1)$ and let $a < (1-r)^{-1}$ be a fixed number. If we take r sufficiently close to 1 and a sufficiently close to $(1-r)^{-1}$, we may guarantee that a satisfies the additional lower bound $a^q > 2$, which will be needed below. Define $f_0 \equiv 1$ and, for $n = 1, 2, \ldots$,

$$f_n(\omega) = \begin{cases} a^n & \text{if } \omega \in [0, (1-r)^n), \\ a^{n-1} & \text{if } \omega \in [(1-r)^n, (1-r)^{n-1}), \\ 0 & \text{otherwise.} \end{cases}$$

Let $(\mathcal{F}_n)_{n\geq 0}$ be the filtration generated by $(f_n)_{n\geq 0}$. Then $g_0 = f_0 \equiv 1$ and, directly from the above formula, we have that if $\omega \in [0, (1-r)^{n-1})$, then

$$g_n(\omega) = \frac{a^n(1-r)^n + a^{n-1}[(1-r)^{n-1} - (1-r)^n]}{(1-r)^{n-1}} = a^{n-1}(r + (1-r)a).$$

Consequently,

$$\begin{aligned} ||g||_{L^1(\ell^q)} &\geq \mathbb{E}\left(\sum_{n=1}^\infty g_n^q 1_{[(1-r)^n,(1-r)^{n-1})}\right)^{1/q} \\ &= \mathbb{E}\sum_{n=1}^\infty g_n 1_{[(1-r)^n,(1-r)^{n-1})} \\ &= r(r+(1-r)a)\sum_{n=1}^\infty a^{n-1}(1-r)^{n-1} = \frac{r(r+(1-r)a)}{1-a(1-r)}. \end{aligned}$$

On the other hand, for $\omega \in [(1-r)^n, (1-r)^{n-1})$ we have

$$||f||_{\ell^q}(\omega) = \Big(1 + a^q + a^{2q} + \ldots + a^{(n-1)q} + a^{(n-1)q}\Big)^{1/q} \leq (2a^{(n-2)q} + 2a^{(n-1)q})^{1/q}.$$

The latter bound is evident for $n = 1$, while for $n \geq 2$ we note that

$$1 + a^q + a^{2q} + \ldots + a^{(n-2)q} = \frac{a^{(n-1)q} - 1}{a^q - 1} \leq \frac{a^{(n-1)q} + a^{(n-2)q}(a^q - 2)}{a^q - 1} = 2a^{(n-2)q}.$$

Therefore, we get

$$||f||_{L^1(\ell^q)} \leq r\sum_{n=1}^\infty (1-r)^{n-1}(2a^{(n-2)q} + 2a^{(n-1)q})^{1/q} = \frac{r(2a^{-q}+2)^{1/q}}{1-a(1-r)},$$

which implies that

$$\frac{||g||_{L^1(\ell^q)}}{||f||_{L^1(\ell^q)}} \geq \frac{r + (1-r)a}{(2a^{-q}+2)^{1/q}}.$$

However, if we take r sufficiently close to 1 and then a sufficiently close to $(1-r)^{-1}$ (which in particular means that a is huge), then the ratio on the right can be made arbitrarily close to $2^{1-1/q}$. This shows that the constant in (1.2) is indeed the best possible. □

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