Sharp norm inequalities for martingales and their differential subordinates

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Abstract

Suppose $f = (f_n)$, $g = (g_n)$ are martingales with respect to the same filtration, satisfying

$$|f_n - f_{n-1}| \leq |g_n - g_{n-1}|, \qquad n = 1, 2, \ldots$$

with probability 1. Under some assumption on f_0, g_0 and an additional condition that one of the processes is nonnegative, some sharp inequalities between the p-th norms of f and g, $0 < p < \infty$, are established. As an application, related sharp inequalities for stochastic integrals and harmonic functions are shown to hold.

Discipline: Probability Theory and Harmonic Analysis.

1 Introduction

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space equipped with a discrete filtration $(\mathcal{F}_n)_{n\geq 0}$. Let $f = (f_n)$, $g = (g_n)$ be two adapted martingales taking values in a certain separable Hilbert space $\mathcal{H}$, with

$$f_n = \sum_{k=0}^{n} df_k, \quad g_n = \sum_{k=0}^{n} dg_k.$$

We say that f is differentially subordinate to g, if for any nonnegative n we have

$$|df_n| \leq |dg_n|$$

almost surely.

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The main interest of this paper is to compare the moments of f and g, if f is differentially subordinate to g. As proved by Burkholder in [1], for $1 < p < \infty$ we have the following sharp estimate

$$||f_n||_p \leq \alpha_p ||g_n||_p, \quad n = 0,\ 1,\ 2,\ \ldots, \tag{1.1}$$

where $\alpha_p = \max\{p, p/(p-1)\} - 1$. Furthermore, if $0 < p \leq 1$, the inequality fails to hold for any finite α_p.

But what happens if we add an extra assumption that one of the martingales f, g is nonnegative? This question was raised and answered by Burkholder in [4] in the case $g \geq 0$. Namely, (1.1) holds for $1 < p < \infty$ and the optimal constant equals

$$\alpha_p' = \begin{cases} 1/(p-1) & \text{if } p \in (1,2], \\ p^{1/p}[(p-1)/2]^{(p-1)/p} & \text{if } p \in (2,\infty). \end{cases}$$

Hence the constant remains the same for $1 < p \leq 2$ and decreases for $p > 2$.

We continue this line of research in two directions. The inequality (1.1) fails to hold if $p \in (0,1)$ and $g \geq 0$. But it turns out that the reverse one is true, if the differential subordination is replaced by a slightly different condition.

Theorem 1.1. *Suppose f is a martingale taking values in $\mathcal{H}$ and g is a nonnegative martingale. Assume that for some deterministic $\beta > 0$ we have*

$$\beta |f_0| \geq g_0 \quad \text{and} \quad |df_n| \leq |dg_n|, \quad n = 1,\ 2,\ \ldots,$$

with probability 1. *Then for $p \in (0,1)$,*

$$||f_n||_p \geq C_{p,\beta} ||g_n||_p, \quad n = 0,\ 1,\ 2\ \ldots, \tag{1.2}$$

where $C_{p,\beta} = 0$ if $\beta \geq 1$ and

$$C_{p,\beta} = \left[\left(\frac{p(1-\beta)}{2(1+\beta-p)} \right)^{1-p} \cdot \frac{2(1+\beta)(1-p)+p^2}{p(1+\beta-p)} \right]^{1/p}$$

if $\beta < 1$. The inequality is sharp if $2\beta > p$.

By sharpness we mean that for any $C > C_{p,\beta}$, there exists a pair (f, g) satisfying the assumptions of the theorem and an integer n for which we have $||f_n||_p < C||g_n||_p$.

The second result we obtain is the following.

Theorem 1.2. *Suppose g is $\mathcal{H}$-valued martingale and f is nonnegative and differentially subordinate to g. Then for $0 < p < \infty$,*

$$||f_n||_p \leq C_p ||g_n||_p, \quad n = 0,\ 1,\ 2\ \ldots, \tag{1.3}$$

where

$$C_p = \begin{cases} \infty & \text{if } p \in (0,1) \\ 1 & \text{if } p = 1, \\ p^{-1/p}[2/(p-1)]^{(p-1)/p} & \text{if } p \in (1,2), \\ p-1 & \text{if } p \in [2,\infty). \end{cases}$$

The inequality is sharp.

Therefore, compared to the general case, the constant decreases for $p \in [1,2)$.

Let us comment upon the method of the proof. In [1] (see also [2]) Burkholder proves the inequality (1.1) for general f, g constructing quite complicated special function U_p satisfying some convex-type properties. It turns out that a certain integration trick is available, which enables to build U_p from much simpler functions and to reduce significantly the complexity of the proof (cf. [5]). In [4], the proof of the inequality (1.1) for nonnegative g follows the same pattern and the special function U_p' is even more complicated than U_p. In this paper we discover integral identity which expresses U_p' in terms of much simpler objects. Related identities yield special functions leading to the inequalities (1.2) and (1.3).

The paper is organized as follows. In the next section we introduce the simple special functions, study their properties and present the crucial integral identities. Section 3 contains the proof of Theorems 1.1 and 1.2. The last two sections are devoted to applications of these theorems to stochastic integrals and harmonic functions on Euclidean domains.

2 The special functions

For a fixed number $s > 1$, consider a set D given by

$$D = \{(x,y) \in \mathbb{R}_+^2 : y \leq \min\big(x+1, \frac{s+1}{s-1} - x\big)\}.$$

Define functions $u_{1,s} : \mathcal{H} \times \mathbb{R}_+ \to \mathbb{R}$, $u_{2,s} : \mathbb{R}_+ \times \mathcal{H} \to \mathbb{R}$, $u_{\infty,s} : \mathcal{H} \times \mathbb{R}_+ \to \mathbb{R}$ by

$$u_{1,s}(x,y) = \begin{cases} \frac{s-1}{s+1}(|x|^2 - y^2) - \frac{2}{s+1}|x| + \frac{2s}{s+1}y & \text{if } (|x|,y) \in D, \\ 1 & \text{if } (|x|,y) \notin D, \end{cases}$$

$$u_{2,s}(x,y) = \begin{cases} \frac{s-1}{s+1}(x^2 - |y|^2) & \text{if } (x,|y|) \in D, \\ \frac{2}{s+1}x - \frac{2s}{s+1}|y| + 1 & \text{if } (x,|y|) \notin D, \end{cases}$$

$$u_{\infty,s}(x,y) = \begin{cases} 0 & \text{if } (y,|x|) \in D, \\ \frac{s-1}{s+1}(|x|^2 - y^2) + \frac{2}{s+1}y - \frac{2s}{s+1}|x| + 1 & \text{if } (y,|x|) \notin D. \end{cases}$$

It is easy to check that these functions are continuous. Furthermore, let $\phi_{1,s}$, $\psi_{1,s}$, $\phi_{2,s}$, $\psi_{2,s}$, $\phi_{\infty,s}$, $\psi_{\infty,s}$ be defined by

$$(\phi_{1,s}(x,y),\psi_{1,s}(x,y)) = \begin{cases} \big(\frac{2(s-1)}{s+1}x - \frac{2}{s+1}x', -\frac{2(s-1)}{s+1}y + \frac{2s}{s+1}\big) & \text{if } (|x|,y) \in D, \\ (0,0) & \text{if } (|x|,y) \in D, \end{cases}$$

$$(\phi_{2,s}(x,y),\psi_{2,s}(x,y)) = \begin{cases} \big(\frac{2(s-1)}{s+1}x, -\frac{2(s-1)}{s+1}y\big) & \text{if } (x,|y|) \in D, \\ \big(\frac{2}{s+1}, -\frac{2s}{s+1}y'\big) & \text{if } (x,|y|) \in D, \end{cases}$$

$$(\phi_{\infty,s}(x,y),\psi_{\infty,s}(x,y)) = \begin{cases} (0,0) & \text{if } (y,|x|) \in D, \\ \big(\frac{2(s-1)}{s+1}x - \frac{2s}{s+1}x', -\frac{2(s-1)}{s+1}y + \frac{2}{s+1}\big) & \text{if } (y,|x|) \in D, \end{cases}$$

where $x' = x/|x|$ for $x \neq 0$ and $x' = 0$ if $x = 0$.

The key properties of the functions above are described in the following lemma.

Lemma 2.1. *Let $s > 1$ be a fixed number.*

(i) We have

$$u_{1,s}(x,y) \leq 1, \tag{2.1}$$

$$u_{2,s}(x,y) \leq \frac{2}{s+1}x - \frac{2s}{s+1}|y| + 1, \tag{2.2}$$

$$u_{\infty,s}(x,y) \leq \frac{s-1}{s+1}(|x|^2 - y^2) + \frac{2}{s+1}y - \frac{2s}{s+1}|x| + 1. \tag{2.3}$$

(ii) Suppose $x,\ h \in \mathcal{H}$, $y,\ y+k \geq 0$ and $|h| \leq |k|$. Then

$$u_{1,s}(x+h,y+k) \leq u_{1,s}(x,y) + \phi_{1,s}(x,y)\cdot h + \psi_{1,s}(x,y)k, \tag{2.4}$$

$$u_{\infty,s}(x+h,y+k) \leq u_{\infty,s}(x,y) + \phi_{\infty,s}(x,y)\cdot h + \psi_{\infty,s}(x,y)k. \tag{2.5}$$

Suppose $x,\ x+h \geq 0$, $y,\ k \in \mathcal{H}$ and $|h| \leq |k|$. Then

$$u_{2,s}(x+h,y+k) \leq u_{2,s}(x,y) + \phi_{2,s}(x,y)h + \psi_{2,s}(x,y)\cdot k. \tag{2.6}$$

Proof. (i) It is easy to see that the inequalities (2.1), (2.2), (2.3) are equivalent and therefore it suffices to prove the first one. To this end, note that for $(|x|,y) \in D$ the partial derivative of $u_{1,s}$ with respect to y equals

$$\frac{2(s-1)}{s+1}\Big(\frac{s}{s-1} - y\Big) \geq 0$$

and the inequality follows by the continuity of $u_{1,s}$.

(ii) This is done by a well-known procedure (cf. [2], [3], [4]). Consider a function

$$G_{1,s}(t) = u_{1,s}(x+th, y+tk),$$

defined on $\{t : y + tk \geq 0\}$. The inequality (2.4) is equivalent to

$$G_{1,s}(1) \leq G_{1,s}(0) + G'_{1,s}(0)$$

(with $(G_{1,s})'_-(0)$, $(G_{1,s})'_+(0)$ or 0 instead of $G'_{1,s}(0)$ if the latter does not exist) and will follow once we have established the concavity of $G_{1,s}$. Consider the sets

$$E_{1,s} = \{t : (|x+th|, y+tk) \notin D\}, \qquad F_{1,s} = \{t : (|x+th|, y+tk) \in D\}. \quad (2.7)$$

On $E_{1,s}$ we have $G_{1,s} \equiv 1$, which is clearly concave, while on $F_{1,s}$, $G_{1,s}(t)$ equals

$$\frac{s-1}{s+1}(|h|^2 - k^2)t^2 + \frac{s-1}{s+1}[|x|^2 + 2tx \cdot h - y^2 - 2tyk] - \frac{2}{s+1}|x+th| + \frac{2s}{s+1}(y+tk)$$

and concavity follows from $|h|^2 \leq k^2$ and concavity of the function $t \mapsto -|x+th|$. It remains to note that $E_{1,s}$, $F_{1,s}$ are intervals and, by (2.1), $G(t) \leq 1$ on $F_{1,s}$.

For the functions $u_{2,s}$, $u_{\infty,s}$ the argument is essentially the same; we introduce the functions $G_{2,s}$ and $G_{\infty,s}$ in the similar manner and reduce the proof of (2.5), (2.6) to the concavity of these functions. The concavity is clear on the sets $E_{2,s}$, $F_{2,s}$ and $E_{\infty,s}$, $F_{\infty,s}$, defined as in (2.7), and the inequality for one-sided derivatives follows from (2.2), (2.3). The sets $E_{2,s}$, $E_{\infty,s}$ may happen to be a sum of two intervals, but this does not change the argument. $\square$

Now let us introduce the special functions corresponding to the moment inequalites. For $p \in (0,1)$, $x \in \mathcal{H}$, $y \geq 0$, let

$$U_{p,s}(x,y) = \frac{p(1-p)(2-p)(s+1)}{2} \int_0^\infty t^{p-1} u_{1,s}(x/t, y/t) dt, \quad (2.8)$$

while for $p \in (1,2)$, $x \geq 0$, $y \in \mathcal{H}$,

$$U_{p,s}(x,y) = \frac{p(p-1)(2-p)(s+1)}{2} \int_0^\infty t^{p-1} u_{2,s}(x/t, y/t) dt. \quad (2.9)$$

Finally, for $p \in (2,\infty)$, $x \in \mathcal{H}$, $y \geq 0$, set

$$U_{p,s}(x,y) = \frac{p(p-1)(p-2)(s+1)}{2} \int_0^\infty t^{p-1} u_{\infty,s}(x/t, y/t) dt. \quad (2.10)$$

The formulas for $U_{p,s}$ are as follows. Suppose $p \in (0,1)$. If $y \leq s|x|$, then

$$U_{p,s}(x,y) = \left(\frac{s-1}{s+1}\right)^{p-1} (|x|+y)^{p-1} \big[y(s-1+p) + |x|(s-sp-1)\big],$$

while for $y \geq s|x|$,

$$U_{p,s}(x,y) = (y-|x|)^{p-1} \big[y(s+1-p) + |x|(sp-s-1)\big].$$

In case $p \in (1,2)$, if $|y| \leq sx$, then

$$U_{p,s}(x,y) = \left(\frac{s-1}{s+1}\right)^{p-1} (x+|y|)^{p-1}\big[|y|(-s-p+1) + x(sp-s+1)\big],$$

while for $|y| \geq sx$,

$$U_{p,s}(x,y) = (|y|-x)^{p-1}\big[|y|(p-s-1) + x(s-sp+1)\big].$$

Finally, let $p \in (2,\infty)$. Then, if $sy \leq |x|$,

$$U_{p,s}(x,y) = (|x|-y)^{p-1}\big[y(sp-s-1) + |x|(s-p+1)\big]$$

and for $sy \geq |x|$,

$$U_{p,s}(x,y) = \left(\frac{s-1}{s+1}\right)^{p-1} (|x|+y)^{p-1}\big[y(s-ps-1) + |x|(s+p-1)\big].$$

The following functions will also play a role. If $p \in (0,1)$ and $s > 1$, let $V_{p,s} : \mathcal{H} \times \mathbb{R}_+ \to \mathbb{R}$ be given by

$$V_{p,s}(x,y) = (s+1-p)\big[y^p - K_{p,s}|x|^p\big]$$

and for $p \in (1,2)$, $s > 1$, define $V_{p,s} : \mathbb{R}_+ \times \mathcal{H} \to \mathbb{R}$ by

$$V_{p,s}(x,y) = (s+1-p)\big[-|y|^p + K_{p,s}x^p\big].$$

Here

$$K_{p,s} = \left(\frac{s-1}{2}\right)^{p-1} \cdot \frac{p}{s+1-p}.$$

We will need the following fact about the functions defined above.

Lemma 2.2. *Suppose $p \in (0,2)$, $p \neq 1$ and $s > 1$. Then*

$$U_{p,s} \geq V_{p,s}. \tag{2.11}$$

Proof. It suffices to prove the inequality in the special case $\mathcal{H} = \mathbb{R}$. Consider the functions F, $G : (0,1) \to \mathbb{R}$ given by

$$F(t) = V_{p,s}(t, 1-t), \quad G(t) = U_{p,s}(t, 1-t).$$

The function F is convex on $(0,t_0)$ and concave on $(t_0,1)$ for some $t_0 \in (0,1)$, while G is concave on $(0,(s+1)^{-1})$ and linear on $((s+1)^{-1},1)$. Moreover,

$$F(0) = G(0),\ F'(0) < G'(0),\ F\Big(\frac{2}{s+1}\Big) = G\Big(\frac{2}{s+1}\Big) \text{ and } F'\Big(\frac{2}{s+1}\Big) = G'\Big(\frac{2}{s+1}\Big).$$

Thus $F \leq G$, which yields (2.11) by homogeneity. □

Remark 2.1. If $x = 0$ or $2|y| = (s-1)|x|$, then we have $U_{p,s}(x,y) = V_{p,s}(x,y)$. This is a consequence of $F(0) = G(0)$ and $F(2/(s+1)) = G(2/(s+1))$.

3 The proofs of the theorems

The inequalities (2.4), (2.5), (2.6) yield the following estimates.

Lemma 3.1. *Let $s > 1$ and suppose f, g are martingales satisfying*

$$|df_n| \leq |dg_n| \quad n = 1,\ 2,\ \ldots$$

with probability 1.

(i) Suppose f is $\mathcal{H}$-valued and g is nonnegative. Then

$$\mathbb{E}u_{1,s}(f_n, g_n) \leq \mathbb{E}u_{1,s}(f_0, g_0), \qquad n = 0,\ 1,\ 2,\ \ldots. \tag{3.1}$$

(ii) Suppose f is $\mathcal{H}$-valued and g is nonnegative. Furthermore, assume that both f and g are square integrable. Then

$$\mathbb{E}u_{\infty,s}(f_n, g_n) \leq \mathbb{E}u_{\infty,s}(f_0, g_0), \quad n = 0,\ 1,\ 2,\ \ldots. \tag{3.2}$$

(iii) Suppose f is nonnegative and g is $\mathcal{H}$-valued. Then

$$\mathbb{E}u_{2,s}(f_n, g_n) \leq \mathbb{E}u_{2,s}(f_0, g_0), \qquad n = 0,\ 1,\ 2,\ \ldots. \tag{3.3}$$

Proof. We will only prove (i), the remaining statements can be established in the same manner. It suffices to show that for any $1 \leq k \leq n$,

$$\mathbb{E}u_{1,s}(f_k, g_k) \leq \mathbb{E}u_{1,s}(f_{k-1}, g_{k-1}). \tag{3.4}$$

Since $|df_k| \leq |dg_k|$ almost surely, the inequality (2.4) gives

$$u_{1,s}(f_k, g_k) \leq u_{1,s}(f_{k-1}, g_{k-1}) + \phi_{1,s}(f_{k-1}, g_{k-1}) \cdot df_k + \psi_{1,s}(f_{k-1}, g_{k-1}) dg_k.$$

Both sides of the inequality above are integrable; taking the conditional expectation with respect to $\mathcal{F}_{k-1}$ gives

$$\mathbb{E}[u_{1,s}(f_k, g_k)|\mathcal{F}_{k-1}] \leq u_{1,s}(f_{k-1}, g_{k-1}).$$

This implies (3.4) and completes the proof. $\square$

Proof. of the inequality (1.2). If $\beta \geq 1$, then $C_{p,\beta} = 0$ and the inequality is trivial. Assume that $\beta < 1$. The identity (2.8) together with Lemmas 2.2 and 3.1 yield

$$(s + 1 - p)\mathbb{E}[g_n^p - K_{p,s}|f_n|^p] = \mathbb{E}V_{p,s}(f_n, g_n) \leq \mathbb{E}U_{p,s}(f_n, g_n) \leq \mathbb{E}U_{p,s}(f_0, g_0) \tag{3.5}$$

for any n. Now set

$$s = \frac{1 + \beta - \beta p}{1 + \beta - p} > 1.$$

Then $\mathbb{E}U_{p,s}(f_0, g_0) \leq 0$, which follows from the fact that for $x \in \mathcal{H}$, $y \in \mathbb{R}_+$ satisfying $\beta|x| \geq y$ we have

$$U_{p,s}(x, y) \leq U_{p,s}(x, \beta|x|) = c[\beta(s - 1 + p) + s - sp - 1] = 0,$$

for a certain nonnegative c. To complete the proof, note that $K_{p,s} = C_{p,\beta}^{-p}$. $\square$

Proof. of the inequality (1.3). It suffices to prove the inequality for $p \in (1, 2)$, as for $p \le 1$ it is trivial and for $p \ge 2$ it holds for general f, g. We proceed as previously. The identity (2.9), Lemmas 2.2 and 3.1 give

$$(s+1-p)\mathbb{E}[-|g_n|^p + K_{p,s} f_n^p] = \mathbb{E}V_{p,s}(f_n, g_n) \le \mathbb{E}U_{p,s}(f_n, g_n) \le \mathbb{E}U_{p,s}(f_0, g_0) \quad (3.6)$$

for any n. Now the choice $s = p$ implies $\mathbb{E}U_{p,s}(f_0, g_0) \le 0$, since $U_{p,p}(x, y) \le 0$ if $x \le |y|$. All that is left is to observe that $C_p^{-p} = K_{p,p}$. $\square$

Remark 3.1. For $p > 2$, the function $U_{p,p}$ can be used to establish the inequality (1.1) for $\mathcal{H}$-valued f differentially subordinate to $g \ge 0$ (with the optimal constant α'_p). In [4], Burkholder uses a slightly different function

$$U'_p(x, y) = \begin{cases} U_{p,p}(x, y) & \text{if } (p-1)y \le 2|x|, \\ p\left(\frac{p-1}{2}\right)^{p-1}|y|^p - |x|^p & \text{if } (p-1)y \ge 2|x| \end{cases}$$

and proves $\mathbb{E}U'_p(f_n, g_n) \le \mathbb{E}U'_p(f_0, g_0) \le 0$ by showing an inequality analogous to (2.4)–(2.6). Our approach (identity (2.10)) enables to avoid technical computations.

Remark 3.2. The inequalities (3.5), (3.6) can be used to obtain variations of (1.2), (1.3), involving the initial variables f_0, g_0. For example, assume that f is $\mathcal{H}$-valued and differentially subordinate to a nonnegative g with $|f_0| = g_0$. If $0 < p < 1$, then (3.5) yields

$$\mathbb{E}g_n^p \le \frac{(s-1)^{p-1}}{s+1-p}\Big[\frac{p}{2^{p-1}}\mathbb{E}|f_n|^p + \frac{2^{p-1}(s-1)(2-p)}{(s+1)^{p-1}}\mathbb{E}|f_0|^p\Big]$$

for any $s > 1$. Take $s \to \infty$ to obtain

$$||g_n||_p \le 2\left(1 - \frac{p}{2}\right)^{1/p} ||f_0||_p.$$

Sharpness. This will be shown in a few steps. Assume $\mathcal{H} = \mathbb{R}$.

Step 1. Let us consider the following process, a modification of the one used by Burkholder in [4]. Let $s > 1$, $\delta \in (0, 1)$ be fixed and set

$$x_n = \left(1 + \frac{2\delta}{s-1}\right)^n, \quad p_n = \left[\frac{(1-\delta)(s-1)}{(1+\delta)(s-1+2\delta)}\right]^n$$

for $n = 0, 1, 2, \ldots$. Consider a Markov chain $H = H(p, s, \delta)$ with values in $\mathbb{R}_+^2$, starting from $(1, s)$, such that for $n = 0, 1, 2 \ldots$,

$$\mathbb{P}(H_{2n+1} = (x_n(1-\delta), x_n(s+\delta)) | H_{2n} = (x_n, sx_n)) = \frac{1}{1+\delta},$$

$$\mathbb{P}(H_{2n+1} = (2x_n, (s-1)x_n) | H_{2n} = (x_n, sx_n)) = \frac{\delta}{1+\delta},$$

$$\mathbb{P}(H_{2n+2} = (0, x_n(s-1+2\delta))|H_{2n+1} = (x_n(1-\delta), x_n(s+\delta))) = \frac{\delta(s+1)}{s-1+2\delta},$$

$$\mathbb{P}(H_{2n+2} = (x_{n+1}, sx_{n+1})|H_{2n+1} = (x_n(1-\delta), x_n(s+\delta))) = \frac{(1-\delta)(s-1)}{s-1+2\delta}$$

with the further condition that all the states lying on the lines $2y = (s-1)x$ and $x = 0$ are absorbing. Then the processes $F = F(p, s, \delta)$, $G = G(p, s, \delta)$, defined by $H_n = (F_n, G_n)$, are martingales such that for $n \geq 1$, $dF_n = \pm dG_n$.

Step 2. Now we will show that the sequence $(\mathbb{E}U_{p,s}(H_n))_{n\geq 0}$ is almost constant. For any nonnegative integer n, let $A_n = \{H_{n+1} \neq H_n\}$. Note that

$$A_{2n} = \{H_{2n} = (x_n, sx_n)\}, \ A_{2n+1} = \{H_{2n+1} = (x_n(1-\delta), x_n(s+\delta))\}.$$

Lemma 3.2. *Let n be a nonnegative integer.*

(i) We have $\mathbb{P}(A_{2n}) = p_n$.

(ii) The following equalities hold true.

$$\mathbb{E}U_{p,s}(H_{2n+2}) = \mathbb{E}U_{p,s}(H_{2n+1}), \tag{3.7}$$

$$\mathbb{E}U_{p,s}(H_{2n+1}) = \mathbb{E}U_{p,s}(H_{2n}) - x_n^p R(\delta) \cdot \mathbb{P}(A_{2n}), \tag{3.8}$$

for some function $R = R_{p,s} : \mathbb{R}_+ \to \mathbb{R}_+$ satisfying $R_{p,s}(\delta)/\delta \to 0$ as $\delta \to 0$.

Proof. (i) We have $P(A_0) = 1 = p_0$ and $\mathbb{P}(A_{2k}|A_{2k-2}) = p_1$ for any $k \geq 1$.

(ii) On the set A_{2n+1}, the variable H_{2n+2} takes values

$$(0, x_n(s-1+2\delta)) \text{ and } (x_{n+1}, sx_{n+1}) = \Big(x_n(1+\frac{2\delta}{s-1}), x_n(s-1+2\delta)+x_n(1+\frac{2\delta}{s-1})\Big).$$

But the function $t \mapsto U_{p,s}(t, x_n(s-1+2\delta)+t)$ is linear on $[0, x_n(1+2\delta/(s-1))]$; this proves the first estimate. For the second one, the argument is similar: on A_{2n},

$$H_{2n+1} \in \{(2x_n, (s-1)x_n), \ (x_n(1-\delta), x_n(s+\delta))\}, \ H_{2n} = (x_n, sx_n)$$

and the function $t \mapsto U_{p,s}(x_n + t, sx_n - t)$ has a continuous derivative on $(-\delta, x_n)$ and is linear on $[0, x_n]$. It remains to use the fact that $U_{p,s}$ is homogeneous of order p to get the special form of the remainder. □

Step 3. Let us study the following estimate.

$$\mathbb{E}V_{p,s}(H_{2n}) + \varepsilon\mathbb{E}F_{2n}^p \geq \mathbb{E}U_{p,s}(H_0). \tag{3.9}$$

Lemma 3.3. *Let $\varepsilon > 0$ be fixed.*

(i) Suppose $p \in (0,1)$ and $s > 1$. Then there exists $\delta > 0$ such that the inequality (3.9) holds for large n.

(ii) Suppose $p \in (1,2)$. Then there exist $s < p$ and $\delta > 0$ such that the inequality (3.9) holds for large n.

Proof. Outside A_{2n}, the variable H_{2n} takes values on one of the lines $2y=(s-1)x$, $x=0$. Since $U_{p,s}$, $V_{p,s}$ coincide on these lines, we have, by Lemma 3.2,

$$\begin{aligned}\mathbb{E}V_{p,s}(H_{2n}) &= \mathbb{E}U_{p,s}(H_{2n})+\mathbb{P}(A_{2n})[V_{p,s}(x_n,sx_n)-U_{p,s}(x_n,sx_n)]\\ &= \mathbb{E}U_{p,s}(H_0)-R(\delta)\sum_{k=0}^{n-1}x_k^pp_k-c\cdot x_n^pp_n,\end{aligned}\tag{3.10}$$

where $c=-V_{p,s}(1,s)+U_{p,s}(1,s)\geq 0$.

On the other hand, we have

$$\mathbb{E}F_{2n}^p\geq\sum_{k=0}^{n-1}(2x_n)^p\cdot\frac{p_n\delta}{1+\delta}\geq 2^{-1}\delta\sum_{k=0}^{n-1}x_n^pp_n=2^{-1}\delta\sum_{k=0}^{n-1}r^k,\tag{3.11}$$

where

$$r=r(\delta)=x_1^pp_1=\Big(1+\frac{2\delta}{s-1}\Big)^{p-1}\cdot\frac{1-\delta}{1+\delta}.\tag{3.12}$$

(i) Fix $\varepsilon>0$, $p\in(0,1)$ and $s>1$. By (3.11), there exists δ such that

$$R(\delta)\sum_{k=0}^{n-1}x_k^pp_k\leq\frac{2R(\delta)}{\delta}\mathbb{E}F_{2n}^p\leq\frac{\varepsilon}{2}\mathbb{E}F_{2n}^p,\tag{3.13}$$

for any n. Furthermore, since $p<1$, we have $r(\delta)<1$; hence $cx_n^pp_n=cr^n\leq\varepsilon\delta/4<2^{-1}\varepsilon\mathbb{E}F_{2n}^p$ for large n. Combining this estimate with (3.10) and (3.13) yields (3.9).

(ii) Fix $\varepsilon>0$ and $p\in(1,2)$. We have $r'(0)=2(p-s)/(s-1)$, so there exists $s\in(1,p)$ and $\delta(\varepsilon)$ such that if $\delta\in(0,\delta(\varepsilon))$, then $1<r(\delta)<1+\varepsilon\delta/8c$. Then, by (3.11),

$$cx_n^pp_n=cr^n\leq c\Big[\frac{2(r-1)}{\delta}\mathbb{E}F_{2n}^p+1\Big]\leq\frac{\varepsilon}{4}\mathbb{E}F_{2n}^p+1<\frac{\varepsilon}{2}\mathbb{E}F_{2n}^p,$$

if n is large enough; the latter inequality follows from $\mathbb{E}F_{2n}^p\to\infty$ as $n\to\infty$. We conclude the proof by the observation that (3.13) holds for sufficiently small δ, and applying (3.10). □

Step 4: the sharpness of (1.2). Let $\beta\in(p/2,1)$, $\delta>0$, $\varepsilon>0$ and set

$$s=\frac{1+\beta-\beta p}{1+\beta-p}>1,\quad a=\frac{2\beta-s+1}{1+\beta}<1.$$

The inequality $p<2\beta$ implies $a>0$. Consider martingales $F=(F_n)_{n\geq-1}$, $G=(G_n)_{n\geq-1}$ satisfying

(I) $F_{-1}=2-a$, $G_{-1}=a+s-1$ almost surely,

(II) $\mathbb{P}((F_0,G_0)=(1,s))=a=1-\mathbb{P}\big((F_0,G_0)=(2,s-1)\big)$,

(III) on $\{F_0=2\}$, the process (F_n,G_n) is constant,

(IV) on $\{F_0=1\}$, the conditional distribution of the process (F_n,G_n) is the distribution of $H(p,s\delta)$ constructed in the Step 1.

By the choice of a, we have $\beta F_{-1}=G_{-1}$ and $\mathbb{E}U_{p,s}(F_0,G_0)=0$. Clearly,

$$\mathbb{E}V_{p,s}(F_{2n},G_{2n})=\mathbb{E}V_{p,s}(F_{2n},G_{2n})\chi_{\{F_0=1\}}+\mathbb{E}V_{p,s}(F_{2n},G_{2n})\chi_{\{F_0=2\}}.$$

On the set $\{F_0=1\}$ we can use Lemma 3.3: a proper choice of δ and n implies

$$\mathbb{E}V_{p,s}(F_{2n},G_{2n})\chi_{\{F_0=1\}}+\varepsilon\mathbb{E}F_{2n}^p\chi_{\{F_0=1\}}\geq \mathbb{E}U_{p,s}(F_0,G_0)\chi_{\{F_0=1\}}.$$

On the set $\{F_0=2\}$ the pair $(F_{2n},G_{2n})=(F_0,G_0)$ lies on the line $2y=(s-1)x$, which implies $V_{p,s}(F_{2n},G_{2n})=U_{p,s}(F_0,G_0)$. Combining these two facts we get

$$\mathbb{E}V_{p,s}(F_{2n},G_{2n})+\varepsilon\mathbb{E}F_{2n}^p\geq \mathbb{E}U_{p,s}(F_0,G_0), \tag{3.14}$$

so

$$\mathbb{E}G_{2n}^p\geq\Big(C_{p,\beta}^{-p}-\frac{\varepsilon}{s+1-p}\Big)\mathbb{E}F_{2n}^p>(C_{p,\beta}^{-p}-\varepsilon)\mathbb{E}F_{2n}^p.$$

This proves that (1.2) is sharp. For the case $\beta\geq 1$, observe that $C_{p,\beta}$ is nonincreasing as a function of β and $C_{p,\beta}\to 0$ as $\beta\uparrow 1$.

Step 5: the sharpness of (1.3). The cases $p\leq 1$, $p=2$ are trivial; for $p\geq 2$, we use the example on page 669 of [1]. The only case left is $p\in(1,2)$.

For $\varepsilon>0$, let $s\in(1,p)$ and $\delta>0$ be the numbers guaranteed by Lemma 3.3. Consider martingales F, G satisfying (I) – (IV) with $a=(3-s)/2$. Using similar arguments as above, (3.9) leads to the inequality (3.14), valid for large n. Since $\mathbb{E}F_{2n}^p\to\infty$, we have $\mathbb{E}U_{p,s}(F_0,G_0)\geq-\varepsilon\mathbb{E}F_{2n}^p$ for large n, which combined with (3.14) implies

$$\mathbb{E}G_{2n}^p\leq(K_{p,s}+\frac{2\varepsilon}{s+p-1})\mathbb{E}F_{2n}^p<\big(C_p^{-p}+2\varepsilon\big)\mathbb{E}F_{2n}^p.$$

Therefore C_p is the best possible in (1.3).

4 Sharp inequalities for stochastic integrals

Suppose $X=(X_t)_{t\geq 0}$ is a martingale on a complete probability space $(\Omega,\mathcal{F},\mathbb{P})$, which is filtered by a nondecreasing right-continuous family $(\mathcal{F}_t)_{t\geq 0}$ of sub-σ-fields of $\mathcal{F}$. In addition, assume that $\mathcal{F}_0$ contains all the events of probability 0. Let Y be the Itô integral of H with respect to X, where H is a predictable process:

$$Y_t=H_0X_0+\int_{(0,t]}H_s dX_s.$$

The continuous-time versions of Theorems 1.1, 1.2 are stated below.

Theorem 4.1. *Suppose $p \in (0,1)$, X is nonnegative and for any $t > 0$, the variable H_t takes values in a closed unit ball of $\mathcal{H}$. If $\beta > 0$ satisfies $\mathbb{P}(\beta|H_0| \geq 1) = 1$, then for any $t > 0$,*

$$||Y_t||_p \geq C_{p,\beta}||X_t||_p \tag{4.1}$$

and the inequality is sharp if $p < 2\beta$.

Theorem 4.2. *Suppose $p \in (0,\infty)$, X is nonnegative and H takes values outside the open unit ball of $\mathcal{H}$. Then for any $t > 0$,*

$$||X_t||_p \leq C_p||Y_t||_p \tag{4.2}$$

and the inequality is sharp.

The proof of the inequalities (4.1), (4.2) follow from (1.2), (1.3) by discretizing argument; see [3], where an analogous submartingale inequality follows from the corresponding discrete-time version. The sharpness follows from the fact that the constants $C_{p,\beta}$, C_p are the best possible in (1.2), (1.3) in the case when f is a transform of g.

5 Inequalities for harmonic functions

In this section we study harmonic extensions of inequalities (1.2), (1.3). Let N be a fixed positive integer and D be an open connected subset of $\mathbb{R}^N$. Fix $\xi \in D$ and consider two harmonic functions u, v on D, taking values in certain Hilbert spaces $\mathcal{H}$, $\mathcal{K}$. Suppose u is differentially subordinate to v, that is

$$|\nabla u| \leq |\nabla v| \quad \text{ on } D.$$

Let D_0 be a bounded subdomain of D with $\xi \in D_0 \subset D_0 \cup \partial D_0 \subset D$. Let $\mu^\xi_{D_0}$ stand for the harmonic measure on ∂D_0 with respect to ξ and

$$||u||_{D_0,p} = \Big[\int_{\partial D_0} |u(z)|^p \mu^\xi_{D_0}(dz)\Big]^{1/p}, \quad 0 < p < \infty.$$

The norm inequalities for smooth functions can be stated as follows.

Theorem 5.1. *Let u, v, D_0 be as above.*

(i) Assume that $p \in (0,1)$ and v is nonnegative. Then

$$||u||_{D_0,p} \geq C_{p,\beta}||v||_{D_0,p}, \tag{5.1}$$

where $\beta = v(\xi)/|u(\xi)|$.

(ii) Assume that $p \in (0,\infty)$, u is nonnegative and $u(\xi) \leq |v(\xi)|$. Then

$$||u||_{D_0,p} \leq C_p||v||_{D_0,p}. \tag{5.2}$$

Proof. We will prove only the first part, the second one can be established similarly. As $C_{p,\beta} = 0$ for $\beta \geq 1$, we may assume that $\beta < 1$. Let

$$s = \frac{1+\beta-\beta p}{1+\beta-p} > 1.$$

It is easy to check that the function $u_{1,s}(u,v)$ is superharmonic. Therefore

$$\int_{D_0} u_{1,s}(u(z),v(z))\mu^{\xi}_{D_0}(dz) \leq u_{1,s}(u(\xi),v(\xi)).$$

Applying the identity (2.8) we obtain

$$\int_{D_0} U_{p,s}(u(z),v(z))\mu^{\xi}_{D_0}(dz) \leq U_{p,s}(u(\xi),v(\xi)) = 0,$$

since $\beta|u(\xi)| = v(\xi)$. It suffices to use the inequality (2.11) to get (5.1). □

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