Sharp L_p estimates for paraproducts on general measure spaces

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Abstract. The paper contains the identification of the L_p-norms of paraproducts, defined on general measure spaces equipped with a dyadic-like structure. The proof exploits the Bellman function method.

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1. Introduction

The purpose of this paper is to study the L_p-boundedness of paraproducts on arbitrary measure spaces equipped with a dyadic-like structure. To present the motivation from the appropriate perspective, we start with the classical, one-dimensional dyadic context. Let $\mathcal{D}$ denote the class of all dyadic intervals contained in $\mathbb{R}$ and, for each integer n, let $\mathcal{D}^{(n)}$ stand for the collection of all elements of $\mathcal{D}$ whose length is equal to 2^{-n}. Given $I \in \mathcal{D}$ and a locally integrable function f on $\mathbb{R}$, we define the associated average by

$$\langle f \rangle_Q = \frac{1}{|Q|} \int_Q f \mathrm{d}x.$$

Then, for each $n \in \mathbb{Z}$, the expectation operator $\mathbb{E}_n$ and the difference operator Δ_n are given by

$$\mathbb{E}_n f = \sum_{Q \in \mathcal{D}^n} \langle f \rangle_Q \chi_Q \qquad \text{and} \qquad \Delta_n f = \mathbb{E}_n f - \mathbb{E}_{n-1} f.$$

Given two locally integrable functions f and b on $\mathbb{R}$, we define the corresponding dyadic paraproduct $\pi_b f$ as the bilinear form

$$\pi_b f = \sum_{n \in \mathbb{Z}} \mathbb{E}_{n-1} f \cdot \Delta_n b.$$

It is easy to check formally that we have the identity

$$bf = \pi_b f + \pi_f b + \sum_{n\in\mathbb{Z}} \Delta_n b \cdot \Delta_n f, \tag{1.1}$$

which explains why the term "paraproduct" is used: $\pi_b f$ can be thought of as a half of the product of b and f. This concept appeared, in the appropriate continuous form, in the context of nonlinear equations (cf. Bony [1]), but probably its most important feature is the deep connection to the theory of singular integral operators. Roughly speaking, T(1) theorem asserts that a large class of singular integral operators T can be decomposed into a part S which is a convolution operator and two additional paraproduct-like terms.

In particular, the above decomposition gives rise to the question about the L_p-boundedness of paraproducts in the range $1 < p < \infty$. As a warm up, motivated by the identity (1.1), one might study this problem first for the usual product operator $f \mapsto bf$. But here the answer is trivial: for a given b, the product is L_p-bounded if and only if $b \in L_\infty$. It turns out that for the paraproducts, the class of admissible b is larger: π_b is bounded on L_p if and only if b belongs to BMO, the class of functions of bounded mean oscillation. The latter amounts to saying that

$$\|b\|_{BMO} = \sup_{I\in\mathcal{D}} \left\langle (b - \langle b\rangle_I)^2 \right\rangle_I^{1/2} < \infty.$$

There is a natural question about the extension of this boundedness property of paraproducts to the wider context in which the real line equipped with the dyadic lattice is replaced with an arbitrary measure space admitting a certain dyadic-like structure. Such a passage from a regular to a non-homogeneous setting has gained a lot of interest in the recent literature and has been studied, for example, in the weighted theory, maximal operators, sparse operators, square functions, Carleson embedding theorems, and many more. See e.g. [2, 4, 3, 6, 7]. We should also emphasize that such generalizations are meaningful from the probabilistic point of view: they amount to extending results valid for regular martingales to general processes.

We continue with the specification of the necessary background and notation. We assume that $(X, \mathcal{F}, \mu)$ is an arbitrary measure space, equipped with the *tree* (or *dyadic-like*) structure $\mathcal{T}$. Namely, we have $\mathcal{T} = \bigcup_{n\in\mathbb{Z}} \mathcal{T}^{(n)}$, where $(\mathcal{T}^{(n)})_{n\in\mathbb{Z}}$ is an increasing family of partitions of X into $\mathcal{F}$-measurable sets of positive and finite measure. Clearly, this concept generalizes the notion of a dyadic lattice (which corresponds to the choice $\mathcal{T} = \mathcal{D}$ and $\mathcal{T}^{(n)} = \mathcal{D}^{(n)}$). It has also a well-established meaning in the probability theory: if $\mu(X) = 1$, then there is a one-to-one correspondence between tree structures $\mathcal{T}$ and atomic filtrations $(\mathcal{F}_n)_{n\in\mathbb{Z}}$ of X, expressed via the identity $\mathcal{F}_n = \sigma(\mathcal{T}^{(n)})$.

For any $(X, \mathcal{F}, \mu)$ and $\mathcal{T}$ as above, one defines the averages, expectation operator and the difference operator using similar formulas as previously. Let $f : X \to \mathbb{R}$ be a $\mathcal{T}$-locally integrable function, i.e., satisfying $\int_Q |f| \mathrm{d}\mu < \infty$ for each $Q \in \mathcal{T}$. Given such an f, we set $\langle f\rangle_{Q,\mu} = \frac{1}{\mu(Q)} \int_Q f \mathrm{d}\mu$ and, for $n \in \mathbb{Z}$, we let $\mathbb{E}_n f = \sum_{Q\in\mathcal{T}^n} \langle f\rangle_{Q,\mu} \chi_Q$ and $\Delta_n f = \mathbb{E}_n f - \mathbb{E}_{n-1} f$. We will also need to work with the associated maximal operator $\mathcal{M}$ given by $\mathcal{M}f = \sup_{n\in\mathbb{Z}} |\mathbb{E}_n f|$, as well as its truncated version: $\mathcal{M}_N f = \max_{n\leq N} |\mathbb{E}_n f|$ for $N \in \mathbb{Z}$. Finally, for any $\mathcal{T}$-locally integrable functions b and f, we define the associated paraproduct by

$$\pi_b f = \sum_{n\in\mathbb{Z}} \mathbb{E}_{n-1} f \cdot \Delta_n b.$$

In analogy to the dyadic context, the function b is said to belong to BMO, if

$$\|b\|_{BMO} = \sup_{Q\in\mathcal{T}} \left\langle (b - \langle b\rangle_{Q,\mu})^2 \right\rangle^{1/2} = \sup_{Q\in\mathcal{T}} \left(\langle b^2\rangle_{Q,\mu} - \langle b\rangle_{Q,\mu}^2 \right)^{1/2} < \infty.$$

In the language of expectation operators, we see that $\|b\|_{BMO} \leq c$ if and only if for any $n \in \mathbb{Z}$ we have the double pointwise estimate $0 \leq \mathbb{E}_n(b^2) - (\mathbb{E}_n b)^2 \leq c^2$, the lower bound being a simple consequence of Schwarz' inequality.

We will prove the following statement.

Theorem 1.1. *Let $1 < p \leq 2$. Then for any $b \in BMO$ and $f \in L_p$ we have*

$$\|\pi_b f\|_{L_p} \leq \frac{p}{p-1} \|b\|_{BMO} \|f\|_{L_p}. \tag{1.2}$$

For any p, the constant $p/(p-1)$ is the best possible: for any $\varepsilon > 0$, there exists a probability space with a tree structure $\mathcal{T}$ and two random variables $b \in BMO$, $f \in L_p$ such that

$$\|\pi_b f\|_{L_p} > \left(\frac{p}{p-1} - \varepsilon \right) \|b\|_{BMO} \|f\|_{L_p}.$$

Actually, as we will see, our approach will yield the sharp inequality

$$\|\pi_b f\|_{L_p} \leq \|b\|_{BMO} \|\mathcal{M} f\|_{L_p},$$

which immediately gives (1.2) by virtue of Doob's maximal estimate for martingales [5]. What might seem a little strange and unexpected, the estimate (1.2) does not hold for $p > 2$ with any finite constant. The reason for this phenomenon is that with no assumptions on the regularity of the tree, the BMO functions of integral zero might have arbitrarily large L_p norms.

The remaining part of the paper is organized as follows. The estimate (1.2) is established in the next section, with the use of the Bellman function method. The final part of the paper contains the construction of examples, which show the optimality of the constant $p/(p-1)$ and the failure of (1.2) in the range $p \in (2, \infty)$.

2. Proof of (1.2)

Throughout this section, $1 < p \leq 2$ is a number and $(X, \mathcal{F}, \mu)$ is a fixed measure space with a tree structure $\mathcal{T}$. Given a positive parameter α, we define the special function $B = B^{(\alpha)} : \mathbb{R}^3 \times [0, \infty) \to [0, \infty)$ by the formula

$$B(u, v, y, z) = \alpha z^p (u^2 - v + 1) + (4\alpha)^{-1} z^{2-p} y^2.$$

This function is a key ingredient of our argument, as we will see now.

Proof of (1.2). Suppose that $b \in BMO$ and $f \in L_p$, $g \in L_{p'}$ are fixed functions, where $p' = p/(p-1)$ is the conjugate to p. By homogeneity, we may and do assume that $\|b\|_{BMO} \leq 1$. Let us split the reasoning into two parts.

Step 1. First we prove that for any $n \in \mathbb{Z}$ we have

$$\begin{aligned} &\int_X B\left(\mathbb{E}_{n+1}b, \mathbb{E}_{n+1}(b^2), \mathbb{E}_{n+1}g, \mathcal{M}_n f\right) \mathrm{d}\mu \\ &\geq \int_X \left(B\left(\mathbb{E}_n b, \mathbb{E}_n(b^2), \mathbb{E}_n g, \mathcal{M}_{n-1}f\right) + \mathbb{E}_n f \cdot \Delta_{n+1}b \cdot \Delta_{n+1}g\right) \mathrm{d}\mu. \end{aligned} \tag{2.1}$$

To this end, we observe that

$$\begin{aligned} &\int_X \left[\left(\mathcal{M}_n f\right)^p \left((\mathbb{E}_{n+1}b)^2 - \mathbb{E}_{n+1}(b^2) + 1\right)\right] \mathrm{d}\mu \\ &= \int_X \left[\left(\mathcal{M}_n f\right)^p \left((\mathbb{E}_n b)^2 - \mathbb{E}_n(b^2) + 1\right)\right] \mathrm{d}\mu + \int_X \left[\left(\mathcal{M}_n f\right)^p (\Delta_{n+1}b)^2\right] \mathrm{d}\mu \\ &\quad + \int_X \left[\left(\mathcal{M}_n f\right)^p \left(2\mathbb{E}_n b \cdot \Delta_{n+1}b - \Delta_{n+1}(b^2)\right)\right] \mathrm{d}\mu. \end{aligned}$$

The latter term vanishes, since $\mathbb{E}_n\left(\Delta_{n+1}b\right) = \mathbb{E}_n\left(\Delta_{n+1}(b^2)\right) = 0$. Furthermore, the functions $\mathcal{M}_n f$ increase along with n and we have $(\mathbb{E}_n b)^2 - \mathbb{E}_n(b^2) + 1 \geq 0$, since $\|b\|_{BMO} \leq 1$. Putting all these facts together, we get

$$\begin{aligned} &\int_X \left[\left(\mathcal{M}_n f\right)^p \left((\mathbb{E}_{n+1}b)^2 - \mathbb{E}_{n+1}(b^2) + 1\right)\right] \mathrm{d}\mu \\ &\geq \int_X \left[\left(\mathcal{M}_{n-1} f\right)^p \left((\mathbb{E}_n b)^2 - \mathbb{E}_n(b^2) + 1\right) + \left(\mathcal{M}_n f\right)^p (\Delta_{n+1}b)^2\right] \mathrm{d}\mu. \end{aligned}$$

A similar argument gives

$$\begin{aligned} &\int_X \left[\left(\mathcal{M}_n f\right)^{2-p} (\mathbb{E}_{n+1}g)^2\right] \mathrm{d}\mu \\ &\geq \int_X \left[\left(\mathcal{M}_{n-1} f\right)^{2-p} (\mathbb{E}_n g)^2 + \left(\mathcal{M}_n f\right)^{2-p} (\Delta_{n+1}g)^2\right] \mathrm{d}\mu. \end{aligned}$$

Combining the above estimates we get

$$\begin{aligned} &\int_X B\left(\mathbb{E}_{n+1}b, \mathbb{E}_{n+1}(b^2), \mathbb{E}_{n+1}g, \mathcal{M}_n f\right) \mathrm{d}\mu \\ &\quad \geq \int_X B\left(\mathbb{E}_n b, \mathbb{E}_n(b^2), \mathbb{E}_n g, \mathcal{M}_{n-1}f\right) \mathrm{d}\mu \\ &\qquad + \int_X \left(\alpha\left(\mathcal{M}_n f\right)^p (\Delta_{n+1}b)^2 + (4\alpha)^{-1}\left(\mathcal{M}_n f\right)^{2-p} (\Delta_{n+1}g)^2\right) \mathrm{d}\mu \end{aligned}$$

and (2.1) follows. We actually get the stronger estimate, in which the last term in (2.1) is replaced by $\mathbb{E}\Big(\mathcal{M}_n f \cdot |\Delta_{n+1}b| \cdot |\Delta_{n+1}g|\Big)$.

Step 2. Observe that for any $\beta > 0$ and any $(u, v, y, z) \in \mathbb{R}^3 \times [0, \infty)$ satisfying $u^2 \leq v$ we have

$$0 \leq B(u, v, y, z) \leq \alpha z^p + (4\alpha)^{-1} z^{2-p} y^2$$
$$\leq \left(\alpha + (4\alpha)^{-1} \frac{2-p}{p} \beta^{p/(2-p)}\right) z^p + (4\alpha)^{-1} \frac{2p-2}{p} \beta^{-p'/2} |y|^{p'}.$$

Indeed, the first and the second estimates are obvious, while the third is a simple consequence of Young's inequality. Therefore, by (2.1), we get that for any $K < N$,

$$\int_X \left(\sum_{n=K+1}^{N} \mathbb{E}_{n-1} f \cdot \Delta_n b\right) g \mathrm{d}\mu$$
$$= \int_X \sum_{n=K+1}^{N} \mathbb{E}_{n-1} f \cdot \Delta_n b \cdot \Delta_n g \mathrm{d}\mu$$
$$\leq \int_X B\big(\mathbb{E}_K b, \mathbb{E}_K(b^2), \mathbb{E}_K g, |\mathbb{E}_K f|\big) \mathrm{d}\mu + \int_X \sum_{n=K+1}^{N} \mathbb{E}_{n-1} f \cdot \Delta_n b \cdot \Delta_n g \mathrm{d}\mu$$
$$\leq \int_X B\left(\mathbb{E}_N b, \mathbb{E}_N(b^2), \mathbb{E}_N g, \mathcal{M}_{N-1} f\right) \mathrm{d}\mu$$
$$\leq \left(\alpha + (4\alpha)^{-1} \frac{2-p}{p} \beta^{p/(2-p)}\right) \|\mathcal{M}_{N-1} f\|_{L_p}^p + (4\alpha)^{-1} \frac{2p-2}{p} \beta^{-p'/2} \|\mathbb{E}_N g\|_{L_{p'}}^{p'}$$
$$\leq \left(\alpha + (4\alpha)^{-1} \frac{2-p}{p} \beta^{p/(2-p)}\right) \|\mathcal{M} f\|_{L_p}^p + (4\alpha)^{-1} \frac{2p-2}{p} \beta^{-p'/2} \|g\|_{L_{p'}}^{p'}.$$

Now we optimize over β: plugging $\beta = \left(\mathbb{E}|g|^{p'} / \mathbb{E}(\mathcal{M} f)^p\right)^{2(p-1)(2-p)/p^2}$ we obtain

$$\mathbb{E}\left(\sum_{n=K+1}^{N} \mathbb{E}_{n-1} f \cdot \Delta_n b\right) g \leq \alpha \|\mathcal{M} f\|_{L_p}^p + (4\alpha)^{-1} \|\mathcal{M} f\|_{L_p}^{2-p} \|g\|_{L_{p'}}^2.$$

Now we optimize over α: setting $\alpha = \frac{1}{2}\left(\mathbb{E}|g|^{p'} / \mathbb{E}(\mathcal{M} f)^p\right)^{1/p'}$ gives

$$\mathbb{E}\left(\sum_{n=K+1}^{N} \mathbb{E}_{n-1} f \cdot \Delta_n b\right) g \leq \|\mathcal{M} f\|_{L_p} \|g\|_{L_{p'}}.$$

Since $g \in L_{p'}$ was arbitrary and we have the Doob inequality $\|\mathcal{M} f\|_{L_p} \leq \frac{p}{p-1} \|f\|_{L_p}$, this yields

$$\left\| \sum_{n=K+1}^{N} \mathbb{E}_{n-1} f \cdot \Delta_n b \right\|_{L_p} \leq \frac{p}{p-1} \|f\|_{L_p} \|b\|_{BMO}.$$

It remains to let $K \to -\infty$, $N \to \infty$ and use Fatou's lemma to get (1.2). $\square$

3. Extremal examples

3.1. Sharpness of (1.2) for $1 < p \leq 2$

Now we will prove that the constant $p/(p-1)$ appearing in (1.2) cannot be improved, by providing an appropriate example. Fix a small $\delta > 0$ and consider the space $X = [0,1)$ with the Lebesgue measure μ and the tree $\mathcal{T}$ given as follows: for $n \leq 0$ we have $\mathcal{T}^{(n)} = \{X\}$, while for positive n, $\mathcal{T}^{(n)}$ is

$$\Big\{ \big[0,(1-\delta)^n\big), \big[(1-\delta)^n,(1-\delta)^{n-1}\big), \big[(1-\delta)^{n-1},(1-\delta)^{n-2}\big), \ldots, \big[1-\delta,1\big) \Big\}.$$

Consider the functions $f,\, b : X \to \mathbb{R}$ defined by

$$f = \sum_{n=0}^{\infty} (1+a\delta)^n \chi_{[(1-\delta)^{n+1},(1-\delta)^n)}, \qquad b = \sum_{n=0}^{\infty} (-1)^n \chi_{[(1-\delta)^{n+1},(1-\delta)^n)},$$

where a is a fixed parameter belonging to $(0, p^{-1})$. Observe that

$$\int_X f^p \mathrm{d}\mu = \sum_{n=0}^{\infty} (1+a\delta)^{np}(1-\delta)^n \delta$$

is finite if δ is sufficiently small: indeed, the ratio of the geometric series equals $(1+a\delta)^p(1-\delta) = 1-(1-pa)\delta + o(\delta)$. Next, note the obvious estimate $\|b\|_{BMO} \leq \|b\|_\infty = 1$. It remains to study the L_p norm of the paraproduct $\pi_b f$. To this end, fix a nonnegative integer m and compute that

$$\langle f \rangle_{[0,(1-\delta)^m),\mu} = (1-\delta)^{-m} \sum_{n=m}^{\infty} (1+a\delta)^n (1-\delta)^n \delta = \frac{(1+a\delta)^m}{1-a+a\delta}$$

and similarly

$$\langle b \rangle_{[0,(1-\delta)^m),\mu} = (1-\delta)^{-m} \sum_{n=m}^{\infty} (-1)^n (1-\delta)^n \delta = \frac{(-1)^m \delta}{2-\delta}.$$

Consequently, on the set $[(1-\delta)^{m+1}, (1-\delta)^m)$ we have

$$f_n = \begin{cases} \dfrac{(1+a\delta)^n}{1-a+a\delta} & \text{if } 0 \leq n \leq m, \\ (1+a\delta)^n & \text{if } n > m, \end{cases} \qquad b_n = \begin{cases} \dfrac{(-1)^n \delta}{2-\delta} & \text{if } 0 \leq n \leq m, \\ (-1)^n & \text{if } n > m \end{cases} \tag{3.1}$$

and hence

$$db_n = \begin{cases} \dfrac{(-1)^n \cdot 2\delta}{2-\delta} & \text{if } 1 \leq n \leq m, \\ \dfrac{2\cdot(-1)^n}{2-\delta} & \text{if } n = m+1, \\ 0 & \text{if } n > m+1. \end{cases}$$

Let us shift b: set $\tilde{b} = b - \langle b \rangle_{[0,1)}$, so that b has integral zero. Then $\|\tilde{b}\|_{BMO} = \|b\|_{BMO} \leq 1$, $d\tilde{b}_n = 0$ for $n \leq 0$ and $d\tilde{b}_n = db_n$ for $n \geq 1$. Putting the above calculations together, we see that on the set $[(1-\delta)^{m+1},(1-\delta)^m)$ we have

$$\begin{aligned}\pi_{\tilde b} f &= \sum_{n=1}^{\infty} f_{n-1} d\tilde b_n \\ &= \sum_{n=1}^{m} f_{n-1} db_n + f_m db_{m+1} \\ &= \sum_{n=1}^{m} \frac{(1+a\delta)^{n-1}}{1-a+a\delta}\cdot\frac{(-1)^n\cdot 2\delta}{2-\delta} + \frac{(1+a\delta)^m}{1-a+a\delta}\cdot\frac{2(-1)^{m+1}}{2-\delta} \\ &= -\frac{2\delta(1-(-1)^m(1+a\delta)^m)}{(2-\delta)(1-a+a\delta)(2+a\delta)} + \frac{(1+a\delta)^m}{1-a+a\delta}\cdot\frac{2(-1)^{m+1}}{2-\delta}.\end{aligned}$$

Now let us analyze separately each term in the last line. We have

$$\left|-\frac{2\delta(1-(-1)^m(1+a\delta)^m)}{(2-\delta)(1-a+a\delta)(2+a\delta)}\right| \leq \frac{2\delta(1+a\delta)^m}{1-a+a\delta}$$

and

$$\left|\frac{(1+a\delta)^m}{1-a+a\delta}\cdot\frac{2(-1)^{m+1}}{2-\delta}\right| \geq \frac{(1+a\delta)^m}{1-a+a\delta}.$$

Plugging this into the preceding identity, we get

$$\left|\sum_{n=1}^{\infty} f_{n-1} db_n\right| \geq (1-2\delta)\cdot\frac{(1+a\delta)^m}{1-a+a\delta} = \frac{1-2\delta}{1-a+a\delta} f.$$

Now recall that f belongs to L_p if δ is chosen small enough. Therefore, the optimal constant in (1.2) cannot be smaller than $(1-2\delta)/(1-a+a\delta)$. The latter expression can be made arbitrarily close to $p/(p-1)$, by picking a sufficiently close to p^{-1} and then taking δ appropriately small. This establishes the desired sharpness.

3.2. The case $p>2$

Now we will prove the failure of (1.2) (with any finite constant) in the range $p>2$. Fix an arbitrary positive constant M. Consider the probability space $[0,1)$ with the tree $\mathcal{T}$ satisfying $\mathcal{T}^{(n)}=\{[0,1)\}$ for $n\leq 0$ and $\mathcal{T}^{(n)}=\{[0,(M^2+1)^{-1}),[(M^2+1)^{-1},1)\}$ for $n\geq 1$. Let b, $f:[0,1)\to\mathbb{R}$ be given by $b=M\chi_{[0,(M^2+1)^{-1})}-M^{-1}\chi_{[(M^2+1)^{-1},1)}$ and $f=\chi_{[0,1)}$. Since b is measurable with respect to the σ-algebra generated by $\mathcal{T}^{(1)}$, we have $\langle(b-\langle b\rangle_Q)^2\rangle_Q=0$, unless $Q=[0,1)$; furthermore, we easily see that $\langle b\rangle_{[0,1)}=0$ and thus

$$\|b\|_{BMO}^2=\langle b^2\rangle_{[0,1)}=\frac{M^2}{M^2+1}+M^{-2}\cdot\frac{M^2}{M^2+1}=1.$$

Consequently, $\|b\|_{BMO}\|f\|_{L_p}=1$; on the other hand, we have

$$\|\pi_b(f)\|_{L_p}=\|\Delta_1 b\|_{L_p}=\|b\|_{L_p}>\frac{M}{(M^2+1)^{1/p}}.$$

The latter expression can be made arbitrarily large. This shows that the estimate (1.2) cannot hold with any finite constant.

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