BEST CONSTANTS IN SOME ESTIMATES FOR THE HARMONIC MAXIMAL OPERATOR ON THE REAL LINE

ŁUKASZ KAMIŃSKI AND ADAM OSĘKOWSKI

ABSTRACT. The paper contains the proofs of strong-type, weak-type, Lorentz-norm and stability estimates for the harmonic maximal operator on the real line, associated with an arbitrary Borel measure. The constants obtained are optimal in the special case of the Lebesgue measure.

1. INTRODUCTION

The purpose of this paper is to provide some tight information about the behavior of harmonic maximal operator, which plays an important role in harmonic analysis. Let us start with the necessary definitions. Suppose that μ is a nonnegative Borel measure on $\mathbb{R}^d$ and assume that $f : \mathbb{R}^d \to \mathbb{R}$ is a Borel measurable function. The maximal harmonic function of f with respect to μ is defined by

$$\mathcal{M}_\mu f(x) = \sup \left(\frac{1}{\mu(Q)} \int_Q |f|^{-1} \mathrm{d}\mu \right)^{-1} \tag{1.1}$$

where the supremum is taken over all cubes Q containing x, with sides parallel to the axes. Here and below, we will use the convention $1/0 = +\infty$, $0 \cdot \infty = 0$ and $1/+\infty = 0$.

The harmonic maximal operator is a companion to the classical maximal operator M_μ of Hardy and Littlewood, which is given by

$$M_\mu f(x) = \sup_Q \frac{1}{\mu(Q)} \int_Q |f| \mathrm{d}\mu,$$

where the supremum is taken over the same class as above. In a sense, the joint behavior of $\mathcal{M}_\mu$ and M_μ is similar to that of the harmonic

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$$\frac{n}{|x_1|^{-1}+|x_2|^{-1}+\ldots+|x_n|^{-1}}, \qquad \frac{|x_1|+|x_2|+\ldots+|x_n|}{n},$$

where x_1, x_2, …, x_n are arbitrary real numbers. The notion of harmonic maximal operator appeared for the first time in [7, 9, 8], actually in a slightly different manner. Namely, the authors studied there another object, the so-called minimal operator

$$\mathfrak{M}_\mu f(x)=\inf\left\{\frac{1}{\mu(Q)}\int_Q |f|\mathrm{d}\mu\right\},$$

where the infimum is taken over the same class as before. This alternative operator is linked to $\mathcal{M}_\mu$ through the identity $\mathcal{M}_\mu f=\mathfrak{M}_\mu(|f|^{-1})^{-1}$. The minimal operator was used to study the fine structure of A_p weights in [7], further applications to weighted norm inequalities and differentiation theory can be found in [8, 9]. See also [11] for certain class of estimates in the weighted context.

From the viewpoint of the applications, it is of interest to investigate sharp, or at least tight estimates for $\mathcal{M}_\mu$. The purpose of this paper is to study this problem in the one-dimensional setting. Given $0<p<\infty$, let us distinguish two constants c_p and C_p: let

$$c_p=\frac{p}{(p+1)^{1+1/p}(2^{p/(p+1)}-1)(2-2^{p/(p+1)})^{1/p}}, \tag{1.2}$$

and define C_p as the unique number in $(1,\infty)$ satisfying

$$(p+1)C_p^p+1=pC_p^{p+1}. \tag{1.3}$$

There are two types of results studied in this paper. First, we will establish the following sharp weak-type, strong-type and Lorentz-norm estimates. Here and below, $L^p(\mu)$ and $L^{p,\infty}(\mu)$ stand for the usual L^p- and weak L^p-spaces associated with the measure μ.

Theorem 1.1. *For an arbitrary Borel measure μ on $\mathbb{R}$ and any $0<p<\infty$, we have*

$$\|\mathcal{M}_\mu\|_{L^p(\mu)\to L^p(\mu)}\leq C_p, \tag{1.4}$$

$$\|\mathcal{M}_\mu\|_{L^{p,\infty}(\mu)\to L^{p,\infty}(\mu)}\leq C_p \tag{1.5}$$

and

$$\|\mathcal{M}_\mu\|_{L^p(\mu)\to L^{p,\infty}(\mu)} \le c_p. \tag{1.6}$$

In general, the constants in the above estimates cannot be improved: for the Lebesgue measure, the equalities hold.

Let us emphasize here that the estimates (1.4) and (1.6), with *some* constants, can be extracted from [8] and [11]; the inequality (1.5) seems to be new. Our main contribution is the identification of the best constants involved.

The second class of results concerns the stability of the L^p estimate (1.4). As we have stated above, the constant C_p is the best possible; its optimality (for the Lebesgue measure) will be demonstrated by constructing appropriate functions ('extremals'). It turns out that given $0 < p < \infty$, such an extremal object is (asymptotically) an eigenfunction of $\mathcal{M}_{|\cdot|}$ corresponding to the eigenvalue C_p (that is, roughly speaking, the almost-equality $\|\mathcal{M}_\mu f\|_{L^p} \approx C_p\|f\|_{L^p}$ holds even pointwise). This gives rise to the following interesting subject, referred to as the stability: given f for which equality in (1.4) is almost attained, how far is f from being such an eigenfunction?

We will provide the following answer to this question.

Theorem 1.2. *Let $\varepsilon > 0$ and suppose that f is a measurable function on $\mathbb{R}$ such that $\|\mathcal{M}_\mu f\|_{L^p} \ge (C_p - \varepsilon)\|f\|_{L^p}$. Then we have*

$$\|\mathcal{M}_\mu f - C_p f\|_{L^p} \le \begin{cases} 2C_p^2\,\varepsilon^{1/2}\,\|f\|_{L^p} & \text{if } 0 < p < 2,\\ (2p)^{1/p}C_p\,\varepsilon^{1/p}\,\|f\|_{L^p} & \text{if } p \ge 2.\end{cases} \tag{1.7}$$

If μ is the Lebesgue measure, the factors $\varepsilon^{1/2}$ and $\varepsilon^{1/p}$ cannot be improved: the exponents $1/2$ for $0 < p < 2$ and $1/p$ for $p \ge 2$ are the biggest possible.

Of course, the inequality (1.7) is of interest when ε is small. This motivates the question about the largest possible power of ε allowed on the right-hand side: this is the decisive factor which actually controls the size of the upper bound. It might seem quite unexpected that the exponents behave differently for $p \le 2$ and $p \ge 2$. However, a similar phenomenon occurs in the context of martingale transforms and wide classes of Fourier multipliers [1]. For various examples of other stability results in geometry and spectral theory, we refer the reader to the work by Brasco and Philippis [3], as well as to Bianchi and Egnell [2], Chen, Frank and Weth [5], Christ [6], Dolbeault and Toscani [10], Fathi, Indrei and Ledoux [12], and the very recent paper of Carlen [4].

The remaining part of the paper is split into three sections. The first of them contains the proof of a certain distributional inequality for $\mathcal{M}_\mu$, which is in the spirit of Riesz' sunrise lemma [13] for harmonic maximal operators. Section 3 is devoted to the proof of Theorem 1.1. In the final part we address the stability of L^p estimates.

2. A distributional inequality

The purpose of this section is to establish a certain special estimate for $\mathcal{M}_\mu$, which will be the main building block for the proofs of the inequalities (1.4), (1.5) and (1.6). Here and below, we use the shortened notation $\{f > \lambda\}$ for the set $\{x \in \mathbb{R} : f(x) > \lambda\}$.

Proposition 2.1. *For any $\lambda > 0$ and any measurable function $f : \mathbb{R} \to [0,\infty)$, we have the inequality*

$$\mu(\{\mathcal{M}_\mu f > \lambda\}) + \mu(\{f > \lambda\}) \geq \lambda \int_{\{\mathcal{M}_\mu f > \lambda\}} f^{-1} d\mu + \lambda \int_{\{f>\lambda\}} f^{-1} d\mu. \tag{2.1}$$

Proof. Fix $\lambda > 0$ and let $E_\lambda = \{\mathcal{M}_\mu f > \lambda\}$. With no loss of generality, we may and do assume that $\mu(\{f > \lambda\}) < \infty$, since otherwise the left-hand side of (2.1) is infinite and the estimate is evident. By the very definition of the harmonic maximal operator, for every $x \in E_\lambda$ there is a closed interval I_x containing x such that

$$\left(\frac{1}{\mu(I_x)} \int_{I_x} f^{-1} d\mu\right)^{-1} > \lambda. \tag{2.2}$$

This inequality shows at once that I_x is automatically contained in E_λ; in other words, we can write $E_\lambda = \bigcup_{x \in E_\lambda} I_x$. The idea is to replace, or rather approximate this union (which in general is taken over an uncountable set) by the union of a finite number of the intervals, enjoying appropriate sparseness.

To this end, first we apply Lindelöf's lemma to get a countable subcollection $\{I_j\}_{j=1}^\infty$ of $\{I_x\}_{x\in E_\lambda}$ for which $\bigcup_{j=1}^\infty I_j = \bigcup_{x\in E_\lambda} I_x$ (the lemma is applicable, even though the sets I_x are not open; this follows from a simple approximation argument). Fix $N \in \mathbb{N}$, let $\mathcal{I} = \{I_j : j = 1, 2, \ldots, N\}$ and set $E^N = \bigcup_{I\in\mathcal{I}} I$. Now we apply a certain inductive procedure to obtain two subcollections $\mathcal{I}_1$ and $\mathcal{I}_2$ of $\mathcal{I}$. The algorithm can be described as follows.

1° Take $J_0 \in \mathcal{I}$ satisfying $\inf J_0 = \inf E^N$ and put into $\mathcal{I}_1$. (If there are two or more intervals $I \in \mathcal{I}$ with $\inf I = \inf E^N$, then take the interval with the biggest measure).

2° Suppose that we have successfully defined J_{2n}. Consider the family of all intervals $I \in \mathcal{I}$ which intersect J_{2n} and satisfy $\sup I > \sup J_{2n}$. If this family is nonempty, choose the interval with largest left-endpoint (if this object is not unique, pick the one with the biggest measure), denote this interval by J_{2n+1} and put it into $\mathcal{I}_2$.

3° If the family in 2° is empty, then consider all intervals $I \in \mathcal{I}$ with $\inf I > \sup J_{2n}$. If this family is nonempty, choose an element with the smallest left-endpoint (again, if this object is not unique, pick the one with the biggest measure). Denote it by J_{2n+1} and put it into $\mathcal{I}_2$.

4° Suppose that we have successfully defined J_{2n+1}. Consider the family of all elements $I \in \mathcal{I}$ which intersect J_{2n+1} and satisfy $\sup I > \sup J_{2n+1}$. If this family is nonempty, choose the interval with largest left-endpoint (if this object is not unique, pick the one with the biggest measure). Denote this interval by J_{2n+2} and put it into $\mathcal{I}_1$.

5° If the family in 4° is empty, then consider all intervals $I \in \mathcal{I}$ with $\inf I > \sup J_{2n+1}$. If this family is nonempty, choose an element with the smallest left-endpoint (again, if this object is not unique, pick the one with the biggest measure). Denote it by J_{2n+2} and put it into $\mathcal{I}_1$.

6° Go to 2°.

Since $\mathcal{I}$ is finite, the above procedure terminates after a finite number of steps. It is easy to check, using the maximality of the sets picked in 1°–5°, that if we define $F_i = \bigcup_{I\in\mathcal{I}_i} I$ for $i = 1, 2$, then $E^N = F_1 \cup F_2$. Furthermore, the requirement of considering the smallest/largest endpoints implies that among each class $\mathcal{I}_1$ and $\mathcal{I}_2$, the elements are pairwise disjoint. Therefore, by (2.2), we obtain

$$\mu(F_i) = \sum_{I\in\mathcal{I}_i} \mu(I) > \lambda \sum_{I\in\mathcal{I}_i} \int_I f^{-1}\mathrm{d}\mu = \lambda \int_{F_i} f^{-1}\mathrm{d}\mu \quad \text{for } i = 1, 2.$$

Consequently, we get

$$\begin{aligned} \mu(E^N) + \mu(F_1 \cap F_2) &= \mu(F_1) + \mu(F_2) \\ &> \lambda \int_{F_1} f^{-1}\mathrm{d}\mu + \lambda \int_{F_2} f^{-1}\mathrm{d}\mu \\ &= \lambda \int_{E^N} f^{-1}\mathrm{d}\mu + \lambda \int_{F_1\cap F_2} f^{-1}\mathrm{d}\mu. \end{aligned} \tag{2.3}$$

Now, observe that we have the estimate

$$\mu(F_1 \cap F_2) - \lambda \int_{F_1\cap F_2} f^{-1}\mathrm{d}\mu \le \mu(\{f > \lambda\}) - \lambda \int_{\{f>\lambda\}} f^{-1}\mathrm{d}\mu. \tag{2.4}$$

Indeed, note that

$$\int_{F_1\cap F_2} (1-\lambda f^{-1})\mathrm{d}\mu \le \int_{F_1\cap F_2\cap\{f\ge\lambda\}} (1-\lambda f^{-1})\mathrm{d}\mu \le \int_{\{f\ge\lambda\}} (1-\lambda f^{-1})\mathrm{d}\mu,$$

which is (2.4). Combining this inequality with (2.3) gives

$$\mu(E^N) + \mu(\{f > \lambda\}) \ge \lambda \int_{E^N} f^{-1}\mathrm{d}\mu + \lambda \int_{\{f>\lambda\}} f^{-1}\mathrm{d}\mu$$

Since $(E^N)_{N=1}^\infty$ is an increasing sequence of μ-measurable sets whose union is E_λ, the assertion follows by letting $N \to \infty$. □

3. Estimates for harmonic maximal operators

Now we will exploit the distribution estimate established in the previous section to obtain our main result.

3.1. **The strong-type estimate.**

Proof of (1.4). Let $f : \mathbb{R} \to \mathbb{R}$ be an arbitrary measurable function. Actually, we may restrict ourselves to nonnegative f, since the passage from f to $|f|$ does not change $\mathcal{M}_\mu f$ or $\|f\|_{L^p(\mu)}$. In addition, we may assume that $\|f\|_{L^p(\mu)} > 0$, since otherwise the claim is obvious. By

Theorem 2.1, we obtain

$$\begin{aligned}
&\int_{\mathbb{R}} (\mathcal{M}_\mu f)^p \mathrm{d}\mu + \int_{\mathbb{R}} f^p \mathrm{d}\mu \\
&\quad = p\int_0^\infty \lambda^{p-1}\mu(\{\mathcal{M}_\mu f > \lambda\})\,\mathrm{d}\lambda + p\int_0^\infty \lambda^{p-1}\mu(\{f > \lambda\})\,\mathrm{d}\lambda \\
&\quad \geq p\int_0^\infty \lambda^p \int_{\{\mathcal{M}_\mu f > \lambda\}} f^{-1}\mathrm{d}\mu\,\mathrm{d}\lambda + p\int_0^\infty \lambda^p \int_{\{f>\lambda\}} f^{-1}\mathrm{d}\mu\,\mathrm{d}\lambda \\
&\quad = \frac{p}{p+1}\int_{\mathbb{R}} (\mathcal{M}_\mu f)^{p+1} f^{-1}\,\mathrm{d}\mu + \frac{p}{p+1}\int_{\mathbb{R}} f^p\,\mathrm{d}\mu
\end{aligned}$$

(recall the convention $0\cdot\infty = 0$ under the first integral), which can be rewritten in the form

$$\int_{\mathbb{R}} (\mathcal{M}_\mu f)^p \mathrm{d}\mu + \frac{1}{p+1}\int_{\mathbb{R}} f^p \mathrm{d}\mu \geq \frac{p}{p+1}\int_{\mathbb{R}} (\mathcal{M}_\mu f)^{p+1} f^{-1}\,\mathrm{d}\mu. \tag{3.1}$$

However, the Hölder inequality implies

$$\left(\int_{\mathbb{R}} (\mathcal{M}_\mu f)^{p+1} f^{-1}\,\mathrm{d}\mu\right)^{\frac{p}{p+1}} \left(\int_{\mathbb{R}} f^p\,\mathrm{d}\mu\right)^{\frac{1}{p+1}} \geq \int_{\mathbb{R}} (\mathcal{M}_\mu f)^p\,\mathrm{d}\mu,$$

or equivalently,

$$\int_{\mathbb{R}} (\mathcal{M}_\mu f)^{p+1} f^{-1}\,\mathrm{d}\mu \geq \left(\int_{\mathbb{R}} (\mathcal{M}_\mu f)^p\,\mathrm{d}\mu\right)^{\frac{p+1}{p}} \left(\int_{\mathbb{R}} f^p\,\mathrm{d}\mu\right)^{-\frac{1}{p}}.$$

Plugging this into (3.1) yields

$$\int_{\mathbb{R}} (\mathcal{M}_\mu f)^p \mathrm{d}\mu + \frac{1}{p+1}\int_{\mathbb{R}} f^p \mathrm{d}\mu \geq \frac{p}{p+1}\left(\int_{\mathbb{R}} (\mathcal{M}_\mu f)^p\,\mathrm{d}\mu\right)^{\frac{p+1}{p}} \left(\int_{\mathbb{R}} f^p\,\mathrm{d}\mu\right)^{-\frac{1}{p}}.$$

In other words, the ratio $\|\mathcal{M}_\mu f\|_{L^p(\mu)}/\|f\|_{L^p(\mu)}$ satisfies the inequality

$$(p+1)\left(\frac{\|\mathcal{M}_\mu f\|_{L^p(\mu)}}{\|f\|_{L^p(\mu)}}\right)^p + 1 \geq p\left(\frac{\|\mathcal{M}_\mu f\|_{L^p(\mu)}}{\|f\|_{L^p(\mu)}}\right)^{p+1}$$

and hence it is not bigger than C_p. This is precisely the desired estimate $\|\mathcal{M}_\mu f\|_{L^p(\mu)} \leq C_p\|f\|_{L^p(\mu)}$. □

Sharpness of (1.4) *for the Lebesgue measure.* Fix an exponent p, an auxiliary parameter $\alpha > p$ and let C_α be given by (1.3). The function $f:\mathbb{R}\to[0,\infty)$, given by $f(x) = |x|^{-1/\alpha}\chi_{[-1,1]}(x)$, belongs to L^p: a direct integration gives $\|f\|_{L^p}^p = 2(1-p/\alpha)^{-1}$. Now we will show that

$$\mathcal{M}_{|\cdot|} f = C_\alpha f \qquad \text{on } \mathbb{R}. \tag{3.2}$$

This will yield the desired sharpness at once, since $C_\alpha \to C_p$ as $\alpha \downarrow p$.

The identity (3.2) is easy for $|x| > 1$: then f vanishes in some neighborhood of x, so $\mathcal{M}_{|\cdot|}f(x) = 0$ by the very definition of the harmonic maximal operator. One checks similarly that $\mathcal{M}f(0) = f(0) = +\infty$. So, suppose that $|x| \in [-1, 1] \setminus \{0\}$; actually, by symmetry, we may assume that $x \in (0, 1]$. Note that f is even, radially decreasing and continuous on $\mathbb{R} \setminus \{0\}$. Then it is easy to see that $\mathcal{M}_\mu f(x)$ can be achieved by looking at the interval $[-r_x, x]$, where $r_x \in (0, x]$ is such that

(3.3)
$$f(-r_x)^{-1} = \frac{1}{x + r_x}\int_{-r_x}^{x} f(s)^{-1}\mathrm{d}s = \frac{1}{x + r_x} \cdot \frac{\alpha}{\alpha + 1}\big(r_x^{1+1/\alpha} + x^{1+1/\alpha}\big).$$

For each $r \in (0, 1]$ we have

$$\frac{1}{x + rx}\int_{-rx}^{x} f(s)^{-1}\mathrm{d}s = x^{1/\alpha} \cdot \frac{1}{1 + r}\int_{-r}^{1} f(s)^{-1}\mathrm{d}s,$$

which implies $r_x = xr_1$ and $\mathcal{M}_\mu f(x) = x^{-1/\alpha}\mathcal{M}_\mu f(1)$. Finally, observe that (3.3) is satisfied with $x = 1$ and $r_x = C_\alpha^{-\alpha}$. Indeed, we have

$$(1 + C_\alpha^{-\alpha})(\alpha + 1)C_\alpha^{-1} = \alpha(1 + C_\alpha^{-(\alpha+1)}),$$

which is equivalent to (1.3) with α in place of p. Thus $\mathcal{M}_\mu f(x) = f(-r_x) = C_\alpha x^{-1/\alpha} = C_\alpha f(x)$. $\square$

Remark 3.1. We could follow a shorter path and simply write that

$$\mathcal{M}_{|\cdot|}f(x) \geq \left(\frac{1}{[-C_\alpha^{-\alpha}x, x]}\int_{-C_\alpha^{-\alpha}x}^{x}(f(s))^{-1}\mathrm{d}s\right)^{-1} = C_\alpha f(x)$$

for $x \geq 0$, with an analogous estimate for $x < 0$. This is sufficient for the bound $\|\mathcal{M}_{|\cdot|}f\|_{L^p} \geq C_\alpha\|f\|_{L^p}$, which gives the sharpness. However, the precise evaluation of $\mathcal{M}_{|\cdot|}f$ presented above will turn out to be useful in Section 4.

3.2. **The Lorentz-norm estimate.**

Proof of (1.5). By homogeneity, we may assume that $\|f\|_{L^{p,\infty}(\mu)} = 1$. By the definition of the Lorentz norm, we have $\mu(\{f > \lambda\}) \leq \lambda^{-p}$ for any $\lambda > 0$, or which is the same, $f_\mu^*(s) \leq s^{-1/p}$ for all $s > 0$. Here $f_\mu^*(t) = \inf\{s > 0 : \mu(\{f > s\}) \leq t\}$ is the nonincreasing rearrangement of f (with respect to the measure μ).

The well-known inequality of Hardy and Littlewood states that if h is a nonnegative function and A is a Borel subset of $\mathbb{R}$, then

$$\int_A h \mathrm{d}\mu \leq \int_0^{\mu(A)} h_\mu^*(s)\mathrm{d}s.$$

Fix an arbitrary $\lambda > 0$. It is easy to check that the above fact implies

$$\int_{\{f>\lambda\}} f^{-1}\mathrm{d}\mu \geq \int_0^{\mu(\{f>\lambda\})} (f_\mu^*(s))^{-1}\mathrm{d}s \geq \int_0^{\mu(\{f>\lambda\})} s^{1/p}\mathrm{d}s$$
$$= \frac{p(\mu(\{f>\lambda\}))^{1+1/p}}{p+1}$$

and similarly

$$\int_{\{\mathcal{M}_\mu f>\lambda\}} f^{-1}\mathrm{d}\mu \geq \int_0^{\mu(\{\mathcal{M}_\mu f>\lambda\})} (f^*(s))^{-1}\mathrm{d}s \geq \frac{p(\mu(\{\mathcal{M}_\mu f>\lambda\}))^{1+1/p}}{p+1}.$$

Combining these two estimates with (2.1) yields

$$pT_\lambda^{p+1} - (p+1)T_\lambda^p \leq t_\lambda^p(p+1-pt_\lambda),$$

where $T_\lambda = \lambda\mu(\{\mathcal{M}_\mu f > \lambda\})^{1/p}$ and $t_\lambda = \lambda\mu(\{f>\lambda\})^{1/p}$. However, the assumption $\|f\|_{L^{p,\infty}(\mu)} \leq 1$ gives $t_\lambda \leq 1$; in addition, the function $t \mapsto t^p(p+1-pt)$ is increasing on $[0,1]$ (its derivative is given by $t \mapsto p(p+1)t^{p-1}(1-t)$). Consequently, the preceding estimate implies $pT_\lambda^{p+1} - (p+1)T_\lambda^p \leq 1$ which, in the light of (1.3), gives the bound $T_\lambda \leq C_p$. Since λ was chosen arbitrarily, we obtain $\|\mathcal{M}_\mu f\|_{L^{p,\infty}(\mu)} \leq C_p$, as claimed. $\square$

Sharpness of (1.5) *for the Lebesgue measure.* Since the best constants in (1.4) and (1.5) are the same, it is natural to try the same extremal function as in the proof of the sharpness of the L^p bound. We will do so and even take the limiting value $\alpha = p$. In other words, consider the function f given by $f(x) = |x|^{-1/p}\chi_{[-1,1]}(x)$. Since $\lambda|\{f>\lambda\}|^{1/p} = \min\{\lambda,1\}2^{1/p} \leq 2^{1/p}$ for any $\lambda > 0$, we get $f \in L^{p,\infty}$. Calculating as in the proof of the sharpness of (1.4), we see that $\mathcal{M}_{|\cdot|}f(x) = C_p f(x)$ for $x \in [-1,1]$ and hence also $\|\mathcal{M}_{|\cdot|}f\|_{L^{p,\infty}} = C_p\|f\|_{L^{p,\infty}}$. $\square$

3.3. **The weak-type bound.**

Proof of (1.6). By homogeneity, it is enough to show that

$$\mu(\mathcal{M}_\mu f > 1) \leq c_p^p \|f\|_{L^p(\mu)}^p. \tag{3.4}$$

We rewrite the inequality (2.1), in the special case $\lambda = 1$, in the form

$$\int_{\mathbb{R}} u(f, \mathcal{M}_\mu f)\mathrm{d}\mu \geq 0,$$

where $u(x,y) = \big(\chi_{\{|x|>1\}} + \chi_{\{|y|>1\}}\big)(1 - x^{-1})$. Note that by a standard application of Lebesgue's differentiation theorem, we have that $\mathcal{M}_\mu f \geq |f|$ μ-almost everywhere. Therefore, (3.4) will follow at once from the above estimate, if we manage to establish the majorization

$$u(x,y) \leq \gamma_p(c_p^p x^p - \chi_{\{y>1\}}), \qquad y \geq x \geq 0, \tag{3.5}$$

for some $\gamma_p > 0$. We will prove that $\gamma_p = (2 - 2^{p/(p+1)})/(2^{p/(p+1)} - 1)$ works. We consider separately three cases. If $y \leq 1$, then also $x \leq 1$ and (3.5) is obvious: $0 \leq \gamma_p c_p^p x^p$. Suppose that $y > 1$ and $x \leq 1$. Then the estimate becomes

$$1 - x^{-1} - \gamma_p(c_p^p x^p - 1) \leq 0. \tag{3.6}$$

This is an easy elementary estimate. Denoting the left-hand side by $F_p(x)$, we check that $F_p(x_0) = F_p'(x_0) = 0$, where $x_0 = (p+1)(2^{p/(p+1)} - 1)/p \in (0,1)$ (note that the estimate $x_0 < 1$ is equivalent to $2^{p/(p+1)} - 1 \leq p/(p+1)$, which is due to the convexity of $s \mapsto 2^s - 1$ on $[0,1]$). It remains to observe that $F(0) = -\infty$, $F(+\infty) = -\infty$ and that the derivative F' is continuous on $(0,\infty)$, vanishing only at x_0.

It remains to check (3.5) for $x > 1$ and $y > 1$. The estimate reads

$$2(1 - x^{-1}) - \gamma_p(c_p^p x^p - 1) \leq 0$$

and can be proved exactly in the same manner as above. We omit the details and just mention that the left-hand side, considered as a function of x, vanishes along with its derivative at the point $x_1 = (p+1)(2 - 2^{1/(p+1)})/p$ (one can also get the inequality $x_1 > 1$ invoking the suitable convexity argument). $\square$

Sharpness of (1.6) *for the Lebesgue measure.* Let us start with some informal arguments showing how to discover the extremal function. First, recall the points (x,y) at which both sides of (3.5) are equal. These special points are of the three types: $(0,y)$, where $y \leq 1$; (x_0,y), where $y > 1$; and (x_1,y), where $y > 1$. This suggests that the extremal function f should take values in the set $\{0, x_0, x_1\}$. Furthermore, motivated by the examples in the L^p bound, it seems plausible to assume that f is even and radially decreasing. So, set

$$f = x_0\chi_{[-b,-a)} + x_1\chi_{[-a,a]} + x_0\chi_{(a,b]},$$

for some parameter $0 < a < b$. By a standard dilation, we may assume that $a = 1$. To guess the value of b, motivated by the above proof of the weak-type bound, we pick b so that $\mathcal{M}_{|\cdot|}f = 1$ on $[-b, b]$. Since

$$\frac{1}{|[-1,b]|}\int_{-1}^{b} f^{-1}\mathrm{d}s = \frac{2x_1^{-1} + (b-1)x_0^{-1}}{b+1},$$

the choice $b = (x_0^{-1} - 2x_1^{-1} + 1)/(x_0^{-1} - 1)$ implies that $\mathcal{M}_{|\cdot|}f \geq 1$ on $[-1, b]$ and hence, by symmetry, on the whole interval $[-b, b]$. Consequently, for any $\lambda < 1$ we have

$$\frac{\lambda^p|\{\mathcal{M}_{|\cdot|}f > \lambda\}|}{\|f\|_{L^p}^p} \geq \frac{\lambda^p \cdot 2b}{2(b-1)x_0^p + 2x_1^p},$$

and the latter expression is precisely $\lambda^p c_p^p$. Letting $\lambda \to 1$, we see that the optimal constant in (1.6) cannot be smaller than c_p. □

4. Stability estimates

Here the reasoning will be more technical and involved. We start with the following two technical lemmas.

Lemma 4.1. *For any $p \geq 2$ and any $t \geq 0$ we have*

$$(p+1)t^p + 1 - pt^{p+1} + C_p|t - C_p|^p - (C_p^{p+1} + 1)(1 - C_p^{-p}t^p) \leq 0.$$

Proof. Denote the left-hand side by $F(t)$. If $t \leq C_p$, then

$$F'(t) = pt^{p-1}\left[(p+1)(1-t) - C_p(C_pt^{-1} - 1)^{p-1} + C_p + C_p^{-p}\right].$$

Introduce the new variable $s = t^{-1} \in [C_p^{-1}, \infty)$ and denote the expression in the square brackets by $G(s)$. We have $G'(s) = (p+1)s^{-2} - (p-1)C_p^2(C_ps - 1)^{p-2}$: obviously, G' is decreasing, $G'(C_p^{-1}) > 0$ and $\lim_{s\to\infty} G'(s) < 0$. Consequently, there is $s_0 > C_p^{-1}$ such that G is increasing on (C_p^{-1}, s_0) and decreasing on (s_0, ∞). But $G(C_p^{-1}) = 0$ and $\lim_{s\to\infty} G(s) = -\infty$, so there exists $s_1 > s_0$ such that $G > 0$ on (C_p^{-1}, s_1) and $G < 0$ on (s_1, ∞). Because $F'(t)$ and $G(t^{-1})$ have the same sign, we see that F is decreasing on $(0, s_1^{-1})$ and increasing on

(s_1^{-1}, C_p). However, we have $F(0) = F(C_p) = 0$, so the desired estimate holds on $[0, C_p]$. For $t \in (C_p, 1]$ we argue similarly: we compute that

$$F'(t) = pt^{p-1}\left[(p+1)(1-t) + C_p(1-C_pt^{-1})^{p-1} + C_p + C_p^{-p}\right]$$

and denote the expression in the square brackets by $G(s)$, where $s = t^{-1} \in (0, C_p^{-1}]$. Observe that $G(C_p^{-1}) = 0$ and

$$G'(s) = (p+1)s^{-2} - (p-1)C_p^2(1-C_ps)^{p-2} \geq (p+1)C_p^2 - (p-1)C_p^2 > 0.$$

This implies that G is negative on $(0, C_p^{-1})$ and hence F is decreasing on (C_p, ∞). Since $F(C_p) = 0$, the assertion holds also for $t > C_p$. □

Lemma 4.2. *For any $0 < p < 2$ and any $t \geq 1$ we have*

$$(p+1)t^p + 1 - pt^{p+1} + \beta_p t^{p-2}(t - C_p)^2 - (C_p^{p+1} + 1)(1 - C_p^{-p}t^p) \leq 0,$$

where $\beta_p = pC_p^{-1}/2$.

Proof. If we substitute $s = t^{-1} \leq 1$, the inequality can be rewritten as

$$G(s) := p + 1 + s^p - ps^{-1} + \beta_p(1 - C_ps)^2 - (C_p^{p+1} + 1)(s^p - C_p^{-p}) \leq 0.$$

We compute directly that $G(C_p^{-1}) = G'(C_p^{-1}) = 0$ and $G''(s) = -2ps^{-3} - p(p-1)C_p^{p+1}s^{p-2} + 2\beta_pC_p^2$. We will prove that $G'' < 0$ on $(0, C_p^{-1})$ and $G' < 0$ on $(C_p^{-1}, 1)$; this will clearly yield the claim.

For $1 \leq p < 2$ the estimate $G''(s) < 0$ for $s < C_p^{-1}$ is trivial (we have $-2ps^{-3} \leq -2pC_p < -2\beta_pC_p^2$ and $-p(p-1)C_p^{p+1}s^{p-2} \leq 0$), for $0 < p < 1$ we write

$$\begin{aligned} G''(s) &= ps^{-3}[-2 + (1-p)(C_ps)^{p+1}] + pC_p \\ &\leq ps^{-3}[-2 + (1-p)] + pC_p \leq -p(p+1)C_p^3 + pC_p < 0. \end{aligned}$$

To show that $G'(s) < 0$ for $s > C_p^{-1}$, we apply the mean-value theorem to obtain

$$\begin{aligned} G'(s) &= -p(C_ps - 1)s^{-2}\left[\frac{(C_ps)^{p+1} - 1}{C_ps - 1} - \frac{2\beta_pC_ps^2}{p}\right] \\ &\leq -p(C_ps - 1)s^{-2}(p + 1 - s^2) < 0. \end{aligned}$$ □

We are ready for the stability estimate.

Proof of (1.7). Let $\varepsilon > 0$ and suppose that f is a measurable function such that $\|\mathcal{M}_\mu f\|_{L^p} > (C_p - \varepsilon)\|f\|_{L^p}$. We rewrite the estimate (3.1) in the equivalent form

$$\int_{\mathbb{R}} w(f, \mathcal{M}_\mu f)\mathrm{d}\mu \geq 0, \tag{4.1}$$

where $w(x,y) = (p+1)y^p + x^p - py^{p+1}x^{-1}$. Now we consider two cases. If $p \geq 2$, then Lemma 4.1 gives

$$w(x,y) \leq \frac{C_p^{p+1}+1}{C_p^p}(C_p^p x^p - y^p) - C_p|y - C_p x|^p$$

(it suffices to divide throughout by x^p and substitute $t = y/x$). Combining this estimate with the previous inequality gives

$$\frac{C_p^{p+1}+1}{C_p^p}\left(C_p^p\|f\|_{L^p}^p - \|\mathcal{M}_\mu f\|_{L^p}^p\right) - C_p\|\mathcal{M}_\mu f - C_p f\|_{L^p}^p \geq 0,$$

and hence, by Bernoulli's inequality,

$$\begin{aligned}\|\mathcal{M}_\mu f - C_p f\|_{L^p}^p &\leq \frac{C_p^{p+1}+1}{C_p}\left(1 - \left(1 - \frac{\varepsilon}{C_p}\right)^p\right)\|f\|_{L^p}^p \\ &\leq \frac{p(C_p^{p+1}+1)\varepsilon}{C_p^2}\|f\|_{L^p}^p \leq 2pC_p^p\varepsilon\|f\|_{L^p}^p.\end{aligned}$$

This gives the assertion. For $0 < p < 2$ the proof is similar, but we need an additional use of the Hölder inequality. The application of Lemma 4.2 gives

$$w(x,y) \leq \frac{C_p^{p+1}+1}{C_p^p}(C_p^p x^p - y^p) - \frac{pC_p^{-1}}{2}y^{p-2}(y - C_p x)^2,$$

which combined with (4.1) (and (1.3)) yields

$$\begin{aligned}&\int_{\mathbb{R}} (\mathcal{M}_\mu f)^{p-2}(\mathcal{M}_\mu f - C_p f)^2\mathrm{d}\mu \\ &\qquad \leq \frac{2(p+1)C_p(C_p-1)}{p}(C_p^p\|f\|_{L^p}^p - \|\mathcal{M}_\mu f\|_{L^p}^p) \\ &\qquad \leq \frac{2(p+1)C_p(C_p-1)}{p}\cdot C_p^p\left(1 - \left(1 - \frac{\varepsilon}{C_p}\right)^p\right)\|f\|_{L^p}^p\end{aligned}$$

(in the first line, we have used the convention $\infty \cdot 0 = 0$). If $p \geq 1$, then $(1 - \varepsilon/C_p)^p \geq 1 - p\varepsilon/C_p$ and $2(p+1)(C_p - 1) \leq 4C_p^2$, so

$$\int_{\mathbb{R}} (\mathcal{M}_\mu f)^{p-2}(\mathcal{M}_\mu f - C_p f)^2\mathrm{d}\mu \leq 4C_p^{p+2}\varepsilon\|f\|_{L^p}^p. \tag{4.2}$$

On the other hand, if $p<1$, then $(1-\varepsilon/C_p)^p \geq 1-\varepsilon/C_p$ and $2(p+1)(C_p-1)/p \leq 4C_p^2$: the latter bound holds since $C_p > (p+1)/p$ (which follows directly from (1.3): $pC_p^{p+1} = (p+1)C_p^p + 1 > (p+1)C_p^p$). Thus, (4.2) holds also for $p<1$. Therefore, by (1.4) and the Hölder inequality with exponents $2/p$ and $2/(2-p)$, we obtain

(4.3)
$$\begin{aligned} \|\mathcal{M}_\mu f - C_p f\|_{L^p} &\leq \left(\int_{\mathbb{R}} (\mathcal{M}_\mu f)^{p-2}(\mathcal{M}_\mu f - C_p f)^2 \mathrm{d}\mu\right)^{1/2} \|\mathcal{M}_\mu f\|_{L^p}^{1-p/2} \\ &\leq 2C_p^2 \varepsilon^{1/2} \|f\|_{L^p}. \end{aligned}$$

This completes the proof. □

Now we will handle the optimality of the exponents in (1.7). We consider the cases $0<p<2$ and $p \geq 2$ separately. From now on, we assume that μ is the Lebesgue measure.

Sharpness of the exponent, the case $p \geq 2$. Let $0<\eta<C_p^{-p}$ be a fixed parameter and consider the function

$$f(x) = \frac{p+1}{p}\eta^{-1/p}\chi_{[-\eta,\eta]}(x) + |x|^{-1/p}\chi_{[-1,1]\setminus[-\eta,\eta]}(x).$$

Note that $\int_{-\eta}^{\eta} f^{-1}\mathrm{d}s = \frac{2p}{p+1}\eta^{1+1/p} = \int_{-\eta}^{\eta} g^{-1}\mathrm{d}s$, where $g(s) = |s|^{-1/p}$. Therefore, if $x \in [C_p^p\eta, 1]$, then $-C_p^{-p}x \in [-1,-\eta]$ and consequently,

$$\begin{aligned} \mathcal{M}_{|\cdot|} f(x) &\geq \left(\frac{1}{|[-C_p^{-p}x, x]|}\int_{-C_p^{-p}x}^{x} f^{-1}\mathrm{d}s\right)^{-1} \\ &= \mathcal{M}_{|\cdot|} g(x) = C_p g(x) = C_p f(x), \end{aligned} \tag{4.4}$$

which by symmetry holds also for $x \in [-1, -C_p^p\eta]$. Clearly, we also have

$$\mathcal{M}_{|\cdot|} f(x) = f(x) = \frac{p+1}{p}\eta^{-1/p} \qquad \text{on } [-\eta,\eta]. \tag{4.5}$$

Therefore, by (4.4) and Bernoulli's inequality,

$$
\begin{aligned}
&\|\mathcal{M}_{|\cdot|}f\|_{L^p} - C_p\|f\|_{L^p} \\
&\geq C_p\left(2\int_{C_p^p\eta}^1 s^{-1}\mathrm{d}s\right)^{1/p} - C_p\left(2\int_\eta^1 s^{-1}\mathrm{d}s + 2\left(\frac{p+1}{p}\right)^p\right)^{1/p} \\
&= 2^{1/p}C_p\left[\left(-\ln(C_p^p\eta)\right)^{1/p} - \left(-\ln\eta + \left(\frac{p+1}{p}\right)^p\right)^{1/p}\right] \\
&\geq -\frac{2^{1/p}C_p}{p}\left(-\ln(C_p^p\eta)\right)^{1/p-1}\left(\ln C_p^p + \left(\frac{p+1}{p}\right)^p\right) \\
&= -\frac{C_p}{p}\left(\ln C_p^p + \left(\frac{p+1}{p}\right)^p\right)\left(\frac{-\ln(C_p^p\eta)}{-\ln\eta + \left(\frac{p+1}{p}\right)^p}\right)^{1/p} \times \\
&\qquad\qquad \times\left(-\ln(C_p^p\eta)\right)^{-1}\cdot\|f\|_{L^p} = \varepsilon_\eta\|f\|_{L^p},
\end{aligned}
$$

with $\varepsilon_\eta = \Theta((-\ln\eta)^{-1})$ as $\eta \to 0$. Furthermore, by (4.5),

$$
\begin{aligned}
&\|\mathcal{M}_{|\cdot|}f - C_pf\|_{L^p} \\
&\geq \left(\int_{-\eta}^{\eta} |\mathcal{M}_{|\cdot|}f - C_pf|^p\mathrm{d}s\right)^{1/p} \\
&= 2^{1/p}\frac{(p+1)(C_p-1)}{p} \\
&= 2^{1/p}\frac{(p+1)(C_p-1)}{p}\left(-2\ln\eta + 2\left(\frac{p+1}{p}\right)^p\right)^{-1/p}\|f\|_{L^p}.
\end{aligned}
$$

The latter expression behaves as $\varepsilon_\eta^{1/p}\|f\|_{L^p}$ as $\eta \to 0$ (in the sense that the ratio of the two quantities converges to a nonzero and finite limit). This proves that the exponent $1/p$ is indeed optimal in the range $2 \leq p < \infty$. $\square$

Sharpness of the exponent, the case $0 < p < 2$. Here the analysis is a little more complicated. Let $\eta \in (0,p)$ be a given number and set $\alpha = p+\eta$, $\beta = p-\eta$. Consider the function

$$f(x) = (1+1/\alpha)^{-1}|x|^{-1/\alpha}\chi_{[-1,1]}(x) + (1+1/\beta)^{-1}|x|^{-1/\beta}\chi_{\mathbb{R}\setminus[-1,1]}(x).$$

Then we have

$$\|f\|_{L_p} = \left[\frac{2}{(1+1/\alpha)^p(1-p/\alpha)} + \frac{2}{(1+1/\beta)^p(p/\beta-1)}\right]^{1/p}, \tag{4.6}$$

which behaves as $\eta^{-1/p}$ as $\eta \to 0$ (as above, this means that the ratio of the two expressions converges to a nontrivial limit). Let us analyze the function $\mathcal{M}_{|\cdot|}f$ precisely. Arguing as in (3.2), we prove that $\mathcal{M}_{|\cdot|}f(x) = C_\alpha f(x)$ for $|x| \leq 1$, for sufficiently small η. Next, if $x > 1$, we compute directly that

$$\begin{aligned}
\mathcal{M}_{|\cdot|}f(x) &\geq \left(\frac{1}{|[-C_\beta^{-\beta}x, x]|} \int_{-C_\beta^{-\beta}x}^{x} f^{-1}(s)\mathrm{d}s \right)^{-1} \\
&= \begin{cases} \left(\dfrac{C_\beta^{-\beta(1+1/\alpha)}x^{1/\alpha} + x^{1/\beta}}{1 + C_\beta^{-\beta}} \right)^{-1} & \text{if } x < C_\beta^\beta, \\ \left(\dfrac{C_\beta^{-\beta-1}x^{1/\beta} + x^{1/\beta}}{1 + C_\beta^{-\beta}} \right)^{-1} & \text{if } x \geq C_\beta^\beta, \end{cases}
\end{aligned}$$

which is essentially $C_\beta f(x)$. Indeed, for $x \geq C_\beta^\beta$ we have equality, while for $x \in (1, C_\beta^\beta)$,

$$\begin{aligned}
&\left(\frac{C_\beta^{-\beta(1+1/\alpha)}x^{1/\alpha} + x^{1/\beta}}{1 + C_\beta^{-\beta}} \right)^{-1} - C_\beta f(x) \\
&= \left(\frac{C_\beta^{-\beta(1+1/\alpha)}x^{1/\alpha} + x^{1/\beta}}{1 + C_\beta^{-\beta}} \right)^{-1} - \left(\frac{C_\beta^{-\beta-1}x^{1/\beta} + x^{1/\beta}}{1 + C_\beta^{-\beta}} \right)^{-1} \\
&= \frac{1 + C_\beta^{-\beta}}{C_\beta^{\beta+1} + 1} \cdot \frac{1 - (C_\beta^{-\beta}x)^{1/\alpha - 1/\beta}}{C_\beta^{-\beta(1+1/\alpha)}x^{1/\alpha} + x^{1/\beta}} = -O(\eta)
\end{aligned}$$

as $\eta \to 0$, and hence

$$\int_1^\infty (\mathcal{M}_{|\cdot|}f(s))^p \geq C_\beta^p \int_1^\infty f(s)^p \mathrm{d}s - \Theta(\eta).$$

Putting all the above observations together, we get

$$
\begin{aligned}
&\|\mathcal{M}_{|\cdot|} f\|_{L^p}^p \\
&\geq 2C_\alpha^p \int_0^1 f^p \mathrm{d}s + 2C_\beta^p \int_1^\infty f^p \mathrm{d}s - \Theta(\eta) \\
&= C_p^p \|f\|_{L^p}^p + 2(C_\alpha^p - C_p^p) \int_0^1 f^p \mathrm{d}s + 2(C_\beta^p - C_p^p) \int_1^\infty f^p \mathrm{d}s - \Theta(\eta) \\
&= C_p^p \|f\|_{L^p}^p + \frac{2(C_\alpha^p - C_p^p)}{(1+1/\alpha)^p (1-p/\alpha)} + \frac{2(C_\beta^p - C_p^p)}{(1+1/\beta)^p (p/\beta - 1)} - \Theta(\eta) \\
&= C_p^p \|f\|_{L^p}^p + \frac{2\alpha(C_\alpha^p - C_p^p)}{(1+1/\alpha)^p \eta} + \frac{2\beta(C_\beta^p - C_p^p)}{(1+1/\beta)^p \eta} - \Theta(\eta).
\end{aligned}
$$

Now, there is $\kappa_p > 0$ depending only on p such that $C_\alpha^p - C_p^p = -\kappa_p \eta + O(\eta^2)$ and $C_\beta^p - C_p^p = \kappa_p \eta + O(\eta^2)$. Plugging this above, we see that

$$
\|\mathcal{M}_{|\cdot|} f\|_{L^p}^p \geq C_p^p \|f\|_{L^p}^p - \Theta(\eta) = (C_p^p - \Theta(\eta^2)) \|f\|_{L^p}^p,
$$

since $\|f\|_{L^p}^p$ behaves as $\Theta(\eta^{-1})$ (see (4.6)). Therefore, we obtain

$$
\|\mathcal{M}_{|\cdot|} f\|_{L^p} \geq (C_p - \varepsilon) \|f\|_{L^p}, \tag{4.7}
$$

with $\varepsilon = \Theta(\eta^2)$. On the other hand,

$$
\begin{aligned}
\|\mathcal{M}_{|\cdot|} f - C_p f\|_{L^p} &\geq \|\mathcal{M}_{|\cdot|} f - C_p f\|_{L^p(-1,1)} \\
&= (C_p - C_\alpha) \left(\int_{-1}^1 f^p \mathrm{d}s \right)^{1/p} = \Theta(\eta) \|f\|_{L^p},
\end{aligned}
$$

and the latter expression behaves as $\varepsilon^{1/2} \|f\|_{L^p}$. This yields the optimality of the exponent. □

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Faculty of Mathematics, Informatics and Mechanics, University of Warsaw, Banacha 2, 02-097 Warsaw, Poland

Email address: `L.Kaminski5@student.uw.edu.pl`

Faculty of Mathematics, Informatics and Mechanics, University of Warsaw, Banacha 2, 02-097 Warsaw, Poland

Email address: `A.Osekowski@mimuw.edu.pl`