INEQUALITIES FOR NONCOMMUTATIVE WEAKLY DOMINATED MARTINGALES AND APPLICATIONS

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Abstract. Motivated by the results from the classical probability theory, we introduce the concepts of tangency and weak domination of noncommutative martingales. Then we establish the weak-type and strong-type estimates arising in this context. The proof rests on a novel Gundy-type decomposition which is of independent interest. We also show the corresponding square function inequalities under the assumption of the weak domination. The results strengthen recent works on noncommutative differentially subordinate martingales ([10, 14]), which in turn, give rise to a new application in harmonic analysis: a weak-type estimate (along with a completely bounded version) for the directional Hilbert transform associated with a quantum tori.

1. Introduction

The purpose of this paper is to study weak-type and strong-type estimates for noncommutative martingales under a certain mild domination relation, and provide some applications in harmonic analysis. To present this topic from the appropriate perspective, we will first discuss some results which were obtained in the commutative setting about twenty-thirty years ago. Suppose that $(\Omega, \mathcal{F}, \mathbb{P})$ is a classical probability space, filtered by $(\mathcal{F}_n)_{n\geq 0}$, a nondecreasing sequence of sub-σ-fields of $\mathcal{F}$. Let $x = (x_n)_{n\geq 0}$, $y = (y_n)_{n\geq 0}$ be two adapted real-valued martingales, with the associated difference sequences $dx = (dx_n)_{n\geq 0}$, $dy = (dy_n)_{n\geq 0}$ defined by $dx_0 = x_0$, $dy_0 = y_0$ and $dx_n = x_n - x_{n-1}$, $dy_n = y_n - y_{n-1}$ for all $n \geq 1$. The problem of comparing the sizes of x and y (measured by norms in various function spaces), under certain domination relations expressed in terms of dx and dy, has a long history and goes back to the classical results for sums of mean-zero independent random variables (cf. [25]). One of fundamental examples is that of the differential subordination. Following Burkholder [4, 5], we say that y is differentially subordinate to x, if for any $n \geq 0$ we have $|dy_n| \leq |dx_n|$ almost surely. This type of domination generalizes another crucial concept of the so-called martingale transforms. Recall that y is a transform of x, if there exists a predictable sequence $v = (v_n)_{n\geq 0}$ such that $dy_n = v_n dx_n$ for each $n \geq 0$. Here

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by predictability we mean that for any n the random variable v_n is $\mathcal{F}_{(n-1)\vee 0}$-measurable. Clearly, if the terms v_n are assumed to be bounded in absolute value by 1, then y is differentially subordinate to x.

The differential subordination implies many important and interesting estimates between x and y. Here is a fundamental result due to Burkholder. For a survey over related estimates we refer the reader to [5, 28, 29].

Theorem 1.1. *Suppose that x, y are martingales such that y is differentially subordinate to x.*

(i) *For any $n \geq 0$ we have*
$$\|y_n\|_{L_{1,\infty}} \leq 2\|x_n\|_{L_1},$$
and the constant 2 is the best possible. It is already optimal in the context of martingale transforms.

(ii) *For any $n \geq 0$ and any $1 < p < \infty$ we have*
$$\|y_n\|_{L_p} \leq (p^* - 1)\|x_n\|_{L_p},$$
where $p^* = \max\{p, p/(p-1)\}$. The inequality is sharp, already for martingale transforms.

This statement is a starting point for many applications, including bounds for wide classes of Fourier multipliers, properties of unconditional constants and the geometry of Banach spaces. We refer the reader to, for instance, [1, 4, 5] and references therein.

One can also study analogous weak-type and strong-type estimates under different dominations. Let us briefly discuss two examples which will serve as the motivation for our research. The concept of tangent sequences appeared in the work of Kwapień and Woyczyński [23] and gave rise to the powerful technique of decoupling which has proved to be a very efficient tool in the study of sums of independent random variables and random chaoses. Recall that two sequences $(u_n)_{n\geq 0}$, $(v_n)_{n\geq 0}$ of real-valued random variables, adapted to a given filtration $(\mathcal{F}_n)_{n\geq 0}$, are said to be tangent, if for each $n \geq 0$ the conditional distributions of u_n and v_n given $\mathcal{F}_{n-1}$ coincide (we set $\mathcal{F}_{-1} = \{\emptyset, \Omega\}$). This is equivalent to saying that for any $n \geq 0$ and any bounded Borel function f, we have the identity
$$\mathcal{E}_{n-1} f(u_n) = \mathcal{E}_{n-1} f(v_n),$$
where $\mathcal{E}_{n-1}$ is the conditional expectation with respect to $\mathcal{F}_{n-1}$. For the tangent sequences, we have the following estimates proved in [27, 28] (see also [9] and [23, 24]).

Theorem 1.2. *Let x, y be martingales with tangent difference sequences.*

(i) *There exists a universal constant C such that*
$$\|y_n\|_{L_{1,\infty}} \leq C\|x_n\|_{L_1}, \qquad n = 0, 1, 2, \ldots. \tag{1.1}$$

(ii) *For any $1 < p < \infty$ there is a constant C_p depending only on p such that*
$$\|y_n\|_{L_p} \leq C_p\|x_n\|_{L_p}, \qquad n = 0, 1, 2, \ldots. \tag{1.2}$$
Furthermore, the orders $C_p = O((p-1))^{-1}$ as $p \to 1+$ and $C_p = O(p)$ as $p \to \infty$ are the best possible.

Theorems 1.1 and 1.2 were generalized significantly in [27]. The type of domination considered there is less restrictive than both the differential subordination and the tangency. Following [24], we say that a martingale y is weakly dominated by x, if for any $n \geq 0$ and any $\lambda > 0$ we have

$$\mathcal{E}_{n-1}(|dy_n| - \lambda)_+ \leq \mathcal{E}_{n-1}(|dx_n| - \lambda)_+.$$

There is an equivalent and perhaps a little more transparent reformulation of this condition: for any $n \geq 0$ and any nondecreasing convex function $f : [0, \infty) \to \mathbb{R}$ with a linear growth at infinity we have

$$\mathcal{E}_{n-1} f\big(|dy_n|\big) \leq \mathcal{E}_{n-1} f\big(|dx_n|\big). \tag{1.3}$$

The aforementioned result of [27] asserts that the estimates (1.1) and (1.2) hold true if y is assumed to be weakly dominated by x; in addition, the optimal orders of C_p as $p \to 1+$ or $p \to \infty$ are the same. Actually, as already pointed out in [27], the domination (1.3) can be further relaxed by restricting to some special functions f. For the sake of clarity, we will formulate this as a separate statement. Let us distinguish the function $\varphi : \mathbb{R} \to [0, \infty)$ given by the formula

$$\varphi(s) = \begin{cases} s^2 & \text{if } |s| \leq 1, \\ 2|s| - 1 & \text{if } |s| > 1. \end{cases} \tag{1.4}$$

Theorem 1.3. [27] *Let x, y be martingales and φ be as defined in (1.4).*

(i) *Suppose that for any $n \geq 0$ and any $\lambda > 0$,*

$$\mathcal{E}_{n-1}\varphi(\lambda|dy_n|) \leq \mathcal{E}_{n-1}\varphi(\lambda|dx_n|).$$

Then for any $n = 0, 1, 2, \ldots$ we have

$$\|y_n\|_{L_{1,\infty}} \leq 6\|x_n\|_{L_1}$$

and

$$\|y_n\|_{L_p} \leq 3(p-1)^{-1}\|x_n\|_{L_p}, \quad 1 < p < 2.$$

(ii) *Suppose that $p \geq 2$ and for any $n \geq 0$ we have*

$$\mathcal{E}_{n-1}|dy_n|^2 \leq \mathcal{E}_{n-1}|dx_n|^2 \quad \text{and} \quad \mathcal{E}_{n-1}|dy_n|^p \leq \mathcal{E}_{n-1}|dx_n|^p.$$

Then there exists an absolute constant C such that for any $n = 0, 1, 2, \ldots$

$$\|y_n\|_{L_p} \leq Cp\|x_n\|_{L_p}.$$

For related results under different types of dominations and their applications to the theory of multiple stochastic integrals, we refer the reader to the monograph [24].

On the other hand, as we know, the theory of noncommutative martingales has been developed significantly and applied in various topics of harmonic analysis. The beginning of the theory dates back to the seminal paper [32] of Pisier and Xu, which contains the description of the general framework and an appropriate form of Burkholder-Gundy estimates. Another paper which turned out to be fundamental to the development of the area is Junge's work [16] devoted to Doob's maximal estimate in the noncommutative context. Since the appearance of these two articles, the topic has gained considerable interest from the mathematical community and has been applied in numerous contexts of harmonic analysis. The literature is extremely extensive here and it is impossible to give even a brief survey. We refer the interested reader to the works [2, 3, 6, 12, 17, 18, 19, 20] and the references therein. Particularly, in a recent paper [10], the notion of differential subordination was generalized to the noncommutative setting and a non-classical version of Theorem 1.1 was formulated. See also the references [11, 13, 14] for other works on noncommutative differentially subordinated martingales and their applications in harmonic analysis.

Following the above line of research, the main purpose of this paper is to study noncommutative weakly dominated martingales and explore some possible applications in harmonic analysis. We should mention that some progress on noncommutative tangent and weakly dominated martingales has been already carried out in [12], which enabled a successful treatment of strong-type estimates in the range $p \geq 2$. Unfortunately, the general approach introduced there has not allowed the study of weak-type estimates or the strong-type bounds for $1 < p < 2$: these inequalities were shown to fail under the tangency or the weak domination condition introduced there (see the next section for the detailed discussion). In the present paper, we overcome this difficulty and complete the study on the topic. It should be emphasized that this development requires the invention of new ideas and techniques. It turns out that in the range $1 < p < 2$, one needs to work with a 'stronger' version of the weak domination. Actually, the phenomenon that one has to work under different assumptions (a weaker for $p \geq 2$, a stronger for $1 \leq p < 2$) has been already discovered and explained by the first three authors in [10]. Working under the 'stronger' weak domination, we show the weak-type estimates and the strong-type bounds for $1 < p < 2$. These results, combined with [12, Theorem 5.5], can be regarded as a noncommutative version of Theorem 1.3. Our approach mainly depends on a new type of Gundy's decomposition which is designed for the study of the context of the weak domination. In addition, we complement the above analysis by showing the weak- and strong-type (p, p) inequalities (with $1 < p < 2$) for noncommutative martingales and the square functions of the weakly dominated processes. The proof mainly relies on a new endpoint inequality of a triangular truncated operator (see Proposition 4.6 below) and the variant of Gundy's decomposition mentioned above. We should mention that our main results strengthen recent works on noncommutative differentially subordinated martingales ([10, 14]).

Similar to the classical case, the weak domination is less restrictive than the differential subordination considered in [10] and hence it should possess a wider range of applications. Exploiting the machinery developed for noncommutative weakly dominated martingales, we establish a weak-type estimate and also a completely bounded version for the directional Hilbert transform associated with a quantum tori. These statements do not seem to follow from weak-type estimate for differentially subordinate martingales and hence the weak domination does provide a wider range of applications.

The paper is organized as follows. In the next section we recall some basic information on operator algebras which will be needed in our further considerations. Section 3 includes a new description of a Gundy type decomposition for weakly dominated martingales which is different from the versions in [14, 30] and is of independent interest. Section 4 contains the study of estimates for noncommutative weakly dominated martingales. The last section is devoted to applications: a weak-type estimate and also a completely bounded version for the directional Hilbert transform associated with a quantum tori.

2. Preliminaries

We start with some basic facts from the operator theory, for the detailed exposition of the subject we refer the reader to [21, 22, 36]. Throughout the paper, $\mathcal{M}$ is a von Neumann algebra equipped with a semifinite normal faithful trace τ. We assume that $\mathcal{M}$ is a subalgebra of the algebra of all bounded operators acting on some Hilbert space $\mathcal{H}$. A closed densely defined operator a on $\mathcal{H}$ is said to be affiliated with $\mathcal{M}$ if $u^*au = a$ for all unitary operators u in the commutant $\mathcal{M}'$ of $\mathcal{M}$. A closed densely defined operator a on $\mathcal{H}$ affiliated with $\mathcal{M}$ is said to be τ-measurable if for any $\varepsilon > 0$ there exists a projection e such that $e(\mathcal{H})$ is contained in the domain of x and $\tau(I-e) < \varepsilon$; here and below, I denotes the identity operator. The set of all τ-measurable operators will be denoted by $L_0(\mathcal{M}, \tau)$. The trace τ can be extended to a positive tracial functional on the positive part $L_0^+(\mathcal{M}, \tau)$ of $L_0(\mathcal{M}, \tau)$ and this extension is still denoted by τ. For a given family $(e_i)_{i\in I}$ of projections, the symbol $\bigwedge_{i\in I} e_i$ will denote the intersection of the family, i.e., the projection onto $\bigcap_{i\in I} e_i(\mathcal{H})$. Next, suppose that a is a self-adjoint τ-measurable operator and let $a = \int_{-\infty}^{\infty} \lambda de_\lambda$ stand for its spectral decomposition. For any Borel subset B of $\mathbb{R}$, the spectral projection of a corresponding to the set B is defined by $I_B(a) = \int_{-\infty}^{\infty} \chi_B(\lambda) de_\lambda$.

For $0 < p < \infty$, we recall that the noncommutative L_p-space (cf. [8, 26]) associated with $(\mathcal{M}, \tau)$ is defined by

$$L_p(\mathcal{M}, \tau) = \{x \in L_0(\mathcal{M}, \tau) : \tau(|x|^p) < \infty\}$$

and equipped with the (quasi-)norm

$$\|x\|_{L_p(\mathcal{M})} = (\tau(|x|^p))^{1/p},$$

where $|x| = (x^*x)^{1/2}$ is the modulus of x. For $p = \infty$, the space $L_p(\mathcal{M}, \tau)$ coincides with $\mathcal{M}$ with its usual operator norm. Recall also that the weak L_p-space $L_{p,\infty}(\mathcal{M}, \tau)$ is defined by

$$L_{p,\infty}(\mathcal{M}, \tau) := \left\{ x \in L_0(\mathcal{M}, \tau) : \sup_\lambda \lambda^p \tau(I_{[\lambda,\infty)}(|x|)) < \infty \right\}$$

and the associated quasi-norm reads

$$\|x\|_{L_{p,\infty}(\mathcal{M})} := \sup_\lambda \lambda \tau(I_{[\lambda,\infty)}(|x|))^{1/p}.$$

For simplicity, we write $\|\cdot\|_p$, $\|\cdot\|_{p,\infty}$ to replace $\|\cdot\|_{L_p(\mathcal{M})}$, $\|\cdot\|_{L_{p,\infty}(\mathcal{M})}$ in the following.

We will also need some basic facts from the theory of Orlicz spaces. Let $\Phi:\mathbb{R}\to\mathbb{R}_+$ be an Orlicz function, i.e., an even convex function such that $\Phi(0)=0$ and $\Phi(\infty)=\infty$. Given $1\leq p\leq\infty$, an Orlicz function Φ is p-convex (respectively, p-concave) if the function $t\mapsto\Phi(t^{1/p})$ is convex (respectively, concave) on $(0,\infty)$. Any Orlicz function Φ gives rise to the corresponding Orlicz space $L_\Phi(0,\alpha)$ (where $\alpha\in(0,\infty]$ is a given parameter), defined as the class of all measurable functions f on $(0,\alpha)$ such that

$$\|f\|_{L_\Phi}=\inf\left\{\lambda>0\,:\,\int_0^\alpha\Phi\left(\frac{|f(t)|}{\lambda}\right)\mathrm{d}t\right\}\leq 1.$$

To define the operator version of this space, recall that for any $t>0$ and any measurable operator x, the associated generalized singular numbers are given by

$$\mu_t(x)=\inf\Big\{s>0\,:\,\tau\big(I_{(s,\infty)}(|x|)\big)\leq t\Big\}.$$

Now, we say that the operator x belongs to the Orlicz space $L_\Phi(\mathcal{M})$, if its generalized singular numbers $t\mapsto\mu_t(x)$ belong to $L_\Phi(0,\tau(I))$.

We now turn our attention to the general setup of noncommutative martingales. Suppose that $(\mathcal{M}_n)_{n\geq0}$ is a filtration, i.e., a nondecreasing sequence of von Neumann subalgebras of $\mathcal{M}$ whose union is weak*-dense in $\mathcal{M}$. Then for any $n\geq0$ there is a normal conditional expectation $\mathcal{E}_n$ from $\mathcal{M}$ onto $\mathcal{M}_n$, satisfying

(i) $\mathcal{E}_n(axb)=a\mathcal{E}_n(x)b$ for all a, $b\in\mathcal{M}_n$ and $x\in\mathcal{M}$;
(ii) $\tau\circ\mathcal{E}_n=\tau$.

It is straightforward to check that the conditional expectations satisfy the tower property $\mathcal{E}_m\mathcal{E}_n=\mathcal{E}_n\mathcal{E}_m=\mathcal{E}_{\min(m,n)}$ for all nonnegative integers m and n. Furthermore, since $\mathcal{E}_n$ is trace preserving, it can be extended to a contractive projection from $L_p(\mathcal{M},\tau)$ onto $L_p(\mathcal{M}_n,\tau_n)$ for all $1\leq p\leq\infty$, where τ_n is the restriction of τ to $\mathcal{M}_n$.

A sequence $x=(x_n)_{n\geq0}$ in $L_1(\mathcal{M})$ is called a noncommutative martingale (with respect, or adapted to $(\mathcal{M}_n)_{n\geq0}$), if for any $n\geq0$ we have the equality

$$\mathcal{E}_n(x_{n+1})=x_n.$$

The associated difference sequence is defined as in the commutative case, with the use of the formulae $dx_0=x_0$ and $dx_n=x_n-x_{n-1}$ for $n\geq1$. If for some given $1\leq p\leq\infty$ we have $x=(x_n)_{n\geq0}\subset L_p(\mathcal{M})$ and

$$\|x\|_p=\sup_{n\geq0}\|x_n\|_p<\infty,$$

then x is said to be a bounded L_p-martingale. An important identification is in order. Suppose that $1\leq p<\infty$ and $x=(x_n)_{n\geq0}$ is a martingale given by $x_n=\mathcal{E}_n(x_\infty)$ for some operator $x_\infty\in L_p(\mathcal{M})$. Then x is a bounded L_p-martingale and $\|x\|_p=\|x_\infty\|_p$. Conversely, if $1<p<\infty$, then every bounded L_p-martingale converges in $L_p(\mathcal{M})$, and so is given by some operator x_∞ as previously. Consequently, one can identify the space of bounded L_p-martingales with the space $L_p(\mathcal{M})$ in the case $1<p<\infty$, with the identification given by $x\mapsto x_\infty$.

Next, let us present some background on noncommutative Hardy spaces. Following [32], we define the column and row versions of square functions relative to a martingale $x = (x_n)_{n\geq 1}$:

$$S_{c,n}(x) = \Big(\sum_{k=1}^{n} |dx_k|^2\Big)^{1/2}, \quad S_c(x) = \Big(\sum_{k=1}^{\infty} |dx_k|^2\Big)^{1/2}.$$

and

$$S_{r,n}(x) = \Big(\sum_{k=1}^{n} |dx_k^*|^2\Big)^{1/2}, \quad S_r(x) = \Big(\sum_{k=1}^{\infty} |dx_k^*|^2\Big)^{1/2}.$$

For $1 \leq p \leq \infty$, the column martingale Hardy space $\mathcal{H}_p^c(\mathcal{M})$ (resp. the row martingale Hardy space $\mathcal{H}_p^r(\mathcal{M})$) is defined to be the space of all martingales x for which $S_c(x) \in L_p(\mathcal{M})$ (resp. $S_r(x) \in L_p(\mathcal{M})$) under the norm, $\|x\|_{\mathcal{H}_p^c} = \|S_c(x)\|_p$ (resp. $\|x\|_{\mathcal{H}_p^r} = \|S_r(x)\|_p$). For $0 < p < 1$, $\mathcal{H}_p^c(\mathcal{M})$ (resp. $\mathcal{H}_p^r(\mathcal{M})$) is the completion of all finite martingale $x \in L_2(\mathcal{M})$ under the quasi-norm $\|\cdot\|_{\mathcal{H}_p^c}$ (resp. $\|\cdot\|_{\mathcal{H}_p^r}$). The noncommutative martingale Hardy spaces $\mathcal{H}_p(\mathcal{M})$ are defined as follows: if $0 < p < 2$, we set

$$\mathcal{H}_p(\mathcal{M}) = \mathcal{H}_p^c(\mathcal{M}) + \mathcal{H}_p^r(\mathcal{M}),$$

and equip it with the (quasi) norm

$$\|x\|_{\mathcal{H}_p} = \inf\big\{\|y\|_{\mathcal{H}_p^c} + \|z\|_{\mathcal{H}_p^r}\big\},$$

where the infimum is taken over all decomposition $x = y + z$ with $y \in \mathcal{H}_p^c$ and $z \in \mathcal{H}_p^r(\mathcal{M})$. When $2 \leq p < \infty$, we put

$$\mathcal{H}_p(\mathcal{M}) = \mathcal{H}_p^c(\mathcal{M}) \cap \mathcal{H}_p^r(\mathcal{M})$$

and consider the norm

$$\|x\|_{\mathcal{H}_p} = \max\big\{\|x\|_{\mathcal{H}_p^c}, \|x\|_{\mathcal{H}_p^r}\big\}.$$

Now we will discuss the notion of tangency, following [12].

Definition 2.1. Two adapted sequences $a = (a_n)_{n\geq 0}$ and $b = (b_n)_{n\geq 0}$ of self-adjoint operators are said to be tangent, if for any $n \geq 0$ and any bounded Borel function f on $\mathbb{R}$ we have

$$\mathcal{E}_{n-1} f(a_n) = \mathcal{E}_{n-1} f(b_n).$$

The following result was established in [12], which can be regarded as a noncommutative version of Theorem 1.3 (ii).

Theorem 2.2. *Let $p \geq 2$ and suppose that x, y are self-adjoint, L_2-bounded martingales such that for each $n = 0, 1, 2, \ldots$ we have*

$$\mathcal{E}_{n-1}|dy_n|^2 \leq \mathcal{E}_{n-1}|dx_n|^2 \quad \text{and} \quad \mathcal{E}_{n-1}|dy_n|^p \leq \mathcal{E}_{n-1}|dx_n|^p. \tag{2.1}$$

Then

$$\|y_n\|_p \leq C_p \|x_n\|_p, \qquad n = 0, 1, 2, \ldots,$$

for some constant C_p depending only on p. Furthermore, $C_p = O(p)$ as $p \to \infty$, which is already optimal in the classical case.

In particular, the L_p estimates hold for martingales with tangent differences: indeed, the condition (2.1) exploits only *two* special convex functions $f(s) = s^2$ and $f(s) = s^p$. From this viewpoint, the tangency assumption is too strong for the moment estimates. On the contrary, the paper [12] contains a striking counterexample showing that under the tangency requirement, the weak-type and the strong-type bounds fail for $1 < p < 2$. More specifically, for any constant κ there are two martingales $x = (x_n)_{n\geq 0}$ and $y = (y_n)_{n\geq 0}$ such that $|dx_n| = |dy_n|$ for each n, and

$$\frac{\tau(I_{[1,\infty)}(|y_N|))}{\|x_N\|_{L_1(\mathcal{M})}} > \kappa, \qquad \frac{\|y_N\|_{L_p(\mathcal{M})}}{\|x_N\|_{L_p(\mathcal{M})}} > \kappa$$

provided N is sufficiently large. This shows that the notion of tangency must invoke some deeper structure hidden in the difference sequences dx and dy. We propose the following.

Definition 2.3. Two sequences $(u_n)_{n\geq 0}$, $(v_n)_{n\geq 0}$ of self-adjoint operators are said to be strongly tangent, if for any n, any projection $R \in \mathcal{M}_{n-1}$ and any bounded Borel function f on $\mathbb{R}$ we have

$$\tau(f(Ru_nR)) = \tau(f(Rv_nR)).$$

In the classical setting, this reduces to the standard tangency. We will prove below the weak-type $(1,1)$ and strong-type (p,p) $(1 < p < 2)$ estimates for martingales whose difference sequences are strongly tangent. Actually, we will study these estimates under less restrictive notion of weak domination, whose classical version was discussed in the introductory section.

Definition 2.4. Let $x = (x_n)_{n\geq 0}$, $y = (y_n)_{n\geq 0}$ be two martingales. Then y is weakly dominated by x, if for any $n \geq 0$, any projection $R \in \mathcal{M}_{n-1}$ and any convex function $f : [0,\infty) \to \mathbb{R}$ with a linear growth at infinity we have

$$\tau(f(Rdy_nR)) \leq \tau(f(Rdx_nR)). \tag{2.2}$$

We will even show more: we will study the weak-type and strong-type estimates assuming that (2.2) holds only for some selected functions f. (Compare to Theorem 1.3).

3. New Gundy-type decomposition

The primary goal of this section is to establish a new type of Gundy's decomposition which enables the efficient study of martingales under the weak domination. This will be one of main tools in our considerations. For the historical perspective, let us mention that Gundy's decomposition for noncommutative martingales was first considered in [30, Theorem 2.1]. However, as we will see below, the version given there does not seem to combine nicely with the notion of weak domination.

Recall that the function φ is given by (1.4). We will prove the following statement.

Theorem 3.1. *Let $x = (x_k)_{k=0}^n$, $y = (y_k)_{k=0}^n$ be self-adjoint martingales such that y is weakly dominated by x. Then for any given number $\lambda > 0$, there exist three martingales α, β and γ satisfying the following properties:*

(i) $y = \alpha + \beta + \gamma$;
(ii) *the martingale α satisfies* $\sum_{k=0}^n \tau\big(\varphi(d\alpha_k/\lambda)\big) \leq 6\sqrt{3}\, \tau(|x_n/\lambda|)$;
(iii) *β and γ are L_1-martingales with*

$$\max\left\{\lambda\tau\Big(\bigvee_{k=0}^n \operatorname{supp}|d\beta_k|\Big),\, \lambda\tau\Big(\bigvee_{k=0}^n \operatorname{supp}|d\gamma_k|\Big)\right\} \leq \tau(|x_n|).$$

In the proof of the above statement, we will work with several different quadratic-linear functions: one of them is φ already defined in (1.4), we will also need ψ, $\eta : \mathbb{R} \to [0, \infty)$ given by the formulas

$$\psi(s) = \begin{cases} s^2 & \text{if } |s| \leq 1, \\ |s| & \text{if } |s| > 1 \end{cases} \qquad \text{and} \qquad \eta(s) = \begin{cases} s^2 & \text{if } |s| \leq 1, \\ 3|s| - 2 & \text{if } |s| > 1. \end{cases}$$

We start with two auxiliary trace inequalities.

Lemma 3.2. *For any self-adjoint operators u, v satisfying $\|u\|_{L_1} < \infty$ and $\|v\|_\infty \leq 1$, we have*

$$2\tau(uv) \leq \tau\big(\varphi(|u|)\big) + \tau(|v|^2) \tag{3.1}$$

and

$$\tau\big(\eta(u)\big) \geq \tau\big(\eta(v)\big) + \tau\big(\eta'(v)(u - v)\big) + \frac{1}{2\sqrt{3}}\tau\big(\varphi(u - v)\big). \tag{3.2}$$

Proof. To show (3.1), we first establish its pointwise version: for $a \in \mathbb{R}$ and $b \in [-1, 1]$ we have

$$2ab \leq \varphi(|a|) + b^2. \tag{3.3}$$

If $|a| \leq 1$, then the estimate is equivalent to the obvious bound $2ab \leq a^2 + b^2$. If $|a| > 1$, then $0 < |a| - 1 \leq |a| - |b|$; squaring both sides we get an inequality equivalent to $2|ab| \leq 2|a| - 1 + b^2$, from which (3.3) follows. Now we turn to the noncommutative setting. By standard approximation, we may and do assume that $u = \sum_{j=1}^m a_j e_j$, $v = \sum_{k=1}^n b_k f_k$ for some scalars $(a_j)_{j=0}^m$, $(b_k)_{k=0}^n$ and some sequences of orthogonal projections $(e_j)_{j=0}^m$ and $(f_k)_{k=0}^n$ satisfying $\sum_{j=0}^m e_j = \sum_{k=0}^n f_k = I$. Using (3.3) and the fact that $\tau(ef) \geq 0$ for any projections e, f, we may write

$$\begin{aligned} 2\tau(uv) &= 2\sum_{j=1}^{m}\sum_{k=1}^{n}\tau(e_jf_k)a_jb_k \\ &\leq \sum_{j=1}^{m}\sum_{k=1}^{n}\tau(e_jf_k)\varphi(|a_j|) + \sum_{j=1}^{m}\sum_{k=1}^{n}\tau(e_jf_k)b_k^2 \\ &= \sum_{j=1}^{m}\tau(e_j)\varphi(|a_j|) + \sum_{k=1}^{n}\tau(f_k)b_k^2 \\ &= \tau\big(\varphi(|u|)\big) + \tau(|v|^2), \end{aligned}$$

so (3.1) is proved.

Now we verify (3.2). Again we start with a certain pointwise estimate: for any $a \in \mathbb{R}$ and $b \in [-1, 1]$ we have

$$\eta(a) \geq \eta(b) + \eta'(b)(a-b) + (1+|a|)^{-1}(a-b)^2. \tag{3.4}$$

(note the slight difference between the last term above and the last term in (3.2); we will comment on it in Remark 3.3 below). If $|a| \leq 1$, then the inequality is equivalent to $(a-b)^2 \geq (1+|a|)^{-1}(a-b)^2$, which is evident. If $|a| > 1$, then the estimate reads

$$3|a| - 2 \geq b^2 + 2b(a-b) + \frac{(a-b)^2}{1+|a|}.$$

However, the right-hand side can be transformed into

$$a^2 - (a-b)^2 + \frac{(a-b)^2}{1+|a|} = |a|\left[|a| - \frac{(a-b)^2}{1+|a|}\right] \leq |a|\left[|a| - \frac{(|a|-1)^2}{1+|a|}\right] = \frac{|a|(3|a|-1)}{1+|a|},$$

which is easily shown to be not bigger than $3|a| - 2$. Therefore, the pointwise bound (3.4) is established; let us rewrite it in the form

$$\eta(a) \geq \eta(b) + \eta'(b)a - \eta'(b)b + (1+|a|)^{-1}a^2 - 2(1+|a|)^{-1}ab + (1+|a|)^{-1}b^2. \tag{3.5}$$

By the approximation of operators u, v by finite linear combinations of projections, the above estimate implies

$$\begin{aligned} \tau\big(\eta(u)\big) &\geq \tau\big(\eta(v)\big) + \tau\big(\eta'(v)(u-v)\big) + \tau\big((1+|u|)^{-1}u^2\big) \\ &\qquad - \tau\big(2(1+|u|)^{-1}uv + (1+|u|)^{-1}v^2\big) \\ &= \tau\big(\eta(v)\big) + \tau\big(\eta'(v)(u-v)\big) + \tau\big((1+|u|)^{-1}(u-v)^2\big), \end{aligned} \tag{3.6}$$

by the tracial property. Now, we have

$$(1+|u|)^2 \leq 3 + \frac{3}{2}|u-v+v|^2 \leq 3 + 3|u-v|^2 + 3|v|^2 \leq 6 + 3|u-v|^2,$$

and hence by the operator monotonicity of the function $t \mapsto t^{-1/2}$, we get

$$(1+|u|)^{-1} \geq (6+3|u-v|^2)^{-1/2}.$$

Consequently,

$$\tau\big((1+|u|)^{-1}(u-v)^2\big) \geq \tau\Big((6+3|u-v|^2)^{-1/2}(u-v)^2\Big)$$

and hence, in order to get (3.2), it suffices to prove a simple pointwise estimate

$$(6+3s^2)^{-1/2}s^2 \geq \frac{1}{2\sqrt{3}}\varphi(s), \qquad s \in \mathbb{R}.$$

If $|s| \leq 1$, then $6+3s^2 \leq 9 < 12$ and $\varphi(s) = s^2$, so the estimate holds. If $|s| > 1$, then the inequality becomes

$$\frac{s^2}{\sqrt{6+3s^2}} \geq \frac{2|s|-1}{2\sqrt{3}},$$

or, after squaring and some simple manipulations, $4|s|^3 + 8|s| \geq 9s^2 + 2$. But

$$4|s|^3 + 8|s| \geq 4|s|\left(s^2 + \frac{3}{2}\right) + 2 \geq 4\sqrt{6}s^2 + 2 \geq 9s^2 + 2,$$

so the claim follows. □

Remark 3.3. Let us explain the reason why we have studied the estimate (3.4) instead of

$$\eta(a) \geq \eta(b) + \eta'(b)(a-b) + \frac{1}{2\sqrt{3}}\varphi(b-a). \tag{3.7}$$

The crucial fact is that (3.4) admits the alternative reformulation (3.5), in which each term is of the form $F(a)G(b)$ for some Borel functions F and G. Such a factorization allows the immediate passage to the operator version (3.6), as we have seen in the proof of (3.1). On the contrary, the more natural bound (3.7) does not have this structural property and hence the direct passage to the operator counterpart seems problematic.

The next lemma describes a simple but powerful control of difference sequence over a martingale.

Lemma 3.4. *For any self-adjoint martingale $x = (x_k)_{k=0}^n$ we have*

$$\tau\big(\psi(x_n)\big) \leq \frac{4}{3}\sum_{k=0}^n \tau\big(\varphi(dx_k)\big).$$

Proof. Consider the operator $z = x_n I_{[-1,1]}(x_n) + x_n|x_n|^{-1} I_{\mathbb{R}\setminus[-1,1]}(x_n)$ and let $(z_k)_{k=0}^n = (\mathcal{E}_k z)_{k=0}^n$ be the associated martingale. Then, using the orthogonality of the martingale difference sequences, we have the identity

$$\tau\big(\psi(x_n)\big) = \tau(x_n z_n) = \sum_{k=0}^n \tau(dx_k dz_k).$$

By Lemma 3.2 and the orthogonality again,

$$\begin{aligned}\sum_{k=0}^{n}\tau(dx_k dz_k) &= 2\sum_{k=0}^{n}\tau(dx_k(dz_k/2))\\ &\leq \sum_{k=0}^{n}\tau\big(\varphi(dx_k)\big)+\frac{1}{4}\sum_{k=0}^{n}\tau(dz_k^2)\\ &= \sum_{k=0}^{n}\tau\big(\varphi(dx_k)\big)+\frac{1}{4}\tau(z^2)\\ &\leq \sum_{k=0}^{n}\tau\big(\varphi(dx_k)\big)+\frac{1}{4}\tau\big(\psi(x_n)\big).\end{aligned}$$

Here the last bound follows from the trivial estimate

$$z^2=|x_n|^2 I_{[-1,1]}(x_n)+I_{\mathbb{R}\setminus[-1,1]}(x_n)\leq\psi(x_n).$$

Putting all the above facts together, we get the desired assertion. □

Now we will introduce the sequences of Cuculescu projections associated with the martingale $x=(x_k)_{k=0}^n$ at level 1 (cf. [7]). Define $R=(R_k)_{k=-1}^n$, $D=(D_k)_{k=0}^n$ and $U=(U_k)_{k=0}^n$ by the following recursive procedure: $R_{-1}=I$ and

$$\begin{aligned}R_k &= R_{k-1}I_{(-1,1)}(R_{k-1}x_kR_{k-1}),\\ U_k &= R_{k-1}I_{(-\infty,-1]}(R_{k-1}x_kR_{k-1}),\\ D_k &= R_{k-1}I_{[1,\infty)}(R_{k-1}x_kR_{k-1}),\end{aligned}$$

for $k=0,1,2,\ldots,n$. Directly from the definition, we see that the operators R_k and $R_{k-1}x_kR_{k-1}$ commute for each k; moreover, $-U_kx_kU_k$ and $D_kx_kD_k$ are positive operators. We will also need the simple estimate

$$\tau(I-R_n)\leq\tau((I-R_n)|x_n|),\qquad n=0,1,2,\cdots$$

(see e.g. [10]).

Now we are going to exploit the following estimate.

Lemma 3.5. *For any L_1-bounded self-adjoint martingale $x=(x_k)_{k=0}^n$ we have*

$$\Big\|\sum_{k=0}^{n}(R_{k-1}-R_k)|R_{k-1}x_kR_{k-1}|\Big\|_{L_1(\mathcal{M})}\leq\tau\big((I-R_n)|x_n|\big).$$

Proof. First, note that by the commuting property of R_k and $R_{k-1}x_kR_{k-1}$ we may write

$$\begin{aligned}(R_{k-1}x_kR_{k-1})^2 &= R_{k-1}x_kR_{k-1}x_kR_{k-1}\\ &= (R_{k-1}-R_k)x_kR_{k-1}x_kR_{k-1}+R_kx_kR_{k-1}x_kR_{k-1}\\ &= (R_{k-1}-R_k)x_k(R_{k-1}-R_k)x_k(R_{k-1}-R_k)+R_kx_kR_kx_kR_k\\ &= \big((R_{k-1}-R_k)x_k(R_{k-1}-R_k)\big)^2+\big(R_kx_kR_k\big)^2,\end{aligned}$$

which implies

$$(R_{k-1}-R_k)|R_{k-1}x_kR_{k-1}|(R_{k-1}-R_k) = |(R_{k-1}-R_k)x_k(R_{k-1}-R_k)|.$$

Now, by the commuting properties of the projections R_n, U_n and D_n, we can write

$$\begin{aligned}(R_{k-1}-R_k)x_k(R_{k-1}-R_k) &= (R_{k-1}-R_k)R_{k-1}x_kR_{k-1}\\ &= (U_k+D_k)R_{k-1}x_kR_{k-1}\\ &= U_kx_kU_k + D_kx_kD_k.\end{aligned}$$

As we have already mentioned above, the operator $U_kx_kU_k$ is negative and $D_kx_kD_k$ is positive. Therefore, the triangle inequality and the martingale property of x give

$$\begin{aligned}\tau\Big(|(R_{k-1}-R_k)x_k(R_{k-1}-R_k)|\Big) &\leq -\tau(U_kx_kU_k) + \tau(D_kx_kD_k)\\ &= -\tau(U_kx_nU_k) + \tau(D_kx_nD_k)\\ &\leq \tau\Big(U_k|x_n|U_k + \tau(D_k|x_n|D_k\Big)\\ &= \tau\Big((U_k+D_k)|x_n|\Big)\\ &= \tau\Big((R_{k-1}-R_k)|x_n|\Big).\end{aligned}$$

From this estimate, we conclude that

$$\begin{aligned}\left\|\sum_{k=0}^n (R_{k-1}-R_k)|R_{k-1}x_kR_{k-1}|\right\|_{L_1(\mathcal{M})} &\leq \sum_{k=0}^n \big\|(R_{k-1}-R_k)|R_{k-1}x_kR_{k-1}|\big\|_{L_1(\mathcal{M})}\\ &= \sum_{k=0}^n \tau\Big(\big|(R_{k-1}-R_k)x_k(R_{k-1}-R_k)\big|\Big)\\ &\leq \sum_{k=0}^n \tau\big((R_{k-1}-R_k)|x_n|\big) = \tau\big((I-R_n)|x_n|\big).\end{aligned}$$

The claim is proved. □

We will also need the following auxiliary result, which is, in a sense, a converse to Lemma 3.4.

Lemma 3.6. *We have*

$$\tau\Big(\big(R_nx_nR_n\big)^2 + (I-R_n)(3|x_n|-2)\Big) \geq \frac{1}{2\sqrt{3}}\sum_{k=0}^n \tau\Big(\varphi\big(R_{k-1}dx_kR_{k-1}\big)\Big).$$

Proof. Pick $k \in \{1, 2, \ldots, n\}$ and plug $u = R_{k-1}x_kR_{k-1}$ and $v = R_{k-1}x_{k-1}R_{k-1}$ into (3.2). We have the identity

$$\tau\big(\eta'(v)(u-v)\big) = 2\tau\Big(R_{k-1}x_{k-1}R_{k-1}dx_kR_{k-1}\Big) = 0,$$

since R_{k-1}, $x_{k-1}\in\mathcal{M}_{k-1}$ and $\mathcal{E}_{k-1}dx_k=0$; consequently, (3.2) gives

$$\tau\Big(\eta\big(R_{k-1}x_kR_{k-1}\big)\Big)\geq\tau\Big(\eta\big(R_{k-1}x_{k-1}R_{k-1}\big)\Big)+\frac{1}{2\sqrt{3}}\tau\Big(\varphi\big(R_{k-1}dx_kR_{k-1}\big)\Big).$$

However, directly from the definition of Cuculescu projections, we have

$$\begin{aligned}\eta\big(R_{k-1}x_kR_{k-1}\big)&=R_k(R_{k-1}x_kR_{k-1})^2+(R_{k-1}-R_k)\big(3|R_{k-1}x_kR_{k-1}|-2\big)\\&=\eta(R_kx_kR_k)+(R_{k-1}-R_k)\big(3|R_{k-1}x_kR_{k-1}|-2\big),\end{aligned}$$

which combined with the preceding estimate yields

$$\begin{aligned}\tau\Big(\eta\big(R_kx_kR_k\big)\Big)\geq\tau\Big(\eta\big(R_{k-1}x_{k-1}R_{k-1}\big)\Big)&+\frac{1}{2\sqrt{3}}\tau\Big(\varphi\big(R_{k-1}dx_kR_{k-1}\big)\Big)\\&-\tau\Big(3(R_{k-1}-R_k)|R_{k-1}x_kR_{k-1}|\Big)+2\tau(R_{k-1}-R_k).\end{aligned}$$

Therefore, summing over k, we get

$$\begin{aligned}\tau\Big(\eta\big(R_nx_nR_n\big)\Big)\geq\tau\Big(\eta\big(R_0x_0R_0\big)\Big)&+\frac{1}{2\sqrt{3}}\sum_{k=1}^n\tau\Big(\varphi\big(R_{k-1}dx_kR_{k-1}\big)\Big)\\&-3\sum_{k=1}^n\tau\Big((R_{k-1}-R_k)|R_{k-1}x_kR_{k-1}|\Big)+2\tau(R_0-R_n).\end{aligned}$$

Now, observe that

$$\begin{aligned}\tau\Big(\eta(R_0x_0R_0)+3(I-R_0)|x_0|\Big)&\geq\tau\big(\psi(dx_0)\big)+2\tau(I-R_0)\\&\geq(2\sqrt{3})^{-1}\tau\big(\varphi(dx_0)\big)+2\tau(I-R_0),\end{aligned}$$

and hence the previous inequality leads to

$$\begin{aligned}\tau\Big(\eta\big(R_nx_nR_n\big)\Big)\geq&\frac{1}{2\sqrt{3}}\sum_{k=0}^n\tau\Big(\varphi\big(R_{k-1}dx_kR_{k-1}\big)\Big)\\&-3\sum_{k=0}^n\tau\Big((R_{k-1}-R_k)|R_{k-1}x_kR_{k-1}|\Big)+2\tau(I-R_n).\end{aligned}$$

It remains to apply Lemma 3.5 and use the identity $\eta\big(R_nx_nR_n\big)=(R_nx_nR_n)^2$ to get the assertion. $\square$

Now we are ready to show Gundy's decomposition.

Proof of Theorem 3.1. By homogeneity, we may and do assume that $\lambda=1$. For each $0\leq k\leq n$, we define

$$\begin{cases}d\alpha_k&:=R_{k-1}dy_kR_{k-1};\\d\beta_k&:=(I-R_{k-1})dy_kR_{k-1};\\d\gamma_k&:=dy_k(I-R_{k-1}).\end{cases}$$

Obviously, $d\alpha$, $d\beta$ and $d\gamma$ are martingale differences relative to $(\mathcal{M}_k)_{k=0}^n$. Denoting the associated martingales by α, β and γ, we easily see that $y = \alpha + \beta + \gamma$, so the required property (i) is satisfied. To check (ii), note that since y is weakly dominated by x, we have

$$\tau(\varphi(d\alpha_k)) = \tau\Big(\varphi\big(R_{k-1}dy_kR_{k-1}\big)\Big) \leq \tau\Big(\varphi\big(R_{k-1}dx_kR_{k-1}\big)\Big) = \tau\Big(\varphi\big|R_{k-1}dx_kR_{k-1}\big|\Big).$$

Based on the fact that $-I \leq R_{k-1}dx_kR_{k-1} \leq I$, we get $|R_{k-1}dx_kR_{k-1}| \leq 1$, so by Lemma 3.6 and homogeneity we obtain

$$\sum_{k=0}^{n} \tau\big(\varphi(d\alpha_k)\big) \leq 2\sqrt{3}\tau\Big((R_nx_nR_n)^2 + (I - R_n)(3|x_n| - 2)\Big).$$

Setting $x_n^+ = x_nI_{[0,\infty)}(x_n)$ and $x_n^- = x_n^+ - x_n$, we see that $x_n^\pm \geq 0$ and hence

$$\begin{aligned}\tau((R_nx_nR_n)^2) &= \tau(R_nx_nR_nx_n)\\ &= \tau(R_nx_nR_nx_n^+) + \tau\big((-R_nx_nR_n)(x_n^-)\big)\\ &\leq \tau(R_nx_n^+) + \tau(R_nx_n^-) = \tau(R_n|x_n|).\end{aligned}$$

Combining this with the previous estimate, we get

$$\begin{aligned}\sum_{k=0}^{n} \tau\big(\varphi(d\alpha_k)\big) &\leq 2\sqrt{3}\tau\Big(3|x_n| - 2(I - R_n)\Big)\\ &= 6\sqrt{3}\tau(|x_n|) - 4\sqrt{3}\tau(I - R_n) \leq 6\sqrt{3}\tau(|x_n|).\end{aligned}$$

This verifies (ii) and it remains to handle (iii). The left support of $d\beta_k$ satisfies

$$\ell(d\beta_k) \leq I - R_{k-1} \leq I - R_n.$$

Therefore,

$$\tau\left(\bigvee_{k=0}^{n} \ell(d\beta_k)\right) \leq \tau(I - R_n) \leq \tau\left((I - R_n)|x_n|\right). \tag{3.8}$$

A similar analysis of the right support of $d\gamma_n$ yields

$$\tau(r(\gamma_n)) \leq \tau\left(\bigvee_{k=0}^{n} \ell(d\gamma_k)\right) \leq \tau(I - R_n) \leq \tau\left((I - R_n)|x_n|\right), \tag{3.9}$$

which completes the proof. □

4. Inequalities for noncommutative weak dominated martingales

Based on our new Gundy-type decomposition, we turn our attention to the investigation of the weak-type and strong-type inequalities. This section is split into two parts, devoted to estimates for weakly dominated martingales and square function inequalities for weakly dominated martingales, respectively.

4.1. **Estimates for weakly dominated martingales.** Our main result can be formulated as follows. Recall the function φ given in (1.4).

Theorem 4.1. *Suppose that $x=(x_n)_{n\geq 0}$, $y=(y_n)_{n\geq 0}$ are two self-adjoint martingales such that for any $n\geq 0$ and any projection $R\in\mathcal{M}_{n-1}$ we have*

$$\tau\big(\varphi(Rdy_nR)\big)\leq \tau\big(\varphi(Rdx_nR)\big).$$

(Here we assume that $\mathcal{M}_{-1}=\mathbb{R}I$.) Then we have the estimate

$$\tau\big(I_{[3,\infty)}(|y_n|)\big)\leq 8\sqrt{3}\tau(|x_n|),\qquad n=0,\,1,\,2,\,\ldots. \tag{4.1}$$

Remark 4.2. The above result give us the noncommutative version of Theorem 1.3 (i): if x and y are self-adjoint martingales such that for any $n\geq 0$, any projection $R\in\mathcal{M}_{n-1}$ and any $\lambda>0$ we have

$$\tau\big(\varphi(\lambda Rdy_nR)\big)\leq \tau\big(\varphi(\lambda Rdx_nR)\big),$$

then

$$\|y_n\|_{1,\infty}\leq 24\sqrt{3}\|x_n\|_1,\qquad n=0,\,1,\,2,\,\ldots.$$

As we mentioned earlier, the noncommutative version of Theorem 1.3 (ii) was already given in the paper [12].

Proof of Theorem 4.1. Fix n and consider the finite martingales $(x_k)_{k=0}^n$, $(y_k)_{k=0}^n$. We apply Theorem 3.1 with $\lambda=1$, obtaining the decomposition $y=\alpha+\beta+\gamma$. By the well-known properties of a distribution function, we have

$$\tau\left(I_{[3,\infty)}(|y_n|)\right)\leq \tau\left(I_{[1,\infty)}(|\alpha_n|)\right)+\tau\left(I_{[1,\infty)}(|\beta_n|)\right)+\tau\left(I_{[1,\infty)}(|\gamma_n|)\right).$$

Inspecting the proof of Theorem 3.1, we may write down the refined estimate

$$\tau\left(I_{[1,\infty)}(|\alpha_n|)\right)\leq \tau\big(\psi(\alpha_n)\big)\leq \frac{4}{3}\sum_{k=0}^n\tau\big(\varphi(d\alpha_k)\big)\leq 8\sqrt{3}\tau(|x_n|)-\frac{16\sqrt{3}}{3}\tau(I-R_n). \tag{4.2}$$

Furthermore, again by Theorem 3.1, we deduce that

$$\tau\Big(I_{[1,\infty)}(|\beta_n|)\Big)\leq \tau\Big(\bigvee_{k=0}^n\ell(d\beta_k)\Big)\leq \tau(I-R_n) \tag{4.3}$$

and

$$\tau\left(I_{[1,\infty)}(|\gamma_n|)\right)\leq \tau(r(\gamma_n))\leq \tau\Big(\bigvee_{k=0}^n\ell(d\gamma_k)\Big)\leq \tau(I-R_n). \tag{4.4}$$

Plugging (4.2), (4.3) and (4.4) into the previous estimate, we get the desired claim. □

To establish the corresponding L_p bound, we need to introduce several auxiliary objects. Let $x=(x_k)_{k=0}^n$, $y=(y_k)_{k=0}^n$ be two self-adjoint martingales. For a fixed $\lambda>0$, let $(R_k^\lambda)_{k=-1}^n$ be the sequence of projections as before, built on the martingale x/λ: that is, we have $R_{-1}^\lambda=I$ and, for any $k=0,\,1,\,2,\,\ldots,\,n$,

$$R_k^\lambda=R_{k-1}^\lambda I_{(-\lambda,\lambda)}(R_{k-1}^\lambda x_kR_{k-1}^\lambda).$$

Next, we recall a slight modification of projections introduced by Randrianantoanina (see e.g., [33, 34]). For any $0 \leq k \leq n$ and $i \in \mathbb{Z}$, we set

$$q_{i,k} := \bigwedge_{\ell=i}^{\infty} R_k^{2^\ell} \quad \text{and} \quad \pi_{i,k} := q_{i,k} - q_{i-1,k}. \tag{4.5}$$

The family $\{q_{i,k}\}_{k,i}$ is decreasing on n and increasing on i. Therefore, for every $k = 0, 1, 2, \ldots, n$, $(\pi_{i,k})_{i\in\mathbb{Z}}$ is a sequence of pairwise disjoint projections satisfying the trivial but crucial identity that for every $k \in \mathbb{Z}$,

$$\sum_{i=-\infty}^{j} \pi_{i,k} = q_{j,k}$$

where the convergence of the series is relative to the strong operator topology. The properties and more details about the Cuculescu projections and the family of projections $\{q_{i,k}\}$ can be found in [33, 34].

Equipped with the above notation, we establish the following auxiliary distributional estimate.

Lemma 4.3. *For any $i \in \mathbb{Z}$, we have*

$$\tau\left(I_{[3\cdot 2^i,\infty)}(|y_n|)\right) \leq \frac{8\sqrt{3}}{3} \cdot 2^{-2i}\tau(q_{i,n}x_nq_{i,n}x_nq_{i,n}) + 24\sqrt{3}\cdot 2^{-i}\tau\big((I-q_{i,n})|x_n|\big).$$

Proof. A careful look at the proof of Theorem 4.1, together with a homogenization argument, reveals that we have actually proved the stronger inequality

$$\tau\left(I_{[3,\infty)}(|y_n|/\lambda)\right) \leq \frac{8\sqrt{3}}{3}\tau\big(\lambda^{-2}(R_n^\lambda x_n R_n^\lambda)^2 + 3\lambda^{-1}(I-R_n^\lambda)|x_n|\big).$$

Applying the above estimate with $\lambda = 2^i$, we get

$$\tau\left(I_{[3\cdot 2^i,\infty)}(|y_n|)\right) \leq \frac{8\sqrt{3}}{3} \cdot 2^{-2i}\tau(R_n^{2^i}x_nR_n^{2^i}x_nR_n^{2^i}) + 8\sqrt{3}\cdot 2^{-i}\tau\big((I-R_n^{2^i})|x_n|\big).$$

Following the argument in the proof of [14, Lemma 2.5(iii)], we obtain

$$\tau\left(R_n^{2^i}x_nR_n^{2^i}x_nR_n^{2^i}\right) \leq \tau(q_{i,n}x_nq_{i,n}x_nq_{i,n}) + 6\cdot 2^i\tau\left((I-q_{i,n})|x_n|\right).$$

Since $q_{i,n}$ is a subprojection of $R_n^{2^i}$, we get

$$\tau\left(I_{[3\cdot 2^i,\infty)}(|y_n|)\right) \leq \frac{8\sqrt{3}}{3} \cdot 2^{-2i}\tau(q_{i,n}x_nq_{i,n}x_nq_{i,n}) + 24\sqrt{3}\cdot 2^{-i}\tau\big((I-q_{i,n})|x_n|\big).$$

This completes the proof. □

Using the above distributional estimate, we can get the following strong-type inequalities by the same reasoning as in [10, 14]. We omit the details.

Corollary 4.4. *Let $1 < p < 2$. Suppose that $x = (x_n)_{n\geq 0}$, $y = (y_n)_{n\geq 0}$ are martingales such that for any $n \geq 0$ and any projection $R \in \mathcal{M}_{n-1}$ and any $\lambda > 0$ we have*

$$\tau(\varphi(\lambda R dy_n R)) \leq \tau(\varphi(\lambda R dx_n R)).$$

Then we have the estimate

$$\|y_n\|_p \leq C_p \|x_n\|_p, \quad n = 0, 1, 2, \cdots,$$

with $C_p = O((p-1)^{-1})$ as $p \to 1$.

4.2. **Estimates for the square functions of weakly dominated martingales.** Next, we proceed to the following weak-type estimate for the square functions of weakly dominated martingales.

Theorem 4.5. *Suppose that x is a self-adjoint L_2-martingale and y is a self-adjoint martingale that is weakly dominated by x. Then there exist two martingales y^r and y^c such that $y = y^c + y^r$ and we have*

$$\left\|S_c(y^c)\right\|_{1,\infty} + \left\|S_r(y^r)\right\|_{1,\infty} \leq K\left\|x\right\|_1,$$

for some absolute constant K.

To prove Theorem 4.5, we need some preparation. Let $\mathsf{P} = \{p_i\}_{i=1}^m$ be a finite sequence of mutually disjoint projections in $\mathcal{M}$. We denote by $\mathcal{T}^{(\mathsf{P})}$ the triangular truncation with respect to P, which acts on an arbitrary operator $a \in L_0(\mathcal{M}, \tau)$ by

$$\mathcal{T}^{(\mathsf{P})} a = \sum_{1\leq i\leq j\leq m} p_i a p_j.$$

The proof of Theorem 4.5 will depend on the following endpoint inequalities for this object.

Proposition 4.6. *Let Φ be a p-convex and q-concave Orlicz function with $1 \leq p \leq q \leq 2$. If $(\mathsf{P}_k)_{k\geq 1}$ is a family of finite sequences of mutually disjoint projections and $(a_k)_{k\geq 1}$ is a sequence in $L_\Phi(\mathcal{M})$, then*

$$\sup_{\lambda>0} \Phi(\lambda)\tau\Big(I_{(\lambda,\infty)}\Big(\sum_{k\geq 1} |\mathcal{T}^{(\mathsf{P}_k)} a_k|^2\Big)^{1/2}\Big) \lesssim_{p,q} \sum_{k\geq 1} \tau\left(\Phi(|a_k|)\right).$$

Before showing Proposition 4.6, we firstly recall and establish several lemmas.

Lemma 4.7. [15, Page 133] *Let $1 \leq p \leq q < \infty$ and Φ be a p-convex and q-concave Orlicz function. Then*

$$\Phi(t) \simeq_{p,q} \int_0^\infty \min\{(ts)^p, (ts)^q\} d\nu(s)$$

for some positive measure ν.

Here and below, $(e_{i,j})_{i,j\geq 1}$ will denote the collection of matrix units of $B(\ell_2)$: for any i, $j \geq 1$, $e_{i,j}$ is the element of $B(\ell_2)$ whose all entries in the matrix representation are equal to zero, except for that standing in the intersection of i-th row and j-th column.

Lemma 4.8. *Let Φ be a p-convex and q-concave Orlicz function with $1 \le p \le q < \infty$. Assume that $\mathcal{M}$ and $\mathcal{N}$ are semifinite von Neumann algebras. For $k \ge 1$, let $T_k : L_p(\mathcal{M}) + L_q(\mathcal{M}) \to L_{p,\infty}(\mathcal{N}) + L_{q,\infty}(\mathcal{N})$ be linear operators such that*

$$\Big\| \sum_{k\ge1} T_k(x_k) \otimes e_{k,1} \Big\|_{L_{r,\infty}(\mathcal{M}\overline{\otimes}B(\ell_2))} \lesssim_r \Big\| \sum_{k\ge1} x_k \otimes e_{k,k} \Big\|_{L_r(\mathcal{M}\overline{\otimes}B(\ell_2))} \ \big(x_k \in L_r,\ r = p, q\big). \tag{4.6}$$

Then for any sequence $(x_k)_{k\ge1} \in L_\Phi(\mathcal{M})$, we have

$$\sup_{\lambda>0} \Phi(\lambda)\tau \otimes \mathrm{tr}\Big(I_{(\lambda,\infty)}\Big(\Big|\sum_{k\ge1} T_k(x_k)\otimes e_{k,1}\Big|\Big)\Big) \lesssim_{p,q} \tau\otimes\mathrm{tr}\Big(\Phi\Big(\Big|\sum_{k\ge1} x_k\otimes e_{k,k}\Big|\Big). \tag{4.7}$$

Proof. Without loss of generality, we assume that x_k are positive operators. Let us first show the result for $\Phi_{p,q}(t) := \min\{t^p, t^q\}$. For each $k \ge 1$, set $p_k = I_{[0,1]}(x_k)$. Let $x = \sum_{k\ge1} x_k \otimes e_{k,k}$ and $\tilde{p} = \sum_{k\ge1} p_k \otimes e_{k,k} = I_{[0,1]}(x)$. Fix $\lambda > 0$. Then we have

$$\begin{aligned}
&\tau\otimes\mathrm{tr}\Big(I_{(2\lambda,\infty)}\Big(\Big|\sum_{k\ge1} T_k(x_k)\otimes e_{k,1}\Big|\Big)\Big)\\
&\le \tau\otimes\mathrm{tr}\Big(I_{(\lambda,\infty)}\Big(\Big|\sum_{k\ge1} T_k(x_kp_k)\otimes e_{k,1}\Big|\Big)\Big) + \tau\otimes\mathrm{tr}\Big(I_{(\lambda,\infty)}\Big(\Big|\sum_{k\ge1} T_k(x_kp_k^\perp)\otimes e_{k,1}\Big|\Big)\Big).
\end{aligned}$$

Using the assumption (4.6), we get

$$\begin{aligned}
\tau\otimes\mathrm{tr}&\Big(I_{(2\lambda,\infty)}\Big(\Big|\sum_{k\ge1} T_k(x_kp_k)\otimes e_{k,1}\Big|\Big)\Big)\\
&\le \lambda^{-q}\Big\|\sum_{k\ge1} x_kp_k\otimes e_{k,k}\Big\|_q^q + \lambda^{-p}\Big\|\sum_{k\ge1} x_kp_k^\perp\otimes e_{k,k}\Big\|_p^p\\
&= \|x\tilde{p}\|_q^q + \|x\tilde{p}^\perp\|_p^p.
\end{aligned}$$

It is clear that $\|x\tilde{p}\|_q^q = \tau\otimes\mathrm{tr}\big(\Phi_{p,q}(x\tilde{p})\big)$ and $\|x\tilde{p}^\perp\|_p^p = \tau\otimes\mathrm{tr}\big(\Phi_{p,q}(x\tilde{p}^\perp)\big)$. Therefore,

$$\Phi_{p,q}(\lambda)\tau\otimes\mathrm{tr}\Big(I_{(2\lambda,\infty)}\Big(\Big|\sum_{k\ge1} T_k(x_k)\otimes e_{k,1}\Big|\Big)\Big) \le \tau\otimes\mathrm{tr}(\Phi_{p,q}(x)) = \tau\otimes\mathrm{tr}\Big(\Phi\Big(\sum_{k\ge1} x_k\otimes e_{k,k}\Big)\Big).$$

Observe that $\Phi_{p,q}(2t) \le 2^{q-1}\Phi_{p,q}(t)$ for $t > 0$. Taking the supremum over λ, we get the estimate (4.7) for $\Phi_{p,q}$:

$$\sup_{\lambda>0} \Phi_{p,q}(\lambda)\tau\Big(I_{(\lambda,\infty)}\Big(\Big|\sum_{k\ge1} T_k(x_k)\otimes e_{k,1}\Big|\Big)\Big) \le 2^{q-1}\tau\otimes\mathrm{tr}\Big(\Phi_{p,q}\Big(\sum_{k\ge1} x_k\otimes e_{k,k}\Big)\Big).$$

For general Φ, by Lemma 4.7 and the above estimate, we have

$$\begin{aligned}
&\Phi(\lambda)\tau\otimes\mathrm{tr}\Big(I_{(\lambda,\infty)}\Big(\Big|\sum_{k\geq 1}T_k(x_k)\otimes e_{k,1}\Big|\Big)\Big)\\
&\simeq_{p,q}\int_0^\infty \Phi_{p,q}(\lambda s)\tau\Big(I_{(\lambda s,\infty)}\Big(\Big|\sum_{k\geq 1}T_k(sx_k)\otimes e_{k,1}\Big|\Big)\Big)d\nu(s)\\
&\lesssim_q \int_0^\infty \tau\otimes\mathrm{tr}\Big(\Phi_{p,q}\Big(\sum_{k\geq 1}sx_k\otimes e_{k,k}\Big)\Big)d\nu(s)\\
&\simeq_{p,q}\tau\otimes\mathrm{tr}\Big(\Phi\Big(\sum_{k\geq 1}x_k\otimes e_{k,k}\Big)\Big).
\end{aligned}$$

The desired assertion follows. □

Proof of Proposition 4.6. It is easy to see that $\mathcal{T}^{(\mathsf{P}_k)}$ is linear for any $k\geq 1$. According to [33, Proposition 1.6],

$$\Big\|\sum_{k\geq 1}\mathcal{T}^{(\mathsf{P}_k)}(x_k)\otimes e_{k,1}\Big\|_{1,\infty}\lesssim_r \Big\|\sum_{k\geq 1}x_k\otimes e_{k,k}\Big\|_1 \quad \big(x_k\in L_1(\mathcal{M})\big).$$

Moreover, since triangular truncations are contractive in $L_2(\mathcal{M})$, it follows that

$$\Big\|\sum_{k\geq 1}\mathcal{T}^{(\mathsf{P}_k)}(x_k)\otimes e_{k,1}\Big\|_{2}\lesssim_r \Big\|\sum_{k\geq 1}x_k\otimes e_{k,k}\Big\|_2 \quad \big(x_k\in L_2(\mathcal{M})\big).$$

The desired result now follows from Lemma 4.8. □

Now we are ready to show the weak-type estimate for square functions of weakly dominated martingales.

Proof of Theorem 4.5. Recall the projections $q_{i,n}$, $\pi_{i,n}$ defined as in (4.5). For $n\geq 0$, we set

$$\begin{cases} p_{0,n} & := q_{0,n}\\ p_{i,n} & := \pi_{i,n} \quad \text{for } i\geq 1.\end{cases}$$

The martingales y^c and y^r are defined from their respective martingale difference sequences as follows:

$$\begin{cases}
dy_0^c & := \sum_{0\leq i\leq j} p_{i,0}dy_0p_{j,0};\\
dy_n^c & := \sum_{0\leq i\leq j} p_{i,n-1}dy_np_{j,n-1} \quad \text{for } n\geq 1;\\
dy_0^r & := \sum_{0\leq j< i} p_{i,0}dy_0p_{j,0};\\
dy_n^r & := \sum_{0\leq j< i} p_{i,n-1}dy_np_{j,n-1} \quad \text{for } n\geq 1.
\end{cases}\tag{4.8}$$

Clearly, $y = y^c + y^r$. Since x is an L_2-martingale, by the weak domination assumption, y is also an L_2-martingale. By the L_2-boundedness of triangular truncations, it is also clear that y^c and y^r are L_2-martingales. However, since y^c and y^r are not self-adjoint martingales, neither of them can be weakly dominated by x and hence we need some additional effort.

We present the argument for $\|S_c(y^c)\|_{1,\infty}$. The argument takes advantage of the new version of Gundy's decomposition that interacts well with the weak domination presented in the previous section. It suffices to show that there exists a constant K such that for any $k \geq 0$,

$$2^k\tau\Big(I_{[2^k,\infty)}\big(S_c(y^c)\big)\Big) \leq K\|x\|_1. \tag{4.9}$$

Fix such a k, pick $0 < \delta < 1$, let $\pi = q_{k,N}$ and write $S_{c,N}(y^c) = S_{c,N}(y^c)\pi + S_{c,N}(y^c)(I - \pi)$. By the well-known properties of a distribution function, we have

(4.10)
$$\begin{aligned}\tau\Big(I_{[2^k,\infty)}\big(S_{c,N}(y^c)\big)\Big) &\leq \tau\Big(I_{[\delta 2^k,\infty)}\big(|S_c(y^c)\pi|\big)\Big) + \tau\Big(I_{[(1-\delta)2^k,\infty)}\big(|S_c(y^c)(I-\pi)|\big)\Big)\\ &\leq \tau\Big(I_{[\delta 2^k,\infty)}\big(|S_{c,N}(y^c)\pi|\big)\Big) + \tau(I-\pi).\end{aligned}$$

The trace $\tau(\mathbf{1} - \pi)$ is controlled by $2^{-k}\|x_N\|_1$. On the other hand, we note that $\pi S^2_{c,N}(y^c)\pi = \pi(S^{(k)}_{c,N}(y^c))^2\pi$, where $S^{(k)}_{c,N}(y^c)$ denotes the truncated square function

$$S^{(k)}_{c,N}(y^c) = \bigg(\Big|\sum_{0\leq i\leq j\leq k} p_{i,1}dy_1p_{j,1}\Big|^2 + \sum_{n=2}^N\Big|\sum_{0\leq i\leq j\leq k} p_{i,n-1}dy_np_{j,n-1}\Big|^2\bigg)^{1/2}.$$

We refer to the proof of [33, Proposition A] for this fact. Next, we consider the decomposition of y according to the Gundy-type decomposition from Lemma 3.1, using the parameter $\lambda = 2^k$. We simply denote $R^{2^k}_n$ by R^k_n. This gives $dy_n = d\alpha_n + d\beta_n + d\gamma_n$, where for each k,

$$d\alpha_n = R^k_{n-1}dy_nR^k_{n-1}, \quad d\beta_n = (I - R^k_{n-1})dy_nR^k_{n-1}, \quad d\gamma_n = dy_n(I - R^k_{n-1}).$$

As in [30], for $n \geq 0$, set $\mathsf{P}^{(k)}_n := (p_{i,n})^k_{i=0}$. Then we see that

$$S^{(k)}_{c,N}(y^c) = \Big(\big|\mathcal{T}^{\mathsf{P}^{(k)}_1}(dy_1)\big|^2 + \sum_{n=2}^N\big|\mathcal{T}^{\mathsf{P}^{(k)}_{n-1}}(dy_n)\big|^2\Big)^{1/2}.$$

We make the following crucial observation. Since $d\beta_n$ is left-supported by $I - R^k_{n-1}$, it is also left-supported by $I - q_{k,n-1} = \sum_{l\geq k+1} p_{k,n-1}$, which implies $\mathcal{T}^{\mathsf{P}^{(k)}_{n-1}}(d\beta_n) = 0$. Similarly, by using the right support projections, we see that $\mathcal{T}^{\mathsf{P}^{(k)}_{n-1}}(d\gamma_n) = 0$. Therefore,

$$S^{(k)}_{c,N}(y^c) = \Big(\big|\mathcal{T}^{\mathsf{P}^{(k)}_1}(d\alpha_1)\big|^2 + \sum_{n=2}^N\big|\mathcal{T}^{\mathsf{P}^{(k)}_{n-1}}(d\alpha_n)\big|^2\Big)^{1/2}.$$

Since the function $\varphi(|\cdot|)$ is 1-convex and 2-concave, applying Proposition 4.6, we get

$$\begin{aligned}\tau\Big(I_{[2^k\delta,\infty)}\big(|S_{c,N}(y^c)\pi|\big)\Big) &= \tau\Big(I_{[\delta 2^k,\infty)}\big(|S^{(k)}_{c,N}(y^c)\pi|\big)\Big)\\ &\leq \tau\Big(I_{[\delta 2^k,\infty)}\Big(\big|\mathcal{T}^{\mathsf{P}^{(k)}_1}(d\alpha_1)\big|^2+\sum_{n=2}^N\big|\mathcal{T}^{\mathsf{P}^{(k)}_{n-1}}(d\alpha_n)\big|^2\Big)^{1/2}\Big)\\ &= \tau\Big(I_{[\delta,\infty)}\Big(\big|\mathcal{T}^{\mathsf{P}^{(k)}_1}(2^{-k}d\alpha_1)\big|^2+\sum_{n=2}^N\big|\mathcal{T}^{\mathsf{P}^{(k)}_{n-1}}(2^{-k}d\alpha_n)\big|^2\Big)^{1/2}\Big)\\ &\lesssim_{p,q} \varphi(\delta)^{-1}\sum_{n=1}^N\tau\Big(\varphi(|2^{-k}d\alpha_n|)\Big).\end{aligned}$$

Using Lemma 3.1 (ii), we get

$$\tau\Big(I_{[2^k\delta,\infty)}\big(|S_{c,N}(y^c)\pi|\big)\Big)\lesssim \varphi(\delta)^{-1}2^{-k}\|x_N\|_1.$$

Combining the above estimates, we conclude that

$$\tau\Big(I_{[2^k,\infty)}\big(|S_{c,N}(y^c)|\big)\Big)\lesssim (\varphi(\delta)^{-1}+1)2^{-k}\|x_N\|_1.$$

Letting $\delta\to 1$ and taking $N\to\infty$, we obtain the desired result. The proof is complete. □

Remark 4.9. Arguing as in the proof of Lemma 4.3, one can show the distributional inequality for $S_c(y^c)$. More precisely, for every $n\geq 0$ and $i\in\mathbb{Z}$ we have

$$\tau\Big(I_{[2^i,\infty)}\big(S_{c,n}(y^c)\big)\Big)\lesssim 2^{-2i}\big\|q_{i,n}x_nq_{i,n}\big\|_2^2+2^{-i}\tau\big(I-q_{i,n})|x_n|\big). \tag{4.11}$$

The same applies to the distribution function of $S_r(y^r)$.

According to the above remark, we can also establish the following strong-type inequality the square functions of weakly dominated martingales.

Corollary 4.10. *Let $1<p<2$. Suppose that x is a self-adjoint L_2-martingale and y is a self-adjoint martingale that is weakly dominated by x. Then there exist two martingales y^r and y^c such that $y=y^c+y^r$ and for every $1<p<2$ we have:*

$$\big\|y^c\big\|_{\mathcal{H}_p^c}+\big\|y^r\big\|_{\mathcal{H}_p^r}\leq C_p\big\|x\big\|_p$$

with $C_p=O((p-1)^{-1})$ as $p\to 1$. The order is best possible.

5. An application: a weak type inequality for the Hilbert transform on a quantum tori

In this section, we will apply the estimates established in the previous section to the study of the Hilbert transform on a quantum tori $\mathbb{T}_\theta^d$. Recall that the L_p and c.b. L_p bounds in this context were given in [6], however, the method used there can not be applied to get the corresponding weak type $(1,1)$ inequality. We will solve this problem with the help of Theorem 4.1.

Let us start with introducing the necessary background and notation. Let $d \geq 2$ be a fixed dimension and let $\theta = (\theta_{kj})_{1\leq j,k\leq d}$ be a real skew-symmetric $d \times d$ matrix. The associated d-dimensional noncommutative torus $\mathcal{A}_\theta$ is defined as the universal C^*-algebra generated by the collection of d unitary operators U_1, U_2, ..., U_d which satisfy the commutation relation

$$U_k U_j = e^{2\pi i\theta_{kj}} U_j U_k, \qquad j,\, k = 1,\, 2,\, \ldots,\, d.$$

It is not difficult to see that since the operators U_k are unitary, we must also have

$$U_j U_k^* = e^{2\pi i\theta_{kj}} U_k^* U_j, \qquad j,\, k = 1,\, 2,\, \ldots,\, d.$$

Set $U = (U_1, U_2, \ldots, U_d)$ and for $m = (m_1, m_2, \ldots, m_d) \in \mathbb{Z}^d$, consider the monomial

$$U^m = U_1^{m_1} U_2^{m_2} \ldots U_d^{m_d}.$$

Any operator of the form

$$x = \sum_{m\in\mathbb{Z}^d} \alpha_m U^m, \qquad \alpha_m \in \mathbb{C}, \tag{5.1}$$

with only a finite number of non-vanishing coefficients α_m, is called a polynomial. The involution algebra of such polynomials is dense in $\mathcal{A}_\theta$. Furthermore, for any polynomial x of the form (5.1), we define $\tau(x) = \alpha_0$. It can be shown that τ extends to a faithful tracial state on $\mathcal{A}_\theta$. The d-dimensional quantum torus $\mathbb{T}_\theta^d$ is defined as the w^*-closure of $\mathcal{A}_\theta$ in the GNS representation of τ; the state τ extends to a normal faithful tracial state on $\mathbb{T}_\theta^d$ and we will use the same letter for this extension. We would like to point out that the choice of the matrix $(\theta_{kj})_{1\leq j,k\leq d}$ with all entries equal to zero gives the classical d-dimensional torus $\mathbb{T}^d$.

We turn our attention to the Hilbert transforms arising in the above context. For a given nonzero vector $a \in \mathbb{R}^d$, we define the Hilbert transform in the direction a by the formula

$$\mathcal{H}^a\left(\sum_m \alpha_m U^m\right) = \sum_m \operatorname{sgn}\langle m, a\rangle \alpha_m U^m,$$

where $\langle\cdot,\cdot\rangle$ denotes the usual scalar product in $\mathbb{R}^d$ and we use the convention $\operatorname{sgn} 0 = 0$. Here we assume that the sequence $(\alpha_m)_{m\in\mathbb{Z}^d}$ possesses only a finite number of nonzero terms. Using the martingale inequalities developed in the previous sections, we will establish the following weak-type bound for $\mathcal{H}^a$; for the L_p and c.b. L_p counterparts of this result, we refer the reader to [6].

Theorem 5.1. *For any $a \in \mathbb{R}^d \setminus \{0\}$, we have the estimate*

$$\|\mathcal{H}^a f\|_{L_{1,\infty}(\mathbb{T}^d_\theta)} \leq 96\sqrt{3}\|f\|_{L_1(\mathbb{T}^d_\theta)}.$$

Proof. By homogeneity and straightforward approximation/decomposition arguments, it is enough to show that for an arbitrary self-adjoint polynomial f we have the estimate

$$\tau\big(I_{[3,\infty)}(|\mathcal{H}^a f|)\big) \leq 8\sqrt{3}\|f\|_{L_1(\mathbb{T}^d_\theta)}.$$

Fix such a polynomial: $f = \sum_{m\in\mathbb{Z}^d} \alpha_m U^m$. We have

$$f^* = \sum_{m\in\mathbb{Z}^d} \overline{\alpha_m}(U^m)^* = \sum_{m\in\mathbb{Z}^d} \overline{\alpha_m} U_d^{-m_d} U_{d-1}^{m_{d-1}} \ldots U_1^{-m_1} = \sum_{m\in\mathbb{Z}^d} \overline{\alpha_m}\sigma_m U^{-m}, \tag{5.2}$$

for some elements σ_m of the unit circle in $\mathbb{C}$ (coming from the commutation relation in $\mathbb{T}^d_\theta$): consequently, $\alpha_{-m} = \overline{\alpha_m}\sigma_m$ for all $m \in \mathbb{Z}^d$. Note that since almost all coefficients α_m are zero, we may assume that $a \in \mathbb{Z}^d$, by a simple stretching and approximation. Suppose further that

$$f^+ = \sum_{m:\langle m,a\rangle\geq 0} \alpha_m U^m \tag{5.3}$$

is the analytic part of f with respect to a: we will compose f^+ with an appropriate probabilistic component which will allow us to pass to martingale theory. We will divide our arguments into several steps.

Step 1. Appropriate martingales x and y. Assume that $(\Omega, \mathcal{G}, \mathbb{P})$ is a classical probability space and let $B = (B_t)_{t\geq 0}$ be a Brownian motion in $\mathbb{C}$, starting from the origin and stopped upon reaching the boundary of the unit disc. Denote by $(\mathcal{G}_t)_{t\geq 0}$ the (classical) filtration generated by B: for any $t \geq 0$, $\mathcal{G}_t$ is the σ-field generated by the random variables B_s, $s \in [0, t]$. Consider the product von Neumann algebra $\mathcal{N} = L_\infty(\Omega, \mathcal{G}, \mathbb{P})\overline{\otimes}\mathbb{T}^d_\theta$, equipped with the filtration $L_\infty(\Omega, \mathcal{G}_t, \mathbb{P})\overline{\otimes}\mathbb{T}^d_\theta$ and the standard tensor trace $\nu = \mathbb{E} \otimes \tau$. The operator f^+ gives rise to the adapted process on this new von Neumann algebra, given by

$$X_t = \frac{1}{2}\sum_{m:\langle m,a\rangle=0} \alpha_m \mathbf{1} \otimes U^m + \sum_{m:\langle m,a\rangle>0} \alpha_m B_t^{\langle m,a\rangle} \otimes U^m, \qquad t \geq 0,$$

where $\mathbf{1}$ is the random variable identically equal to 1. Next, we introduce the sequence $(\tau_n)_{n\geq 0}$ of (classical) stopping times, given by $\tau_0 = 0$ and

$$\tau_{n+1} = \inf\{t > \tau_n \,:\, \|X_t - X_{\tau_n}\|_{L_\infty(\mathcal{N})} \geq 1/4\}, \qquad n = 0, 1, 2, \ldots,$$

with the usual convention $\inf \emptyset = +\infty$. Finally, for any $n = 0, 1, 2, \ldots$, we define the self-adjoint operators $x_n = X_{\tau_n} + X^*_{\tau_n}$ and $y_n = i(X_{\tau_n} - X^*_{\tau_n})$. Observe that the definition makes perfect sense also if $\tau_n = \infty$, since B_∞ is a well-defined random variable.

Let us study certain important properties of x and y. Note that if n is an arbitrary positive integer, then B^n is a martingale, as the composition of the holomorphic function $z \mapsto z^n$ and the analytic martingale B. This has two important consequences. First, we conclude that x and y are self-adjoint martingales relative to the filtration $(\mathcal{F}_n)_{n\geq 0} =$

$(\mathcal{N}_{\tau_n})_{n\geq 0}$. Second, note that for any $n \geq 1$ and any k, $m \in \mathbb{Z}^d$ with $\langle k, a\rangle \geq 0$ and $\langle m, a\rangle \geq 0$, we have

$$\mathbb{E}\Big(\big(B^{\langle k,a\rangle}_{\tau_n} - B^{\langle k,a\rangle}_{\tau_{n-1}}\big)\big(B^{\langle m,a\rangle}_{\tau_n} - B^{\langle m,a\rangle}_{\tau_{n-1}}\big)|\mathcal{G}_{\tau_{n-1}}\Big) = 0, \tag{5.4}$$

where $\mathbb{E}$ stands for the classical expectation with respect to the probability measure $\mathbb{P}$. This implies that for any projection $R \in \mathcal{F}_{n-1}$ we have

$$\nu(R(X_{\tau_n} - X_{\tau_{n-1}})R(X_{\tau_n} - X_{\tau_{n-1}})R) = \nu(R(X^*_{\tau_n} - X^*_{\tau_{n-1}})R(X^*_{\tau_n} - X^*_{\tau_{n-1}})R) = 0. \tag{5.5}$$

Indeed, the trace $\nu(R(X_{\tau_n} - X_{\tau_{n-1}})R(X_{\tau_n} - X_{\tau_{n-1}})R)$ is the (possibly infinite) sum of the elements of the form

$$\beta_{j,k,m}\nu\Big(\big(B^{\langle k,a\rangle}_{\tau_n} - B^{\langle k,a\rangle}_{\tau_{n-1}}\big)\big(B^{\langle m,a\rangle}_{\tau_n} - B^{\langle m,a\rangle}_{\tau_{n-1}}\big)\xi_{j,k,m} \otimes U^j\Big),$$

where $\beta_{j,k,m}$ are scalars and $\xi_{j,k,m}$ are random variables measurable with respect to $\mathcal{G}_{\tau_{n-1}}$. By (5.4), each such term vanishes and hence $\nu(R(X_{\tau_n} - X_{\tau_{n-1}})R(X_{\tau_n} - X_{\tau_{n-1}})R) = 0$. The second equality in (5.5) follows by passing to the adjoint operator.

Step 2. Weak domination between martingales x and y. Now we will verify that martingales x, y constructed above satisfy the weak domination condition of Theorem 4.1. First, since B starts from zero, we have $dy_0 = 0$ and hence the requirement

$$\tau(\varphi(dy_0)) \leq \tau(\varphi(dx_0))$$

is obvious. Next, observe that for each $n \geq 1$, the differences dx_n and dy_n are bounded by 1 (this is actually the purpose of the introduction of the stopping times τ_0, τ_1, τ_2, $\ldots$). Therefore, if R is an arbitrary projection in $\mathcal{F}_{n-1}$, then $\|Rdx_nR\|_{L_\infty(\mathcal{N})} \leq 1$ and $\|Rdy_nR\|_{L_\infty(\mathcal{N})} \leq 1$, which in particular implies that

$$\nu(\varphi(Rdx_nR)) = \nu(Rdx_nRdx_nR) \quad \text{and} \quad \nu(\varphi(Rdy_nR)) = \nu(Rdy_nRdy_nR).$$

However, directly from (5.5), we infer that

$$\begin{aligned}
&\nu(Rdx_nRdx_nR)\\
&\quad = \nu\Big(R(X_{\tau_n} - X_{\tau_{n-1}} + X^*_{\tau_n} - X^*_{\tau_{n-1}})R(X_{\tau_n} - X_{\tau_{n-1}} + X^*_{\tau_n} - X^*_{\tau_{n-1}})R\Big)\\
&\quad = -\nu\Big(R\big(X_{\tau_n} - X_{\tau_{n-1}} - \big(X^*_{\tau_n} - X^*_{\tau_{n-1}}\big)\big)R\big(X_{\tau_n} - X_{\tau_{n-1}} - \big(X^*_{\tau_n} - X^*_{\tau_{n-1}}\big)\big)R\Big)\\
&\quad = \nu(Rdy_nRdy_nR),
\end{aligned}$$

and hence the weak-type estimate (4.1) gives

$$\nu\Big(I_{[3,\infty)}(|X_{\tau_n} - X^*_{\tau_n}|)\Big) \leq 8\sqrt{3}\nu\Big(|X_{\tau_n} + X^*_{\tau_n}|\Big). \tag{5.6}$$

Step 3. The relation between x, y and f, $\mathcal{H}^a f$. Note that

$$\begin{aligned}
&\left\| \sum_{m:\langle m,a\rangle>0} \alpha_m B_{\tau_n}^{\langle m,a\rangle} \otimes U^m - \sum_{m:\langle m,a\rangle>0} \alpha_m B_{\infty}^{\langle m,a\rangle} \otimes U^m \right\|_{L^2(\mathcal{N})} \\
&= \nu \left(\Big(\sum_{m:\langle m,a\rangle>0} \alpha_m (B_{\tau_n} - B_\infty)^{\langle m,a\rangle} \otimes U^m \Big)^* \Big(\sum_{m:\langle m,a\rangle>0} \alpha_m (B_{\tau_n} - B_\infty)^{\langle m,a\rangle} \otimes U^m \Big) \right) \\
&= \nu \left(\Big(\sum_{m:\langle m,a\rangle>0} \overline{\alpha_m} \alpha_m \overline{(B_{\tau_n} - B_\infty)^{\langle m,a\rangle}} (B_{\tau_n} - B_\infty)^{\langle m,a\rangle} \otimes (U^m)^* U^m \right) \\
&= \sum_{m:\langle m,a\rangle>0} \overline{\alpha_m} \sigma_m \mathbb{E} \left(\overline{(B_{\tau_n} - B_\infty)^{\langle m,a\rangle}} (B_{\tau_n} - B_\infty)^{\langle m,a\rangle} \right) \tau(U^{-m} U^m),
\end{aligned}$$

where σ_m are as in (5.2). Recall that only a finite number of the coefficients α_m are nonzero. Letting $n \to \infty$, since B_{τ_n} converges to B_∞ in $L^2(\Omega, \mathcal{G}, \mathbb{P})$, we have the $L^2(\mathcal{N})$ convergence

$$\begin{aligned}
X_{\tau_n} - X_{\tau_n}^* &\to \sum_{m:\langle m,a\rangle>0} \alpha_m B_\infty^{\langle m,a\rangle} \otimes U^m - \sum_{m:\langle m,a\rangle>0} \overline{\alpha_m} \overline{B_\infty^{\langle m,a\rangle}} \otimes (U^m)^* \\
&= \sum_{m:\langle m,a\rangle>0} \alpha_m B_\infty^{\langle m,a\rangle} \otimes U^m - \sum_{m:\langle m,a\rangle>0} \overline{\alpha_m} \sigma_m \overline{B_\infty^{\langle m,a\rangle}} \otimes U^{-m}.
\end{aligned}$$

However, as we have observed above, we have $\overline{\alpha_m}\sigma_m = \alpha_{-m}$, since the polynomial is self-adjoint. Furthermore, the random variable B_∞ is uniformly distributed on the unit circle, so $\overline{B_\infty} = B_\infty^{-1}$. Therefore, we may rewrite the above convergence in the more concise form

$$X_{\tau_n} - X_{\tau_n}^* \to \sum_{m\in\mathbb{Z}^d} \alpha_m \operatorname{sgn}\langle m,a\rangle B_\infty^{\langle m,a\rangle} \otimes U^m.$$

A similar argument gives

$$X_{\tau_n} + X_{\tau_n}^* \to \sum_{m\in\mathbb{Z}^d} \alpha_m B_\infty^{\langle m,a\rangle} \otimes U^m$$

and hence (5.6) yields

(5.7)
$$\nu \left(I_{[3,\infty)} \left(\left| \sum_{m\in\mathbb{Z}^d} \alpha_m \operatorname{sgn}\langle m,a\rangle B_\infty^{\langle m,a\rangle} \otimes U^m \right| \right) \right) \le 8\sqrt{3}\nu \left(\left| \sum_{m\in\mathbb{Z}^d} \alpha_m B_\infty^{\langle m,a\rangle} \otimes U^m \right| \right).$$

It remains to prove that this is the desired weak-type bound. Pick an arbitrary positive integer k and observe that

$$\nu\left(\left(\sum_{m\in\mathbb{Z}^d}\alpha_m B_\infty^{\langle m,a\rangle}\otimes U^m\right)^k\right) = \sum_{m_1,m_2,\ldots,m_k\in\mathbb{Z}^d}\alpha_{m_1}\alpha_{m_2}\ldots\alpha_{m_k}\mathbb{E}(B_\infty^{\langle m_1+m_2+\ldots+m_k,a\rangle})\tau(U^{m_1}U^{m_2}\ldots U^{m_k}).$$

Now if $m_1+m_2+\ldots+m_k\neq 0$, then the trace $\tau(U^{m_1}U^{m_2}\ldots U^{m_k})$ vanishes; otherwise, we have $\mathbb{E}(B_\infty^{\langle m_1+m_2+\ldots+m_k,a\rangle})=1$. In both cases, we see that

$$\mathbb{E}(B_\infty^{\langle m_1+m_2+\ldots+m_k,a\rangle})\tau(U^{m_1}U^{m_2}\ldots U^{m_k})=\tau(U^{m_1}U^{m_2}\ldots U^{m_k})$$

and hence, plugging this above, we get

$$\nu\left(\left(\sum_{m\in\mathbb{Z}^d}\alpha_m B_\infty^{\langle m,a\rangle}\otimes U^m\right)^k\right)=\tau\left(\left(\sum_{m\in\mathbb{Z}^d}\alpha_m\otimes U^m\right)^k\right).$$

Consequently, for any polynomial P on $\mathbb{C}$ we have

$$\nu\left(P\left(\sum_{m\in\mathbb{Z}^d}\alpha_m B_\infty^{\langle m,a\rangle}\otimes U^m\right)\right)=\tau\left(P\left(\sum_{m\in\mathbb{Z}^d}\alpha_m\otimes U^m\right)\right)$$

and hence $\sum_{m\in\mathbb{Z}^d}\alpha_m B_\infty^{\langle m,a\rangle}\otimes U^m$ and $\sum_{m\in\mathbb{Z}^d}\alpha_m\otimes U^m$ have the same distribution functions. Similarly, it can be checked that the distributions of $\sum_{m\in\mathbb{Z}^d}\alpha_m\operatorname{sgn}\langle m,a\rangle B_\infty^{\langle m,a\rangle}\otimes U^m$ and $\sum_{m\in\mathbb{Z}^d}\alpha_m\operatorname{sgn}\langle m,a\rangle\otimes U^m$ also coincide.

Step 4. Conclusion. Putting all the steps above, we deduce that

$$\begin{aligned}\tau\big(I_{[3,\infty)}(|\mathcal{H}^a f|)\big)&=\tau\left(I_{[3,\infty)}\Big|\sum_m \operatorname{sgn}\langle m,a\rangle\alpha_m U^m\Big|\right)\\&\leq 8\sqrt{3}\tau\left(\Big|\sum_m\alpha_m U^m\Big|\right)=8\sqrt{3}\tau(|f|).\end{aligned}$$

The proof is complete. □

As proved in [6], the Hilbert transform on quantum tori is completely L_p bounded for $1<p<\infty$. That is, for any such p, the map $\mathrm{id}_{S_p}\otimes\mathcal{H}^a : L_p(B(\ell_2)\overline{\otimes}\mathbb{T}_\theta^d)\to L_p(B(\ell_2)\overline{\otimes}\mathbb{T}_\theta^d)$ is bounded:

$$\|\mathcal{H}^a\|_{\mathrm{cb}}:=\|\mathrm{id}_{S_p}\otimes\mathcal{H}^a\|_{L_p(B(\ell_2)\overline{\otimes}\mathbb{T}_\theta^d)\to L_p(B(\ell_2)\overline{\otimes}\mathbb{T}_\theta^d)}<\infty.$$

Here S_p stands for the Schatten p-class, i.e., the noncommutative L_p -space associated with $B(\ell_2)$ with the standard trace. See [31] for more on the subject. We would like to point out that essentially the same reasoning as in the proof of Theorem 5.1 leads to the following weak-type estimate.

Theorem 5.2. *For any $a \in \mathbb{R}^d \setminus \{0\}$ we have the estimate*

$$\|\mathrm{id}_{S_1} \otimes \mathcal{H}^a\|_{L_1(B(\ell_2)\overline{\otimes}\mathbb{T}_\theta^d) \to L_{1,\infty}(B(\ell_2)\overline{\otimes}\mathbb{T}_\theta^d)} \leq 96\sqrt{3}.$$

Proof. We need to implement an additional matrix component. Recall that $(e_{i,j})_{i,j\geq 1}$ denotes the collection of matrix units in $B(\ell_2)$. Consider the self-adjoint operator

$$f = \sum_{i,j\geq 1} \sum_{m\in\mathbb{Z}^d} \alpha_m^{i,j} e_{i,j} \otimes U^m \in B(\ell_2)\overline{\otimes}\mathbb{T}_\theta^d,$$

with only a finite number of nonzero coefficients α_m. We will be done if we manage to show that

$$\tau\Big(I_{[3,\infty)}\big(\big|\big(\mathrm{id}_{S_1} \otimes \mathcal{H}^a\big) f\big|\big)\Big) \leq 8\sqrt{3}\|f\|_{L_1(B(\ell_2)\overline{\otimes}\mathbb{T}_\theta^d)}. \tag{5.8}$$

As before, we may assume that $a \in \mathbb{Z}^d$. Consider the product von Neumann algebra $\mathcal{N} = L_\infty(\Omega, \mathcal{G}, \mathbb{P})\overline{\otimes}B(\ell_2)\overline{\otimes}\mathbb{T}_\theta^d$, equipped with the filtration $L_\infty(\Omega, \mathcal{G}_t, \mathbb{P})\overline{\otimes}B(\ell_2)\overline{\otimes}\mathbb{T}_\theta^d$ and the standard tensor trace $\nu = \mathbb{E} \otimes tr \otimes \tau$. Define the associated 'analytic' process as follows

$$X_t = \sum_{i,j=1}^n \left\{ \frac{1}{2} \sum_{m:\langle m,a\rangle=0} \alpha_m^{i,j} \mathbf{1} \otimes e_{i,j} \otimes U^m + \sum_{m:\langle m,a\rangle>0} \alpha_m^{i,j} B_t^{\langle m,a\rangle} \otimes e_{i,j} \otimes U^m \right\}, \qquad t \geq 0.$$

Following the argument in the proof of Theorem 5.1, we introduce the stopping times with respect to the new von Neumann algebra $\mathcal{N}$, given by $\tau_0 = 0$ and

$$\tau_{n+1} = \inf\{t > \tau_n \,:\, \|X_t - X_{\tau_n}\|_{L_\infty(\mathcal{N})} \geq 1/4\}, \qquad n = 0, 1, 2, \ldots.$$

Define self-adjoint operators $x_n = X_{\tau_n} + X_{\tau_n}^*$ and $y_n = i(X_{\tau_n} - X_{\tau_n}^*)$. Then x, y are martingales relative to $(\mathcal{F}_n)_{n\geq 0} = (\mathcal{N}_{\tau_n})_{n\geq 0}$ and the identity (5.4) gives

$$\nu(R(X_{\tau_n} - X_{\tau_{n-1}})R(X_{\tau_n} - X_{\tau_{n-1}})R) = \nu(R(X_{\tau_n}^* - X_{\tau_{n-1}}^*)R(X_{\tau_n}^* - X_{\tau_{n-1}}^*)R) = 0,$$

for any projection $R \in \mathcal{F}_{n-1}$. By the definition of $(\tau_n)_{n\geq 0}$, this implies the weak domination between x and y. Consequently, we have

$$\nu\Big(I_{[3,\infty)}(|X_{\tau_n} - X_{\tau_n}^*|)\Big) \leq 8\sqrt{3}\nu\Big(|X_{\tau_n} + X_{\tau_n}^*|)\Big).$$

Now the same calculation as before yields the $L^2(\mathcal{N})$ convergence

$$X_{\tau_n} - X_{\tau_n}^* \to \sum_{m\in\mathbb{Z}^d} \alpha_m^{i,j} \operatorname{sgn}\langle m, a\rangle B_\infty^{\langle m,a\rangle} \otimes e_{i,j} \otimes U^m$$

and

$$X_{\tau_n} + X_{\tau_n}^* \to \sum_{m\in\mathbb{Z}^d} \alpha_m^{i,j} B_\infty^{\langle m,a\rangle} \otimes e_{i,j} \otimes U^m.$$

But for any positive integer k, a straightforward expansion reveals that

$$\nu\left(\left(\sum_{m\in\mathbb{Z}^d}\alpha_m^{i,j}B_\infty^{\langle m,a\rangle}\otimes e_{i,j}\otimes U^m\right)^k\right)=(tr\otimes\tau)\left(\left(\sum_{m\in\mathbb{Z}^d}\alpha_m^{i,j}e_{i,j}\otimes U^m\right)^k\right),$$

and

$$\nu\left(\left(\sum_{m\in\mathbb{Z}^d}\alpha_m^{i,j}\operatorname{sgn}\langle m,a\rangle B_\infty^{\langle m,a\rangle}\otimes e_{i,j}\otimes U^m\right)^k\right)$$
$$=(tr\otimes\tau)\left(\left(\sum_{m\in\mathbb{Z}^d}\alpha_m^{i,j}\operatorname{sgn}\langle m,a\rangle e_{i,j}\otimes U^m\right)^k\right).$$

Therefore, the distributions of $\sum_{m\in\mathbb{Z}^d}\alpha_m^{i,j}B_\infty^{\langle m,a\rangle}\otimes e_{ij}\otimes U^m$ and $\sum_{m\in\mathbb{Z}^d}\alpha_m^{i,j}e_{ij}\otimes U^m$ coincide; the same is true for the pair $\sum_{m\in\mathbb{Z}^d}\alpha_m^{i,j}\operatorname{sgn}\langle m,a\rangle B_\infty^{\langle m,a\rangle}\otimes e_{ij}\otimes U^m$ and $\sum_{m\in\mathbb{Z}^d}\alpha_m^{i,j}e_{ij}\operatorname{sgn}\langle m,a\rangle\otimes U^m$. Thus, the proof of (5.8) is finished. $\square$

Remark 5.3. As we mentioned before, the weak domination is less restrictive than the differential subordination considered in [10] and hence it should possess a wider range of applications. The above two theorems may serve as a nice illustration of this phenomenon: these statements do not seem to follow from weak-type estimate for differentially subordinate martingales.

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