

NONCOMMUTATIVE MAXIMAL INEQUALITIES ASSOCIATED WITH CONVEX FUNCTIONS

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ABSTRACT. We prove several noncommutative maximal inequalities associated with convex functions, including a Doob type inequality for a convex function of maximal operators on noncommutative martingales, noncommutative Dunford-Schwartz and Stein maximal ergodic inequalities for a convex function of positive and symmetric positive contractions. The key ingredient in our proofs is a Marcinkiewicz type interpolation theorem for a convex function of maximal operators in the noncommutative setting, which we establish in this paper. These generalize the results of Junge and Xu in the L^p case to the case of convex functions.

1. INTRODUCTION

Noncommutative martingale theory has received considerable progress since the seminal paper by Pisier and Xu [40] in 1997, thanks to interactions with several fields of mathematics such as operator spaces (e.g. [18, 38]) and free probability (e.g. [19, 45]). Many classical martingale and ergodic inequalities have been successfully transferred to the noncommutative setting (cf. e.g. [2, 5, 8, 20, 22, 24, 26, 27, 30, 31, 35, 36, 42, 43, 44]). These inequalities of quantum probabilistic nature have, in return, applications to operator spaces, quantum stochastic analysis and noncommutative harmonic analysis. We refer to [21, 23, 28, 39, 47] for some illustrations of applications to operator space theory.

We continue this line of investigation. The aim of this paper is to prove several noncommutative maximal inequalities associated with convex functions. But the study of maximal inequalities is one of the most delicate and difficult parts in the noncommutative setting. Maximal martingale and ergodic inequalities in noncommutative L_p -spaces have been established respectively by Junge [20] and Junge and Xu [26], with the use of the techniques developed for operator space theory and theory of interpolation of Banach spaces. Their method can be generalized to obtain the corresponding maximal inequalities on noncommutative symmetric spaces, including noncommutative Lorentz and Orlicz spaces (see e.g. [12]). However, their argument, relying heavily on Banach space properties such as duality and homogeneousness of norms, cannot be used directly to establish the corresponding inequalities associated with convex functions. Indeed, Xu inspired us to generalize their maximal inequalities to the case of convex functions. To this end, we need to establish a Marcinkiewicz type interpolation theorem for a convex function of maximal operators in the noncommutative setting, which we will prove in this paper based on some ideas of [3]. In the meanwhile, our argument provides an alternative and simpler proof for the results of Junge and Xu mentioned above. Recently, Dirksen [13] showed that our interpolation theorem (Theorem 3.2) holds true yet when the upper control index is finite (see Remark 3.2 (2) below).

The paper is organized as follows. In Section 2, we present some preliminaries and notations on noncommutative martingales and noncommutative maximal operators. In Section 3, we will define a convex function of maximal operators in the noncommutative setting and present several basic properties. Then, a noncommutative Marcinkiewicz type interpolation theorem for a convex function of maximal operators is proved, which is the key ingredient for the proofs of our main results. In Section 4, we prove the main results of this paper, including a Doob type inequality for a convex function of maximal operators noncommutative martingales, noncommutative Dunford-Schwartz and Stein maximal ergodic inequalities for a convex function of positive and symmetric positive contractions. As a consequence, we obtain the noncommutative Burkholder-Davis-Gundy inequality associated with a

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convex function. Finally, in Section 5, the results obtained in the previous sections are extended to cover weak type maximal inequalities associated with convex functions.

In what follows, C always denotes a constant, which may be different in different places. For two nonnegative (possibly infinite) quantities X and Y by $X \lesssim Y$ we mean that there exists a constant $C > 0$ such that $X \leq CY$, and by $X \approx Y$ that $X \lesssim Y$ and $Y \lesssim X$.

2. PRELIMINARIES

2.1. Noncommutative Orlicz spaces. We use standard notions from theory of noncommutative L_p -spaces. Our main references are [41] and [48] (see also [41] for more bibliography). Let \mathcal{N} be a semifinite von Neumann algebra acting on a Hilbert space \mathbb{H} with a normal semifinite faithful trace ν . Let $L_0(\mathcal{N})$ denote the topological $*$ -algebra of measurable operators with respect to (\mathcal{N}, ν) . The topology of $L_0(\mathcal{N})$ is determined by the convergence in measure. The trace ν can be extended to the positive cone $L_0^+(\mathcal{N})$ of $L_0(\mathcal{N})$:

$$\nu(x) = \int_0^\infty \lambda d\nu(E_\lambda(x)),$$

where $x = \int_0^\infty \lambda dE_\lambda(x)$ is the spectral decomposition of x . Given $0 < p < \infty$, let

$$L_p(\mathcal{N}) = \{x \in L_0(\mathcal{N}) : \nu(|x|^p)^{\frac{1}{p}} < \infty\}.$$

We define

$$\|x\|_p = \nu(|x|^p)^{\frac{1}{p}}, \quad x \in L_p(\mathcal{N}).$$

Then $(L_p(\mathcal{N}), \|\cdot\|_p)$ is a Banach (or quasi-Banach for $p < 1$) space. This is the noncommutative L_p -space associated with (\mathcal{N}, ν) , denoted by $L_p(\mathcal{N}, \nu)$ or simply by $L_p(\mathcal{N})$. As usual, we set $L_\infty(\mathcal{N}, \nu) = \mathcal{N}$ equipped with the operator norm.

For $x \in L_0(\mathcal{N})$ we define

$$\lambda_s(x) = \tau(e_s^\perp(|x|)) \quad (s > 0) \quad \text{and} \quad \mu_t(x) = \inf\{s > 0 : \lambda_s(x) \leq t\} \quad (t > 0),$$

where $e_s^\perp(|x|) = e_{(s, \infty)}(|x|)$ is the spectral projection of $|x|$ associated with the interval (s, ∞) . The function $s \mapsto \lambda_s(x)$ is called the *distribution function* of x and $\mu_t(x)$ is the *generalized singular number* of x . We will denote simply by $\lambda(x)$ and $\mu(x)$ the functions $s \mapsto \lambda_s(x)$ and $t \mapsto \mu_t(x)$, respectively. It is easy to check that both are decreasing and continuous from the right on $(0, \infty)$. For further information we refer the reader to [17].

For $0 < p < \infty$, we have the Kolmogorov inequality

$$(2.1) \quad \lambda_s(x) \leq \frac{\|x\|_p^p}{s^p}, \quad \forall s > 0,$$

for any $x \in L_p(\mathcal{N})$. If x, y in $L_0(\mathcal{N})$, then

$$(2.2) \quad \lambda_{2s}(x + y) \leq \lambda_s(x) + \lambda_s(y), \quad \forall s > 0.$$

Let Φ be an Orlicz function on $[0, \infty)$, i.e., a continuous increasing and convex function satisfying $\Phi(0) = 0$ and $\lim_{t \rightarrow \infty} \Phi(t) = \infty$. Recall that Φ is said to satisfy the Δ_2 -condition if there is a constant C such that $\Phi(2t) \leq C\Phi(t)$ for all $t > 0$. In this case, we write $\Phi \in \Delta_2$. It is easy to check that $\Phi \in \Delta_2$ if and only if for any $a > 0$ there is a constant $C_a > 0$ such that $\Phi(at) \leq C_a\Phi(t)$ for all $t > 0$.

For any $x \in L_0(\mathcal{N})$, by means of functional calculus applied to the spectral decomposition of $|x|$, we have

$$(2.3) \quad \nu(\Phi(|x|)) = \int_0^\infty \lambda_s(|x|) d\Phi(s) = \int_0^\infty \Phi(\mu_t(x)) dt,$$

(see e.g. [17]). Recall that for any $x, y \in L_0(\mathcal{N})$ there exist two partial isometries $u, v \in \mathcal{N}$ such that

$$(2.4) \quad |x + y| \leq u^*|x|u + v^*|y|v,$$

(cf. [1]). Then, we have

$$\nu(\Phi(|\alpha x + (1 - \alpha)y|)) \leq \alpha\nu(\Phi(|x|)) + (1 - \alpha)\nu(\Phi(|y|))$$

for any $0 \leq \alpha \leq 1$ and $x, y \in L_0(\mathcal{N})$. In addition, if $\Phi \in \Delta_2$, then

$$\nu(\Phi(|x + y|)) \leq C_\Phi [\nu(\Phi(|x|)) + \nu(\Phi(|y|))].$$

We will frequently use these two inequalities in what follows.

We will work with some standard indices associated to an Orlicz function. Given an Orlicz function Φ , let

$$M(t, \Phi) = \sup_{s>0} \frac{\Phi(ts)}{\Phi(s)}, \quad t > 0.$$

Define

$$p_\Phi = \lim_{t \searrow 0} \frac{\log M(t, \Phi)}{\log t}, \quad q_\Phi = \lim_{t \nearrow \infty} \frac{\log M(t, \Phi)}{\log t}.$$

Note the following properties:

- (1) $1 \leq p_\Phi \leq q_\Phi \leq \infty$.
- (2) The following characterizations of p_Φ and q_Φ hold

$$p_\Phi = \sup \left\{ p > 0 : \int_0^t s^{-p} \Phi(s) \frac{ds}{s} = O(t^{-p} \Phi(t)), \forall t > 0 \right\};$$

$$q_\Phi = \inf \left\{ q > 0 : \int_t^\infty s^{-q} \Phi(s) \frac{ds}{s} = O(t^{-q} \Phi(t)), \forall t > 0 \right\}.$$

- (3) $\Phi \in \Delta_2$ if and only if $q_\Phi < \infty$, or equivalently, $\sup_{t>0} t\Phi'(t)/\Phi(t) < \infty$. ($\Phi'(t)$ is defined for each $t > 0$ except for a countable set of points in which we take $\Phi'(t)$ as the derivative from the right.)

See [33, 34] for more information on Orlicz functions and Orlicz spaces.

For an Orlicz function Φ , the noncommutative Orlicz space $L_\Phi(\mathcal{N})$ is defined as the space of all measurable operators x with respect to (\mathcal{N}, ν) such that

$$\nu\left(\Phi\left(\frac{|x|}{c}\right)\right) < \infty$$

for some $c > 0$. The space $L_\Phi(\mathcal{N})$, equipped with the norm

$$\|x\|_\Phi = \inf \{c > 0 : \nu(\Phi(|x|/c)) < 1\},$$

is a Banach space. If $\Phi(t) = t^p$ with $1 \leq p < \infty$ then $L_\Phi(\mathcal{N}) = L_p(\mathcal{N})$. Note that if $\Phi \in \Delta_2$, then for $x \in L_0(\mathcal{N})$, $\nu(\Phi(|x|)) < \infty$ if and only if $x \in L_\Phi(\mathcal{N})$. Noncommutative Orlicz spaces are symmetric spaces of measurable operators as defined in [15, 46].

Let $a = (a_n)$ be a finite sequence in $L_\Phi(\mathcal{N})$. We define

$$\|a\|_{L_\Phi(\mathcal{N}, \ell_C^2)} = \left\| \left(\sum_n |a_n|^2 \right)^{\frac{1}{2}} \right\|_\Phi \quad \text{and} \quad \|a\|_{L_\Phi(\mathcal{N}, \ell_R^2)} = \left\| \left(\sum_n |a_n^*|^2 \right)^{\frac{1}{2}} \right\|_\Phi,$$

respectively. This gives two norms on the family of all finite sequences in $L_\Phi(\mathcal{N})$ (see [3] for details). The corresponding completion $L_\Phi(\mathcal{N}, \ell_C^2)$ is a Banach space. Since Φ is a continuous increasing function, by (2.3) we have that a sequence $a = (a_n)_{n \geq 0}$ in $L_\Phi(\mathcal{N})$ belongs to $L_\Phi(\mathcal{N}, \ell_C^2)$ if and only if

$$\sup_{n \geq 0} \left\| \left(\sum_{k=0}^n |a_k|^2 \right)^{\frac{1}{2}} \right\|_\Phi < \infty.$$

If this is the case, $(\sum_{k=0}^\infty |a_k|^2)^{\frac{1}{2}}$ can be appropriately defined as an element of $L_\Phi(\mathcal{N})$. Similarly, $\|\cdot\|_{L_\Phi(\mathcal{N}, \ell_R^2)}$ is also a norm on the family of all finite sequence in $L_\Phi(\mathcal{N})$, and the corresponding completion $L_\Phi(\mathcal{N}, \ell_R^2)$ is a Banach space, which is isometric to the row subspace of $L_\Phi(\mathcal{N} \otimes \mathcal{B}(\ell^2))$ consisting of matrices whose nonzero entries lie only in the first row. Observe that the column and row subspaces of $L_\Phi(\mathcal{N} \otimes \mathcal{B}(\ell^2))$ are 1-complemented by Theorem 3.4 in [16].

In what follows, unless otherwise specified, we always denote by Φ an Orlicz function.

2.2. Noncommutative martingales. Let \mathcal{M} be a finite von Neumann algebra with a normalized normal faithful trace τ . Let $(\mathcal{M}_n)_{n \geq 0}$ be an increasing sequence of von Neumann subalgebras of \mathcal{M} such that $\cup_{n \geq 0} \mathcal{M}_n$ generates \mathcal{M} (in the w^* -topology). $(\mathcal{M}_n)_{n \geq 0}$ is called a filtration of \mathcal{M} . The restriction of τ to \mathcal{M}_n is still denoted by τ . Let $\mathcal{E}_n = \mathcal{E}(\cdot | \mathcal{M}_n)$ be the conditional expectation of \mathcal{M} with respect to \mathcal{M}_n . Then \mathcal{E}_n is a norm 1 projection of $L_\Phi(\mathcal{M})$ onto $L_\Phi(\mathcal{M}_n)$ (cf. [16, Theorem 3.4]) and $\mathcal{E}_n(x) \geq 0$ whenever $x \geq 0$.

A noncommutative L_Φ -martingale with respect to $(\mathcal{M}_n)_{n \geq 0}$ is a sequence $x = (x_n)_{n \geq 0}$ such that $x_n \in L_\Phi(\mathcal{M}_n)$ and

$$\mathcal{E}_n(x_{n+1}) = x_n$$

for any $n \geq 0$. Let $\|x\|_\Phi = \sup_{n \geq 0} \|x_n\|_\Phi$. If $\|x\|_\Phi < \infty$, then x is said to be a bounded L_Φ -martingale.

Remark 2.1. Let \mathcal{M} be a semifinite von Neumann algebra with a semifinite normal faithful trace τ . Let $(\mathcal{M}_n)_{n \geq 0}$ be a filtration of \mathcal{M} such that the restriction of τ to each \mathcal{M}_n is still semifinite. Then we can define noncommutative martingales with respect to $(\mathcal{M}_n)_{n \geq 0}$. All results on noncommutative martingales that will be presented below can be extended to this semifinite setting.

Let x be a noncommutative martingale. The martingale difference sequence of x , denoted by $dx = (dx_n)_{n \geq 0}$, is defined as

$$dx_0 = x_0, \quad dx_n = x_n - x_{n-1}, \quad n \geq 1.$$

Set

$$S_n^C(x) = \left(\sum_{k=0}^n |dx_k|^2 \right)^{\frac{1}{2}} \quad \text{and} \quad S_n^R(x) = \left(\sum_{k=0}^n |dx_k^*|^2 \right)^{\frac{1}{2}}.$$

By the preceding discussion, dx belongs to $L_\Phi(\mathcal{M}, \ell_C^2)$ (resp. $L_\Phi(\mathcal{M}, \ell_R^2)$) if and only if $(S_n^C(x))_{n \geq 0}$ (resp. $(S_n^R(x))_{n \geq 0}$) is a bounded sequence in $L_\Phi(\mathcal{M})$; in this case,

$$S^C(x) = \left(\sum_{k=0}^{\infty} |dx_k|^2 \right)^{\frac{1}{2}} \quad \text{and} \quad S^R(x) = \left(\sum_{k=0}^{\infty} |dx_k^*|^2 \right)^{\frac{1}{2}}$$

are elements in $L_\Phi(\mathcal{M})$. These are noncommutative analogues of the usual square functions in the commutative martingale theory. It should be pointed out that the two sequences $S_n^C(x)$ and $S_n^R(x)$ may not be bounded in $L_\Phi(\mathcal{M})$ at the same time.

We define $\mathcal{H}_\Phi^C(\mathcal{M})$ (resp. $\mathcal{H}_\Phi^R(\mathcal{M})$) to be the space of all L_Φ -martingales such that $dx \in L_\Phi(\mathcal{M}, \ell_C^2)$ (resp. $dx \in L_\Phi(\mathcal{M}, \ell_R^2)$), equipped with the norm

$$\|x\|_{\mathcal{H}_\Phi^C(\mathcal{M})} = \|dx\|_{L_\Phi(\mathcal{M}, \ell_C^2)} \quad (\text{resp. } \|x\|_{\mathcal{H}_\Phi^R(\mathcal{M})} = \|dx\|_{L_\Phi(\mathcal{M}, \ell_R^2)}).$$

$\mathcal{H}_\Phi^C(\mathcal{M})$ and $\mathcal{H}_\Phi^R(\mathcal{M})$ are Banach spaces. Note that if $x \in \mathcal{H}_\Phi^C(\mathcal{M})$,

$$\|x\|_{\mathcal{H}_\Phi^C(\mathcal{M})} = \sup_{n \geq 0} \|S_n^C(x)\|_{L_\Phi(\mathcal{M})} = \|S^C(x)\|_{L_\Phi(\mathcal{M})}.$$

Similar equalities hold for $\mathcal{H}_\Phi^R(\mathcal{M})$.

Now, we define the Orlicz-Hardy spaces of noncommutative martingales as follows: If $q_\Phi < 2$, then

$$\mathcal{H}_\Phi(\mathcal{M}) = \mathcal{H}_\Phi^C(\mathcal{M}) + \mathcal{H}_\Phi^R(\mathcal{M}),$$

equipped with the norm

$$\|x\| = \inf \left\{ \|y\|_{\mathcal{H}_\Phi^C(\mathcal{M})} + \|z\|_{\mathcal{H}_\Phi^R(\mathcal{M})} : x = y + z, y \in \mathcal{H}_\Phi^C(\mathcal{M}), z \in \mathcal{H}_\Phi^R(\mathcal{M}) \right\}.$$

If $2 \leq p_\Phi$,

$$\mathcal{H}_\Phi(\mathcal{M}) = \mathcal{H}_\Phi^C(\mathcal{M}) \cap \mathcal{H}_\Phi^R(\mathcal{M}),$$

equipped with the norm

$$\|x\| = \max \left\{ \|x\|_{\mathcal{H}_\Phi^C(\mathcal{M})}, \|x\|_{\mathcal{H}_\Phi^R(\mathcal{M})} \right\}.$$

We refer to [3] for more information on $\mathcal{H}_\Phi(\mathcal{M})$.

2.3. The space $L_p(\mathcal{M}; \ell^\infty)$. Given $1 \leq p < \infty$, recall that $L_p(\mathcal{M}; \ell^\infty)$ is defined as the space of all sequences $(x_n)_{n \geq 1}$ in $L_p(\mathcal{M})$ for which there exist $a, b \in L_{2p}(\mathcal{M})$ and a bounded sequence $(y_n)_{n \geq 1}$ in \mathcal{M} such that $x_n = ay_nb$ for all $n \geq 1$. For such a sequence, set

$$(2.5) \quad \|(x_n)_{n \geq 1}\|_{L_p(\mathcal{M}, \ell^\infty)} := \inf \left\{ \|a\|_{2p} \sup_n \|y_n\|_\infty \|b\|_{2p} \right\},$$

where the infimum runs over all possible factorizations of $(x_n)_{n \geq 1}$ as above. This is a norm and $L_p(\mathcal{M}; \ell^\infty)$ is a Banach space. These spaces were first introduced by Pisier [37] in the case when \mathcal{M} is hyperfinite and by Junge [20] in the general case, and studied extensively by Junge and Xu [26].

As in [26], we usually write

$$\left\| \sup_n^+ x_n \right\|_p = \|(x_n)_{n \geq 1}\|_{L_p(\mathcal{M}, \ell^\infty)}.$$

We warn the reader that this suggestive notation should be treated with care. It is used for possibly nonpositive operators and

$$\left\| \sup_n^+ x_n \right\|_p \neq \left\| \sup_n^+ |x_n| \right\|_p$$

in general. However it has an intuitive description in the positive case, as observed in [26, p.329]: A positive sequence $(x_n)_{n \geq 1}$ of $L_p(\mathcal{M})$ belongs to $L_p(\mathcal{M}; \ell^\infty)$ if and only if there exists a positive $a \in L_p(\mathcal{M})$ such that $x_n \leq a$ for any $n \geq 1$ and in this case,

$$(2.6) \quad \left\| \sup_n^+ x_n \right\|_p = \inf \left\{ \|a\|_p : a \in L_p(\mathcal{M}), x_n \leq a, \forall n \geq 1 \right\}.$$

In particular, it was proved in [26] that the spaces $L_p(\mathcal{M}; \ell^\infty)$ for all $1 \leq p \leq \infty$ form interpolation scales with respect to complex interpolation. However, this result is no longer true for the real interpolation. This is one of the difficulties one will encounter for dealing with Marcinkiewicz type interpolation theorem on maximal operators in the noncommutative setting.

3. INTERPOLATION

In this section, we will establish a noncommutative Marcinkiewicz type interpolation theorem for a convex function of maximal operators, which plays a crucial role in the proofs of our main results in the next section.

To this end, we introduce the following definition.

Definition 3.1. Let $1 \leq p_0 < p_1 \leq \infty$. Let $S = (S_n)_{n \geq 1}$ be a sequence of maps from $L_{p_0}^+(\mathcal{M}) + L_{p_1}^+(\mathcal{M}) \mapsto L_0^+(\mathcal{M})$.

(1) S is said to be subadditive, if for any $n \geq 1$,

$$S_n(x + y) \leq S_n(x) + S_n(y), \quad \forall x, y \in L_{p_0}^+(\mathcal{M}) + L_{p_1}^+(\mathcal{M}).$$

(2) S is said to be of weak type (p, p) ($p_0 \leq p < p_1$) if there is a positive constant C such that for any $x \in L_p^+(\mathcal{M})$ and any $\lambda > 0$ there exists a projection $e \in \mathcal{M}$ such that

$$\tau(e^\perp) \leq \left(\frac{C \|x\|_p}{\lambda} \right)^p \quad \text{and} \quad e S_n(x) e \leq \lambda, \quad \forall n \geq 1.$$

(3) S is said to be of type (p, p) ($p_0 \leq p \leq p_1$) if there is a positive constant C such that for any $x \in L_p^+(\mathcal{M})$ there exists $a \in L_p^+(\mathcal{M})$ satisfying

$$\|a\|_p \leq C \|x\|_p \quad \text{and} \quad S_n(x) \leq a, \quad \forall n \geq 1.$$

In other words, S is of type (p, p) if and only if $\|S(x)\|_{L_p(\mathcal{M}; \ell^\infty)} \leq C \|x\|_p$ for all $x \in L_p^+(\mathcal{M})$.

This definition of subadditive operators in the noncommutative setting is due to Junge and Xu [26], who proved a noncommutative analogue of the classical Marcinkiewicz interpolation theorem as follows.

Theorem 3.1. (cf. [26, Theorem 3.1]) Let $1 \leq p_0 < p_1 \leq \infty$. Let $S = (S_n)_{n \geq 1}$ be a sequence of maps from $L_{p_0}^+(\mathcal{M}) + L_{p_1}^+(\mathcal{M}) \mapsto L_0^+(\mathcal{M})$. Assume that S is subadditive. If S is of weak type (p_0, p_0)

with constant C_0 and of type (p_1, p_1) with constant C_1 , then for any $p_0 < p < p_1$, S is of type (p, p) with constant C_p satisfying

$$C_p \leq C C_0^{1-\theta} C_1^\theta \left(\frac{1}{p_0} - \frac{1}{p} \right)^{-2}$$

where θ is determined by $1/p = (1-\theta)/p_0 + \theta/p_1$ and C is an absolute constant.

To state our results, we need to define a convex function of maximal operators in the noncommutative setting as follows.

Definition 3.2. Let Φ be an Orlicz function. Let (x_n) be a sequence in $L_\Phi(\mathcal{M})$. We define $\tau[\Phi(\sup_n^+ x_n)]$ by

$$(3.1) \quad \tau[\Phi(\sup_n^+ x_n)] := \inf \left\{ \frac{1}{2} \left(\tau[\Phi(|a|^2)] + \tau[\Phi(|b|^2)] \right) \sup_n \|y_n\|_\infty \right\}$$

where the infimum is taken over all decompositions $x_n = ay_n b$ for $a, b \in L_0(\mathcal{M})$ and $(y_n) \subset L_\infty(\mathcal{M})$ with $|a|^2, |b|^2 \in L_\Phi(\mathcal{M})$, and $\|y_n\|_\infty \leq 1$ for all n .

To understand $\tau[\Phi(\sup_n^+ x_n)]$, let us consider a positive sequence $x = (x_n)$ in $L_\Phi(\mathcal{M})$. We then note that

$$(3.2) \quad \tau[\Phi(\sup_n^+ x_n)] \leq \tau[\Phi(a)],$$

if $a \in L_\Phi^+(\mathcal{M})$ such that $x_n \leq a$ for all n . Indeed, for every n there exists a contraction u_n such that $x_n^{\frac{1}{2}} = u_n a^{\frac{1}{2}}$ and hence $x_n = a^{\frac{1}{2}} u_n^* u_n a^{\frac{1}{2}}$. This concludes (3.2). Moreover, the converse to (3.2) also holds true provided $\Phi \in \Delta_2$.

Proposition 3.1. Let Φ be an Orlicz function satisfying the Δ_2 -condition.

(1) If $x = (x_n)$ is a positive sequence in $L_\Phi(\mathcal{M})$, then

$$\tau[\Phi(\sup_n^+ x_n)] \approx \inf \left\{ \tau[\Phi(a)] : a \in L_\Phi^+(\mathcal{M}) \text{ such that } x_n \leq a, \forall n \geq 1 \right\}.$$

(2) For any two sequences $x = (x_n), y = (y_n)$ in $L_\Phi(\mathcal{M})$ one has

$$\tau[\Phi(\sup_n^+(x_n + y_n))] \lesssim \tau[\Phi(\sup_n^+ x_n)] + \tau[\Phi(\sup_n^+ y_n)].$$

Proof. (1). Let (x_n) be a sequence of positive elements in $L_\Phi(\mathcal{M})$. Suppose $x_n = ay_n b$ with $|a|^2, |b|^2 \in L_\Phi(\mathcal{M})$ and $\sup_n \|y_n\|_\infty \leq 1$. Without loss of generality, we can assume $a, b \geq 0$. Set $c = (a^2 + b^2)^{1/2}$. Then there exist two partial isometries $u, v \in \mathcal{M}$ such that

$$a = cu \quad \text{and} \quad b = vc,$$

i.e., $x_n = c u y_n v c$ for all n , and $\sup_n \|u y_n v\|_\infty \leq 1$. Thus, $x_k \leq c^2 \sup_n \|y_n\|_\infty$ for all k . By the Δ_2 -condition, one has

$$\begin{aligned} \tau[\Phi(c^2 \sup_n \|y_n\|_\infty)] &\leq \sup_n \|y_n\|_\infty \tau[\Phi(c^2)] \\ &\leq C_\Phi \sup_n \|y_n\|_\infty \frac{1}{2} \left(\tau[\Phi(|a|^2)] + \tau[\Phi(|b|^2)] \right). \end{aligned}$$

Combining this with (3.2) completes the proof of (1).

(2). We have the following useful description of $\tau[\Phi(\sup_n^+ x_n)]$:

$$(3.3) \quad \tau[\Phi(\sup_n^+ x_n)] = \inf \left\{ \frac{1}{2} \left(\tau[\Phi(|a|^2)] + \tau[\Phi(|b|^2)] \right) \right\},$$

where the infimum is taken over all decompositions $x_n = ay_n b$ for $a, b \in L_0(\mathcal{M})$ and $(y_n) \subset L_\infty(\mathcal{M})$ with $|a|^2, |b|^2 \in L_\Phi(\mathcal{M})$, and $\sup_n \|y_n\|_\infty = 1$. Indeed, for a decomposition $x_n = ay_n b$ with $\sup_n \|y_n\|_\infty \leq 1$, we set $\tilde{a} = \lambda^{1/2} a$, $\tilde{b} = \lambda^{1/2} b$, and $\tilde{y}_n = y_n / \lambda$ with $\lambda = \sup_n \|y_n\|_\infty$. Then $x_n = \tilde{a} \tilde{y}_n \tilde{b}$ for all n and $\sup_n \|\tilde{y}_n\|_\infty = 1$, so that

$$\tau[\Phi(|\tilde{a}|^2)] + \tau[\Phi(|\tilde{b}|^2)] \leq \lambda \left(\tau[\Phi(|a|^2)] + \tau[\Phi(|b|^2)] \right).$$

This concludes (3.3).

Now, to obtain the required inequality, it suffices to repeat the proof of the first part of [11, Theorem 3.2] through using (3.3). We omit the details. \square

Remark 3.1. For a sequences $x = (x_n)$ in $L_\Phi(\mathcal{M})$, set

$$\|\sup_n^+ x_n\|_\Phi := \inf \left\{ \lambda > 0 : \tau \left[\Phi \left(\sup_n^+ \frac{x_n}{\lambda} \right) \right] \leq 1 \right\}.$$

One can check that $\|\sup_n^+ x_n\|_\Phi$ is a norm in $x = (x_n)$. Define

$$L_\Phi(\mathcal{M}; \ell^\infty) := \left\{ (x_n) \subset L_\Phi(\mathcal{M}) : \tau \left[\Phi \left(\sup_n^+ \frac{x_n}{\lambda} \right) \right] < \infty \text{ for some } \lambda > 0 \right\},$$

equipped with $\|(x_n)\|_{L_\Phi(\mathcal{M}; \ell^\infty)} = \|\sup_n^+ x_n\|_\Phi$. Then $L_\Phi(\mathcal{M}; \ell^\infty)$ is a Banach space. For $1 \leq p < \infty$, if $\Phi(t) = t^p$ then $L_\Phi(\mathcal{M}; \ell^\infty) = L_p(\mathcal{M}; \ell^\infty)$ with equivalent norms. The details are left to the interested readers.

We are ready to state and prove the main result of this section.

Theorem 3.2. *Let $S = (S_n)_{n \geq 0}$ be a sequence of maps from $L_1^+(\mathcal{M}) + L_\infty^+(\mathcal{M}) \mapsto L_0^+(\mathcal{M})$. Let $1 \leq p < \infty$. Assume that S is subadditive, and order preserving in the sense that for all $n \geq 1$, $S_n(x) \leq S_n(y)$ whenever $x \leq y$ in $L_0^+(\mathcal{M})$. If S is simultaneously of weak type (p, p) with constant C_p and of type (∞, ∞) with constant C_∞ , then for an Orlicz function Φ with $p < p_\Phi \leq q_\Phi < \infty$, there exists a positive constant C depending only on C_p, C_∞, p_Φ and q_Φ , such that*

$$(3.4) \quad \tau \left[\Phi \left(\sup_n^+ S_n(x) \right) \right] \leq C \tau \left[\Phi(x) \right],$$

for all $x \in L_\Phi^+(\mathcal{M})$.

Proof. Since S is of weak type (p, p) with constant C_p , for any $x \in L_p^+(\mathcal{M})$ and each $\lambda > 0$ there is a projection $q^{(\lambda)} \in \mathcal{M}$ such that

$$\tau(1 - q^{(\lambda)}) \leq \frac{C_p^p \tau(x^p)}{\lambda^p} \quad \text{and} \quad q^{(\lambda)} S_n(x) q^{(\lambda)} \leq \lambda q^{(\lambda)}, \quad \forall n \geq 1.$$

For any $k \in \mathbb{Z}$ we set

$$q_k = \bigwedge_{j \geq k} q^{(2^j)} \quad \text{and} \quad p_k = q_k - q_{k-1}.$$

We claim the following two facts.

(i) $q_k S_n(x) q_k \leq 2^k q_k$ and

$$(3.5) \quad \tau(1 - q_k) \leq \frac{C_p^p}{1 - 2^{-p}} \frac{\tau(x^p)}{2^{kp}}, \quad \forall k \in \mathbb{Z}.$$

(ii) Suppose in addition, that $x \in \mathcal{M}$. Fix an integer N and a sequence $(\alpha_k)_{k=-\infty}^N$ of positive numbers for which $\sum_{k \leq N} \frac{2^k}{\alpha_k} < \infty$. Then the operator

$$(3.6) \quad a = 2C_\infty \|x\| (1 - q_N) + 2 \left(\sum_{k \leq N} \frac{2^k}{\alpha_k} \right) \sum_{k \leq N} \alpha_k p_k.$$

is a majorant of $S(x)$, i.e., $S_n(x) \leq a$ for all $n \geq 1$.

To prove these two statements, note that

$$\tau(1 - q_k) \leq \sum_{j \geq k} \tau(1 - q^{(2^j)}) \leq C_p^p \tau(x^p) \sum_{j \geq k} 2^{-jp} = \frac{C_p^p}{1 - 2^{-p}} \frac{\tau(x^p)}{2^{kp}},$$

which proves (3.5). On the other hand, for a fixed $\xi \in \mathbb{H}$ we have

$$\begin{aligned} (q_N S_n(x) q_N \xi, \xi) &= \left(\sum_{k, m \leq N} p_k S_n(x) p_m \xi, \xi \right) \leq \sum_{k, m \leq N} \|p_k S_n(x) p_m\| \|p_k \xi\| \|p_m \xi\| \\ &\leq \sum_{k, m \leq N} \|p_k S_n(x) p_k\|^{1/2} \|p_m S_n(x) p_m\|^{1/2} \|p_k \xi\| \|p_m \xi\| \\ &= \left(\sum_{k \leq N} \|p_k S_n(x) p_k\|^{1/2} \|p_k \xi\| \right)^2. \end{aligned}$$

Since $p_k S_n(x) p_k \leq 2^k p_k$ and so $\|p_k S_n(x) p_k\| \leq 2^k$, one concludes that

$$(q_N S_n(x) q_N \xi, \xi) \leq \left(\sum_{k \leq N} \frac{2^k}{\alpha_k} \right) \sum_{k \leq N} \alpha_k \|p_k \xi\|^2 = (a_N \xi, \xi),$$

where $a_N = \left(\sum_{k \leq N} \frac{2^k}{\alpha_k} \right) \sum_{k \leq N} \alpha_k p_k$. Note that

$$S_n(x) \leq 2q_N S_n(x) q_N + 2(1 - q_N) S_n(x) (1 - q_N).$$

Thus, a is a majorant of $S(x)$.

Take $x \in L_{\Phi}^+(\mathcal{M})$ and introduce

$$\tilde{x} = \sum_{i \in \mathbb{Z}} 2^{i+1} E_{(2^i, 2^{i+1}]}(x) = \sum_{i \in \mathbb{Z}} 2^i e_i,$$

where $e_i = E_{(2^i, \infty)}(x)$. For a fixed e_i , we will construct a suitable majorant of the sequence $S(e_i) = (S_n(e_i))_{n \geq 1}$. To this end, we take $p < \beta < p_{\Phi}$ and set

$$\delta = \frac{1}{2} \left(\frac{1}{p} + \frac{1}{\beta} \right) \quad \text{and} \quad \alpha_k = 2^{-(N-k)p\delta},$$

where N is an integer and will be fixed later on such that it only depends on p and C_p . By (3.6) we obtain the corresponding majorant of the sequence $S(e_i) = (S_n(e_i))_{n \geq 1}$, denoted by a_i . We claim that there exists a constant $C > 0$ depending only on C_p, C_{∞}, p and β such that

$$(3.7) \quad \mu_t(a_i) \leq Ch \left(\frac{t}{\tau(e_i)} \right) \quad \text{and} \quad \tilde{\mu}_t(a_i) \leq \frac{C}{1-\delta} h \left(\frac{t}{\tau(e_i)} \right), \quad \forall t > 0,$$

where $h(t) = \min\{t^{-\delta}, 1\}$ and, $\tilde{\mu}_t(x) = \frac{1}{t} \int_0^t \mu_s(x) ds$ for any $x \in L_0(\mathcal{M})$ and all $t > 0$.

Indeed, an immediate computation yields that

$$\mu(a_i) \leq C_{p,\beta,N,\infty} \left(\chi_{[0, \tau(1-q_N)]} + \sum_{k \leq N} 2^{-\delta(N-k)p} \chi_{(\tau(1-q_k), \tau(1-q_{k-1}))} \right),$$

where $C_{p,\beta,N,\infty} = 2(C_{\infty} + \sum_{k \leq N} 2^k / \alpha_k)$. By (3.5) one has

$$\mu(a_i) \leq C_{p,\beta,N,\infty} \left(\chi_{[0, C'_p 2^{-pN} \tau(e_i)]} + \sum_{k \leq N} 2^{-\delta(N-k)p} \chi_{(C'_p 2^{-kp} \tau(e_i), C'_p 2^{-(k+1)p} \tau(e_i))} \right),$$

where $C'_p = C_p^p / (1 - 2^{-p})$. Since for any $t \in (C'_p 2^{-kp} \tau(e_i), C'_p 2^{-(k+1)p} \tau(e_i))$,

$$h(2^{(N-k)p}) \leq h \left(2^{-p} \frac{2^{pN} t}{C'_p \tau(e_i)} \right),$$

it follows that

$$\begin{aligned} \mu_t(a_i) &\leq C_{p,\beta,N,\infty} \left(\chi_{[0, C'_p 2^{-pN} \tau(e_i)]}(t) + h \left(2^{-p} \frac{2^{pN} t}{C'_p \tau(e_i)} \right) \chi_{(C'_p 2^{-pN} \tau(e_i), \infty)}(t) \right) \\ &= C_{p,\beta,N,\infty} \left[\chi_{[0,1]} \left(\frac{2^{pN} t}{C'_p \tau(e_i)} \right) + h \left(2^{-p} \frac{2^{pN} t}{C'_p \tau(e_i)} \right) \chi_{(1, \infty)} \left(\frac{2^{pN} t}{C'_p \tau(e_i)} \right) \right] \\ &\leq C_{p,\beta,N,\infty} h \left(2^{-p} \frac{2^{pN} t}{C'_p \tau(e_i)} \right) \leq Ch \left(\frac{t}{\tau(e_i)} \right), \end{aligned}$$

provided we take N to be the least integer satisfying $N \geq \frac{1}{p} \log_2 C'_p + 1$, which implies that $2^{pN-p} / C'_p \geq 1$, as h is decreasing. This proves the first inequality in (3.7), from which the second one follows.

Since $x \mapsto \tilde{\mu}(x)$ is sublinear, we have

$$\begin{aligned} \tau \left[\Phi \left(\sum_{i \in \mathbb{Z}} 2^i a_i \right) \right] &\leq \int_0^{\infty} \Phi \left(\sum_{i \in \mathbb{Z}} 2^i \tilde{\mu}_t(a_i) \right) dt \lesssim \int_0^{\infty} \Phi \left[\sum_{i \in \mathbb{Z}} 2^i h \left(\frac{t}{\tau(e_i)} \right) \right] dt \\ &= \int_0^{\infty} \Phi \left[\int_0^{\infty} \sum_{i \in \mathbb{Z}} 2^i \chi_{(0, \tau(e_i)]} \left(\frac{t}{s} \right) (-h'(s)) ds \right] dt. \end{aligned}$$

Note that $\mu_t(\tilde{x}) = \sum_{i \in \mathbb{Z}} 2^i \chi_{(0, \tau(e_i)]}(t)$, we have

$$\tau \left[\Phi \left(\sum_{i \in \mathbb{Z}} 2^i a_i \right) \right] \lesssim \int_0^\infty \Phi \left[\int_0^\infty \mu_{\frac{t}{s}}(\tilde{x})(-h'(s)) ds \right] dt \leq \int_0^\infty \Phi \left[\int_0^\infty \tilde{\mu}_{\frac{t}{s}}(\tilde{x})(-h'(s)) ds \right] dt.$$

Define $T : L_1(\mathcal{M}) + L_\infty(\mathcal{M}) \mapsto L_1(0, \infty) + L_\infty(0, \infty)$ by

$$(Tx)(t) = \int_0^\infty \tilde{\mu}_{\frac{t}{s}}(x)(-h'(s)) ds, \quad \forall t > 0.$$

Then

$$\|Tx\|_\beta \leq \int_0^\infty \|\tilde{\mu}_{\frac{\cdot}{s}}(x)\|_\beta (-h'(s)) ds = C_{p,\beta} \|\tilde{\mu}(x)\|_\beta \leq C_{p,\beta} \|x\|_{L_\beta(\mathcal{M})},$$

where the last inequality is obtained by the classical Hardy-Littlewood inequality: the mapping $f \mapsto \frac{1}{t} \int_0^t |f(s)| ds$ is bounded in $L_\beta(0, \infty)$ provided $1 < \beta \leq \infty$. Also, it is easy to check that T is of type (∞, ∞) . Thus, by Theorem 2.1 in [3] we conclude that

$$\tau \left[\Phi \left(\sum_{i \in \mathbb{Z}} 2^i a_i \right) \right] \lesssim \int_0^\infty \Phi \left[\int_0^\infty \tilde{\mu}_{\frac{t}{s}}(\tilde{x})(-h'(s)) ds \right] dt \lesssim \tau [\Phi(\tilde{x})].$$

Since $\tilde{x} \leq 2x$, S is order preserving, and so

$$S_n(x) \leq S_n(\tilde{x}) \leq \sum_{i \in \mathbb{Z}} 2^i a_i, \quad \forall n \geq 1,$$

we conclude (3.4). \square

Remark 3.2. (1) The classical Marcinkiewicz interpolation theorem has been extended to include Orlicz spaces as interpolation classes by A. Zygmund, A. P. Calderón *et al.* (cf. e.g. [34]). The noncommutative analogue of this associated with a convex function was recently obtained in [3]. Theorem 3.2 can be considered as a noncommutative Marcinkiewicz type interpolation theorem for a convex function of maximal operators.

(2) Should Theorem 3.2 be true whenever S is simultaneously of weak type (p_0, p_0) and of type (p_1, p_1) and Φ an Orlicz function such that $1 \leq p_0 < p_\Phi \leq q_\Phi < p_1 \leq \infty$ (i.e., the case $p_1 < \infty$ is included). This was raised as an open question in the preliminary version of this paper, and affirmatively answered by Dirksen [13] recently through extending the argument presented here to the case of restricted weak type inequalities for noncommutative maximal operators. However, Theorem 3.2 is sufficient for our purpose of proving noncommutative maximal inequalities associated with convex functions (see Theorems 4.1 and 4.2 below).

The argument presented above can be used to obtain the corresponding interpolation theorem for maximal operators on noncommutative symmetric spaces, which is different from that of [12] and slightly simpler. Let E be a rearrangement invariant (r.i., in short) Banach space with the Boyd indices $p_E \leq q_E$ (for details on r.i. spaces we refer to [32]). Further, let $E(\mathcal{M}, \tau)$ be the associated noncommutative symmetric space (see e.g. [15, 16, 29, 46]).

Theorem 3.3. *Let $S = (S_n)_{n \geq 0}$ be a sequence of maps from $L_1^+(\mathcal{M}) + L_\infty^+(\mathcal{M}) \mapsto L_0^+(\mathcal{M})$. Assume that S is subadditive and order preserving. Let $1 \leq p < \infty$. Let E be a rearrangement invariant space such that $p_E > p$. If S is simultaneously of weak type (p, p) with constant C_p and of type (∞, ∞) with constant C_∞ , then there exists a positive constant C_E depending only on C_p, C_∞, p and p_E , such that for any $x \in E^+(\mathcal{M}, \tau)$ there exists $a \in E^+(\mathcal{M}, \tau)$ satisfying*

$$(3.8) \quad \|a\|_{E(\mathcal{M}, \tau)} \leq C_E \|x\|_{E(\mathcal{M}, \tau)} \quad \text{and} \quad S_n(x) \leq a, \quad \forall n \geq 0.$$

Proof. Indeed, the construction of the majorant of $S = (S_n(\tilde{x}))$ in the proof for Theorem 3.2 is clearly valid. Hence, we have

$$\begin{aligned} \left\| \sum_{i \in \mathbb{Z}} 2^i a_i \right\|_{E(\mathcal{M}, \tau)} &\lesssim \left\| \int_0^\infty \mu_{\frac{\cdot}{s}}(\tilde{x})(-h'(s)) ds \right\|_E \leq \int_0^\infty \|D_s \mu(\tilde{x})\|_E (-h'(s)) ds \\ &\leq \int_0^\infty \|D_s\|_E (-h'(s)) ds \|\tilde{x}\|_{E(\mathcal{M}, \tau)}. \end{aligned}$$

Here D_s ($0 < s < \infty$) are linear operators acting on measurable functions f on $(0, \infty)$ defined by

$$(D_s f)(t) = f(t/s), \quad 0 < t < \infty.$$

It is known (cf. [32, Sect. 2.b]) that for $1 < \beta < p_E$ there is a constant $C_{E,\beta} > 0$ such that

$$\|D_s\|_E \leq C_{E,\beta} s^{\frac{1}{\beta}}, \quad \forall 1 < s < \infty.$$

Thus

$$\int_0^\infty \|D_s\|_E (-h'(s)) ds \leq C_{E,\beta} \int_1^\infty s^{\frac{1}{\beta} - \delta - 1} ds = C_{p_E, \beta, p} < \infty.$$

This completes the proof. \square

With the help of Theorem 3.3, the associated maximal inequalities on noncommutative symmetric spaces are in order, including Doob's inequality, Dunford-Schwartz and Stein maximal ergodic inequalities, as well as the corresponding pointwise convergence theorems (see [26] for detailed information). We omit the details.

4. MAIN RESULTS

Let Φ be an Orlicz function. As noted in [3, Remark 1.1], if $1 < p_\Phi \leq q_\Phi < \infty$, then for any noncommutative L_Φ -martingale $x = (x_n)$, there exists a unique $x_\infty \in L_\Phi(\mathcal{M})$ such that $x_n = \mathcal{E}_n(x_\infty)$ for all n . We simply write $x_\infty = x$ in this case.

Our first main result is the following noncommutative Doob inequality associated with a convex function, generalizing Junge's noncommutative Doob inequality in L_p [20].

Theorem 4.1. *Let \mathcal{M} be a finite von Neumann algebra with a normalized normal faithful trace τ , equipped with a filtration $(\mathcal{M}_n)_{n \geq 0}$ of von Neumann subalgebras of \mathcal{M} . Let Φ be an Orlicz function and $x = (x_n)$ be a noncommutative L_Φ -martingale with respect to (\mathcal{M}_n) . If $1 < p_\Phi \leq q_\Phi < \infty$, then*

$$(4.1) \quad \tau \left[\Phi \left(\sup_n^+ x_n \right) \right] \approx \tau \left[\Phi(|x|) \right].$$

Proof. Decomposing an operator into a linear combination of four positive ones, by Proposition 3.1 (2) we can assume that $x = (x_n)$ is a positive martingale in $L_\Phi(\mathcal{M})$. Let $S = (\mathcal{E}_n)$. By Cuculescu's weak type $(1, 1)$ maximal martingale inequality [9], we see that S is of weak type $(1, 1)$. Also, S is trivially of type (∞, ∞) , due to the well known fact that conditional expectations are contractions for the operator norm. Thus, by Theorem 3.2 we conclude that

$$\tau \left[\Phi \left(\sup_n^+ x_n \right) \right] \lesssim \tau \left[\Phi(|x|) \right].$$

To prove the converse inequality, consider a decomposition $x_n = ay_n b$ for all n and $\sup_n \|y_n\|_\infty \leq 1$. One has

$$\begin{aligned} \tau \left[\Phi(|x|) \right] &= \int_0^\infty \Phi(\mu_t(x)) dt \\ &\leq 2 \sup_n \|y_n\|_\infty \int_0^\infty \Phi[\mu_t(|a|)\mu_t(|b|)] dt \\ &\leq 2 \sup_n \|y_n\|_\infty \int_0^\infty \Phi \left[\frac{1}{2} (\mu_t(|a|)^2 + \mu_t(|b|)^2) \right] dt \\ &\leq 2 \sup_n \|y_n\|_\infty \frac{1}{2} \left(\tau \left[\Phi(|a|^2) \right] + \tau \left[\Phi(|b|^2) \right] \right). \end{aligned}$$

Thus,

$$\tau \left[\Phi(|x|) \right] \leq 2\tau \left[\Phi \left(\sup_n^+ x_n \right) \right].$$

This completes the proof. \square

Remark 4.1. Let Φ be an Orlicz function. We define the Hardy-Orlicz maximal space of noncommutative martingales as

$$\mathcal{H}_\Phi^{\max}(\mathcal{M}) := \left\{ x \in L_\Phi(\mathcal{M}) : \|x\|_{\mathcal{H}_\Phi^{\max}} = \left\| \sup_n^+ \mathcal{E}_n(x) \right\|_\Phi < \infty \right\}.$$

(See [25, Sect. 4] for the case $\Phi(t) = t^p$.) Then, Theorem 4.1 implies that $\mathcal{H}_\Phi^{\max}(\mathcal{M}) = L_\Phi(\mathcal{M})$ with equivalent norms provided $1 < p_\Phi \leq q_\Phi < \infty$.

As a consequence of Theorem 4.1, we obtain the following noncommutative Burkholder-Davis-Gundy inequality associated with a convex function.

Corollary 4.1. *Let \mathcal{M} be a finite von Neumann algebra with a normalized normal faithful trace τ , equipped with a filtration $(\mathcal{M}_n)_{n \geq 0}$ of von Neumann subalgebras of \mathcal{M} . Let Φ be an Orlicz function, and let $x = (x_n)_{n \geq 0}$ be a noncommutative L_Φ -martingale with respect to $(\mathcal{M}_n)_{n \geq 0}$. If $1 < p_\Phi \leq q_\Phi < 2$, then*

$$(4.2) \quad \tau\left(\Phi\left[\sup_n^+ x_n\right]\right) \approx \inf \left\{ \tau\left(\Phi\left[\left(\sum_{k=0}^{\infty} |dy_k|^2\right)^{\frac{1}{2}}\right]\right) + \tau\left(\Phi\left[\left(\sum_{k=0}^{\infty} |dz_k^*|^2\right)^{\frac{1}{2}}\right]\right) \right\},$$

where the infimum runs over all decomposition $x_n = y_n + z_n$ with y_n in $\mathcal{H}_\Phi^C(\mathcal{M})$ and z_n in $\mathcal{H}_\Phi^R(\mathcal{M})$; and if $2 < p_\Phi \leq q_\Phi < \infty$, then

$$(4.3) \quad \tau\left(\Phi\left[\sup_n^+ x_n\right]\right) \approx \tau\left(\Phi\left[\left(\sum_{k=0}^{\infty} |dx_k|^2\right)^{\frac{1}{2}}\right]\right) + \tau\left(\Phi\left[\left(\sum_{k=0}^{\infty} |dx_k^*|^2\right)^{\frac{1}{2}}\right]\right).$$

Remark 4.2. The classical case of Corollary 4.1 was originally proved by Burkholder, Davis, and Gundy in [7] (see also [6]). Note that, the classical case holds even if $p_\Phi = 1$ (e.g. [10]). However, the noncommutative case is surprisingly different. Indeed, it was shown in [25, Corollary 14] that $\mathcal{H}_1 \neq \mathcal{H}_1^{\max}$. This implies that (4.2) does not hold when $\Phi(t) = t$ for which $p_\Phi = 1$.

Proof. It is proved in [3] that if $1 < p_\Phi \leq q_\Phi < 2$, then

$$(4.4) \quad \tau(\Phi(|x|)) \approx \inf \left\{ \tau\left(\Phi\left[\left(\sum_{k=0}^{\infty} |dy_k|^2\right)^{\frac{1}{2}}\right]\right) + \tau\left(\Phi\left[\left(\sum_{k=0}^{\infty} |dz_k^*|^2\right)^{\frac{1}{2}}\right]\right) \right\},$$

where the infimum runs over all decomposition $x_n = y_n + z_n$ with y_n in $\mathcal{H}_\Phi^C(\mathcal{M})$ and z_n in $\mathcal{H}_\Phi^R(\mathcal{M})$; and if $2 < p_\Phi \leq q_\Phi < \infty$, then

$$(4.5) \quad \tau(\Phi(|x|)) \approx \tau\left(\Phi\left[\left(\sum_{k=0}^{\infty} |dx_k|^2\right)^{\frac{1}{2}}\right]\right) + \tau\left(\Phi\left[\left(\sum_{k=0}^{\infty} |dx_k^*|^2\right)^{\frac{1}{2}}\right]\right).$$

An appeal to (4.1) yields the required inequalities (4.2) and (4.3). \square

Remark 4.3. We remark that there is a gap in the proof of (4.5) in [3], as pointed out to the first two named authors by Q. Xu. This was recently resolved by Dirksen and Ricard [14].

Now we turn to noncommutative maximal ergodic inequalities associated with convex functions. To state our results, we need some more notation.

Let \mathcal{M} be a semifinite von Neumann algebra with a normal semifinite faithful trace τ , and let $L_p(\mathcal{M})$ be the associated noncommutative L_p -space. Consider a linear map $T : \mathcal{M} \mapsto \mathcal{M}$ which may satisfy the following conditions:

- (I) T is a contraction on \mathcal{M} , that is, $\|Tx\| \leq \|x\|$ for all $x \in \mathcal{M}$.
- (II) T is positive, i.e., $Tx \geq 0$ if $x \geq 0$.
- (III) $\tau \circ T \leq \tau$, that is, $\tau(Tx) \leq \tau(x)$ for all $x \in L_1(\mathcal{M}) \cap \mathcal{M}_+$.
- (IV) T is symmetric relative to τ , i.e., $\tau((Ty)^*x) = \tau(y^*Tx)$ for all $x, y \in L_2(\mathcal{M}) \cap \mathcal{M}$.

Under conditions (I)-(III), T naturally extends to a contraction on $L_p(\mathcal{M})$ for every $1 \leq p < \infty$. The extension will be still denoted by T .

Theorem 4.2. *Let Φ be an Orlicz function with $1 < p_\Phi \leq q_\Phi < \infty$. If $T : \mathcal{M} \mapsto \mathcal{M}$ is a linear map satisfying (I) – (III), then*

$$(4.6) \quad \tau\left(\Phi\left[\sup_n^+ M_n(x)\right]\right) \lesssim \tau(\Phi(|x|)), \quad \forall x \in L_\Phi(\mathcal{M}),$$

where $M_n := \frac{1}{n+1} \sum_{k=0}^n T^k$ for any $n \geq 1$. If, in addition, T satisfies (IV), then

$$(4.7) \quad \tau\left(\Phi\left[\sup_n^+ T^n(x)\right]\right) \lesssim \tau(\Phi[|x|]), \quad \forall x \in L_\Phi(\mathcal{M}).$$

The inequalities (4.6) and (4.7) are the noncommutative forms of the classical Dunford-Schwartz and Stein maximal ergodic inequality for a convex function of positive and symmetric positive contractions. These generalize the noncommutative Dunford-Schwartz and Stein maximal ergodic inequalities of Junge and Xu [26] in the L^p case to the case of convex functions. The proofs of (4.6) and (4.7) are again based on Theorem 3.2.

Proof. Decomposing an operator into a linear combination of four positive ones, we can assume $x \in L_\Phi^+(\mathcal{M})$. Let $S = (M_n)$. Each M_n is considered to be a map on $L_1^+(\mathcal{M}) + L_\infty^+(\mathcal{M})$, positive and additive (and so subadditive too). Yeadon's weak type (1, 1) maximal ergodic inequality in [49] says that S is of weak type (1, 1). Also, S is evidently of type (∞, ∞) . Then, we deduce (4.6) from Theorem 3.2.

On the other hand, let $S = (T^n)$. Then S is additive and so subadditive. By [26, Theorem 5.1], S is of type (p, p) for every $1 < p \leq \infty$. An appeal to Theorem 3.2 immediately yields (4.7). \square

Let us present two examples illustrating applications of the inequalities obtained above.

Example 4.1. Let $\Phi(t) = t^a \ln(1 + t^b)$ with $a > 1$ and $b > 0$. It is easy to check that Φ is an Orlicz function and

$$p_\Phi = a \quad \text{and} \quad q_\Phi = a + b.$$

Thus, both Theorems 4.1 and 4.2 can be applied to this function. Furthermore, if $1 < a < a + b < 2$, then (4.2) holds true; if $a > 2$, then (4.3) is valid. Unfortunately, when $1 < a \leq 2 \leq a + b$, then Corollary 4.1 gives no information.

Example 4.2. Let $\Phi(t) = t^p(1 + c \sin(p \ln t))$ with $p > 1/(1 - 2c)$ and $0 < c < 1/2$. Then Φ is an Orlicz function and

$$p_\Phi = q_\Phi = p.$$

Therefore, Theorems 4.1 and 4.2 can be applied to this function, and so does Corollary 4.1 except the case $p = 2$.

5. WEAK TYPE MAXIMAL INEQUALITIES

All the results continue to hold if we replace the noncommutative maximal operator $\tau[\Phi(\sup_n^+ x_n)]$ associated with a convex function by a certain weak maximal operator. The required modifications are not difficult and left to the interested reader. However, for the sake of convenience, we write the corresponding definitions and results, and some main points of Theorem 5.1. We refer to [4] for noncommutative weak Orlicz spaces and for the terminology used here.

Let Φ be an Orlicz function. For $x \in L_\Phi^w(\mathcal{M})$, we set

$$\|x\|_{\Phi, \infty} = \sup_{t > 0} t \Phi[\mu_t(x)].$$

When $\Phi(t) = t^p$, $\|x\|_{\Phi, \infty}$ is just the usual weak L_p -norm $\|x\|_{p, \infty}$.

The following is the definition of a weak type maximal operator associated with a convex function.

Definition 5.1. Let (x_n) be a sequence in $L_\Phi^w(\mathcal{M})$. We define $\|\sup_n^+ x_n\|_{\Phi, \infty}$ by

$$(5.1) \quad \left\| \sup_n^+ x_n \right\|_{\Phi, \infty} := \inf \left\{ \frac{1}{2} \left(\| |a|^2 \|_{\Phi, \infty} + \| |b|^2 \|_{\Phi, \infty} \right) \sup_n \|y_n\|_\infty \right\}$$

where the infimum is taken over all decompositions $x_n = ay_n b$ for $a, b \in L_0(\mathcal{M})$ and $(y_n) \subset L_\infty(\mathcal{M})$ with $|a|^2, |b|^2 \in L_\Phi^w(\mathcal{M})$, and $\|y_n\|_\infty \leq 1$ for all n .

We have the noncommutative Marcinkiewicz type interpolation theorem for the weak type maximal operator associated with a convex function as follows. To this end, recall that

$$a_\Phi = \inf_{t>0} \frac{t\Phi'(t)}{\Phi(t)} \quad \text{and} \quad b_\Phi = \sup_{t>0} \frac{t\Phi'(t)}{\Phi(t)}.$$

Note that $1 \leq a_\Phi \leq p_\Phi \leq q_\Phi \leq b_\Phi$, but they do not coincide in general (see [33, 34] for details).

Theorem 5.1. *Suppose $1 \leq p_0 < p_1 \leq \infty$. Let $S = (S_n)_{n \geq 0}$ be a sequence of maps from $L_{p_0}^+(\mathcal{M}) + L_{p_1}^+(\mathcal{M}) \mapsto L_0^+(\mathcal{M})$. Assume that S is subadditive. If S is of weak type (p_0, p_0) with constant C_0 and of type (p_1, p_1) with constant C_1 , then for an Orlicz function Φ with $p_0 < a_\Phi \leq b_\Phi < p_1$, there exists a positive constant C depending only on p_0, p_1, C_0, C_1 and Φ , such that*

$$(5.2) \quad \left\| \sup_n^+ S_n(x) \right\|_{\Phi, \infty} \leq C \|x\|_{\Phi, \infty},$$

for all $x \in L_\Phi^w(\mathcal{M})_+$.

Proof. We give the main point of the proof. Indeed, modifying slightly the proof of [26, Theorem 3.1] we conclude that for $p_0 < p'_0 < a_\Phi \leq b_\Phi < p'_1 < p_1 \leq \infty$,

$$\left\| \sup_n^+ S_n(x) \right\|_{p'_i, \infty} \leq C_{p'_i} \|x\|_{p'_i, \infty}, \quad i = 0, 1,$$

that is, for each $x_i \in L_{p'_i}^+(\mathcal{M})$ there exists $a_i \in L_{p'_i}^+(\mathcal{M})$ such that

$$(5.3) \quad \|a_i\|_{p'_i, \infty} \leq C \|x_i\|_{p'_i, \infty} \quad \text{and} \quad S_n(x_i) \leq a_i, \quad \forall n \geq 1.$$

(This can be also obtained by Theorem 3.3 above.)

Now, take $x \in L_\Phi^w(\mathcal{M})_+$. For any $\alpha > 0$ let $x = x_0^\alpha + x_1^\alpha$, where $x_0^\alpha = x e_{(\alpha, \infty)}(x)$. By (5.3), for x_i^α there exists a corresponding a_i ($i = 0, 1$). The remainder of the proof is the same as that of [4, Theorem 4.2]. \square

The following is a noncommutative Doob weak type inequality associated with a convex function.

Theorem 5.2. *Let \mathcal{M} be a finite von Neumann algebra with a normalized normal faithful trace τ , equipped with a filtration (\mathcal{M}_n) of von Neumann subalgebras of \mathcal{M} . Let Φ be an Orlicz function and let $x = (x_n)$ be a noncommutative L_Φ^w -martingale with respect to (\mathcal{M}_n) . If $1 < a_\Phi \leq b_\Phi < \infty$, then*

$$(5.4) \quad \left\| \sup_n^+ x_n \right\|_{\Phi, \infty} \approx \|x\|_{\Phi, \infty}.$$

Combining this with [4, Theorem 5.8] and the associated result in [14] yields the noncommutative Burkholder-Davis-Gundy weak type inequality associated with a convex function as follows.

Corollary 5.1. *Let \mathcal{M} be a finite von Neumann algebra with a normalized normal faithful trace τ , equipped with a filtration (\mathcal{M}_n) of von Neumann subalgebras of \mathcal{M} . Let Φ be an Orlicz function and let $x = (x_n)_{n \geq 0}$ be a noncommutative L_Φ -martingale with respect to $(\mathcal{M}_n)_{n \geq 0}$. If $1 < a_\Phi \leq b_\Phi < 2$, then*

$$(5.5) \quad \left\| \sup_n^+ x_n \right\|_{\Phi, \infty} \approx \inf \left\{ \left\| \left(\sum_{k=0}^{\infty} |dy_k|^2 \right)^{\frac{1}{2}} \right\|_{\Phi, \infty} + \left\| \left(\sum_{k=0}^{\infty} |dz_k^*|^2 \right)^{\frac{1}{2}} \right\|_{\Phi, \infty} \right\},$$

where the infimum runs over all decomposition $x_n = y_n + z_n$ with (y_n) in $L_\Phi^w(\mathcal{M}; \ell_C^2)$ and (z_n) in $L_\Phi^w(\mathcal{M}; \ell_R^2)$; and if $2 < a_\Phi \leq b_\Phi < \infty$, then

$$(5.6) \quad \left\| \sup_n^+ x_n \right\|_{\Phi, \infty} \approx \left\| \left(\sum_{k=0}^{\infty} |dx_k|^2 \right)^{\frac{1}{2}} \right\|_{\Phi, \infty} + \left\| \left(\sum_{k=0}^{\infty} |dx_k^*|^2 \right)^{\frac{1}{2}} \right\|_{\Phi, \infty}.$$

The weak type analogue of Theorem 4.2 concerning maximal ergodic inequalities associated with a convex function is similar and omitted.

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