SHARP LOCALIZED INEQUALITIES FOR FOURIER MULTIPLIERS

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Abstract. In the paper we study sharp localized $L^q \to L^p$ estimates for Fourier multipliers resulting from modulation of the jumps of Lévy processes. The proofs of these estimates rest on probabilistic methods and exploit related sharp bounds for differentially subordinated martingales, which are of independent interest. The lower bounds for the constants involve the analysis of laminates, a family of certain special probability measures on 2×2 matrices. As an application, we obtain new sharp bounds for the real and imaginary parts of the Beurling-Ahlfors operator.

1. Introduction

This paper is devoted to sharp versions of localized $L^q \to L^p$ estimates for a large class of Fourier multipliers. Recall that for any bounded, complex-valued function m on $\mathbb{R}^d$, there is a unique bounded linear operator T_m on $L^2(\mathbb{R}^d)$, called *the Fourier multiplier with the symbol* m, which is given by $\widehat{T_m f} = m\hat{f}$. Obviously, the norm of T_m on $L^2(\mathbb{R}^d)$ is equal to $||m||_{L^\infty(\mathbb{R}^d)}$. There is an interesting question about the class of those m, for which the corresponding Fourier multipliers extend to bounded linear operators on $L^p(\mathbb{R}^d)$, $1 < p < \infty$. While the full characterization of such a class seems to be hopeless, much work has been done in the literature to construct examples and study their properties (cf. [19], [22], [23], [25]). It will be convenient for us to consider the following class of symbols, studied by Bañuelos and Bogdan [4] and Bañuelos, Bielaszewski and Bogdan [5]. Let ν be a Lévy measure on $\mathbb{R}^d$, i.e., a nonnegative Borel measure on $\mathbb{R}^d$ which does not charge the origin and satisfies

$$\int_{\mathbb{R}^d} \min\{|x|^2, 1\}\nu(\mathrm{d}x) < \infty.$$

Next, assume that μ is a finite Borel measure on the unit sphere $\mathbb{S}$ of $\mathbb{R}^d$ and fix two Borel functions ϕ on $\mathbb{R}^d$ and ψ on $\mathbb{S}$ which take values in the unit ball of $\mathbb{C}$. We define the associated multiplier $m = m_{\phi,\psi,\mu,\nu}$ on $\mathbb{R}^d$ by the formula

$$m(\xi) = \frac{\frac{1}{2}\int_{\mathbb{S}}\langle \xi, \theta\rangle^2 \psi(\theta)\mu(\mathrm{d}\theta) + \int_{\mathbb{R}^d}[1 - \cos\langle \xi, x\rangle]\phi(x)\nu(\mathrm{d}x)}{\frac{1}{2}\int_{\mathbb{S}}\langle \xi, \theta\rangle^2 \mu(\mathrm{d}\theta) + \int_{\mathbb{R}^d}[1 - \cos\langle \xi, x\rangle]\nu(\mathrm{d}x)} \tag{1.1}$$

if the denominator is not 0, and $m(\xi) = 0$ otherwise. Here $\langle \cdot, \cdot \rangle$ denotes the usual scalar product in $\mathbb{R}^d$. The Fourier multipliers corresponding to these symbols can be given a martingale representation by the use of transformations of jumps of Lévy processes; see [4] and [5] for details. Combining this representation with

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Burkholder's moment inequality (see Theorem 3.1 below), Bañuelos, Bielaszewski and Bogdan proved the following L^p estimate.

Theorem 1.1. *Let $1 < p < \infty$ and let $m = m_{\phi,\psi,\mu,\nu}$ be given by (1.1). Then for any $f \in L^p(\mathbb{R}^d)$ we have*

$$||T_m f||_{L^p(\mathbb{R}^d)} \leq (p^* - 1)||f||_{L^p(\mathbb{R}^d)}, \tag{1.2}$$

where $p^* = \max\{p, p/(p-1)\}$.

It turns out that the constant $p^* - 1$ appearing above is the best possible, which again can be shown with the use of probabilistic tools. See Geiss, Montgomery-Smith and Saksman [17] and the paper [6] by Bañuelos and the author.

The martingale approach can be used to establish other tight estimates for Fourier multipliers with symbols from the class (1.1) (see e.g. [30] and [31] for logarithmic and weak-type inequalities). In the present paper we continue this line of research and provide a significant improvement of (1.2). Namely, we study the action of Fourier multipliers, with symbols of the form (1.1), as operators from L^q to L^p, for any p, $q \in [1, \infty)$, $p < q$. It can be easily shown that for essentially all m we have $||T_m||_{L^q(\mathbb{R}^d)\to L^p(\mathbb{R}^d)} = \infty$. However, after an appropriate localization, we obtain non-trivial results. We will study bounds of the form

$$||T_m f||_{L^p(A)} \leq C||f||_{L^q(A)}|A|^{1/p-1/q},$$

where $A \subset \mathbb{R}^d$ is a fixed Borel subset and f is assumed to vanish outside A. Our primary goal is to determine the optimal constants C in the above inequality. Let us introduce some auxiliary notation. For any $1 \leq p < q \leq 2$, let $h : [0, \infty) \to [0, \infty)$ be a special function described in Theorem 2.1 and put

$$L_{p,q} = \frac{1}{2}(2 - p)h(0)^p. \tag{1.3}$$

Furthermore, for $1 \leq p < q < \infty$, define

$$C_{p,q} = \begin{cases} L_{p,q}^{(q-p)/pq} \left(\frac{q-p}{p}\right)^{1/q} \left(\frac{q}{q-p}\right)^{1/p} & \text{if } 1 \leq p < q < 2, \\ C_{q',p'} & \text{if } 2 < p < q < \infty, \\ 1 & \text{otherwise.} \end{cases} \tag{1.4}$$

Here $p' = p/(p-1)$, $q' = q/(q-1)$ denote the harmonic conjugates to p and q respectively. Our main result can be stated as follows.

Theorem 1.2. *Suppose that T_m is a Fourier multiplier with a symbol m belonging to the class (1.1). Let $1 \leq p < q < \infty$ and let A be a Borel subset of $\mathbb{R}^d$. Then for any $f \in L^q(\mathbb{R}^d)$ which vanishes on the compliment of A we have*

$$||T_m f||_{L^p(A)} \leq C_{p,q}||f||_{L^q(\mathbb{R}^d)}|A|^{1/p-1/q}. \tag{1.5}$$

The constant $C_{p,q}$ is the best possible.

Here by sharpness we mean that for any $1 \leq p < q < \infty$ and any $\varepsilon > 0$ there is a Borel subset A of $\mathbb{R}^d$, a function $f \in L^q(\mathbb{R}^d)$ and a symbol m from the class (1.1) for which $||T_m f||_{L^p(A)} > (C_{p,q} - \varepsilon)||f||_{L^q(\mathbb{R}^d)}|A|^{1/p-1/q}$.

We refer the reader to the papers [4] and [5] for various explicit examples of multipliers which have symbols of the form (1.1). We will only mention here two very important examples, strictly related to the so-called Beurling-Ahlfors transform $\mathcal{BA}$ on $\mathbb{C}$. This operator is a Fourier multiplier with the symbol $m(\xi) = \overline{\xi}/\xi$, $\xi \in \mathbb{C}$; alternatively, it can be defined by the singular integral

$$\mathcal{BA}f(z) = -\frac{1}{\pi}\,\text{p.v.}\int_{\mathbb{C}} \frac{f(w)}{(z-w)^2}\mathrm{d}w.$$

The Beurling-Ahlfors transform is of fundamental importance in the study of partial differential equations and quasiconformal mappings, since it changes the complex derivative $\overline{\partial}$ to ∂. Precisely, we have $\mathcal{BA}(\overline{\partial}f) = \partial f$ for any f from the Sobolev space $W^{1,2}(\mathbb{C},\mathbb{C})$ of complex valued locally integrable functions on $\mathbb{C}$ whose distributional first derivatives are in L^2 on the plane. For more on this interplay, consult e.g. the monograph [2] by Astala, Iwaniec and Martin.

The Beurling-Ahlfors operator can be decomposed as $\mathcal{BA} = R_2^2 - R_1^2 - 2iR_1R_2$, where R_1, R_2 are planar Riesz transforms (i.e., Fourier multipliers with the symbols $-i\xi_1/|\xi|$ and $-i\xi_2/|\xi|$, respectively). This follows at once from the identity

$$\frac{\overline{\xi}}{\xi} = \frac{\xi_1^2 - \xi_2^2}{\xi_1^2+\xi_2^2} + i\frac{2\xi_1\xi_2}{\xi_1^2+\xi_2^2}.$$

Note that both $R_2^2 - R_1^2$ and $2R_1R_2$ can be represented as the Fourier multipliers with the symbols of the form (1.1). For example, the choice $d = 2$, $\mu = \delta_{(1,0)} + \delta_{(0,1)}$, $\psi(1,0) = -1 = -\psi(0,1)$ and $\nu = 0$ gives rise to $T_m = \mathrm{Re}\,(\mathcal{BA})$; likewise, $d = 2$, $\mu = \delta_{(1/\sqrt{2},1/\sqrt{2})} + \delta_{(1/\sqrt{2},-1/\sqrt{2})}$, $\psi(1/\sqrt{2},1/\sqrt{2}) = 1 = \psi(1/\sqrt{2},-1/\sqrt{2})$ and $\nu = 0$ leads to $T_m = \mathrm{Im}\,(\mathcal{BA})$. Thus, Theorem 1.2 provides new information on the local behavior of the Beurling-Ahlfors operator, as well as its real and imaginary parts. Actually, we will prove that the optimality of the constants $C_{p,q}$ in (1.5) is achieved on these particular operators. In fact, we will manage to establish a more general, higher dimensional result, which is of interest in the theory of elliptic differential operators and potential theory.

Theorem 1.3. *Suppose that f is of class C^2, supported on a Borel set $A \subset \mathbb{R}^d$. Then for $1 \le p < q < \infty$ and any distinct j, $k \in \{1, 2, \ldots, d\}$ we have*

$$\left|\left|\frac{\partial^2 f}{\partial x_j^2} - \frac{\partial^2 f}{\partial x_k^2}\right|\right|_{L^p(A)} \le C_{p,q}||\Delta f||_{L^q(\mathbb{R}^d)}|A|^{1/p-1/q} \tag{1.6}$$

and

$$\left|\left|2\frac{\partial^2 f}{\partial x_j \partial x_k}\right|\right|_{L^p(A)} \le C_{p,q}||\Delta f||_{L^q(\mathbb{R}^d)}|A|^{1/p-1/q}. \tag{1.7}$$

Both estimates are sharp for each d, j and k.

One easily sees that (1.6) and (1.7) follow from (1.5). Indeed, a similar choice of μ, ν and ψ as above shows that for any d and any distinct j, $k \in \{1, 2, \ldots, d\}$ the multipliers $R_j^2 - R_k^2$, $2R_jR_k$ have the symbols from the class (1.1). Thus, it suffices to use (1.5) and apply $R_j^2 - R_k^2$ and $2R_jR_k$ to Δf (a straightforward comparison of Fourier transforms gives the identity $R_jR_k\Delta f = -\frac{\partial^2 f}{\partial x_j \partial x_k}$ for all j, k). The difficult part is to establish the sharpness of the two estimates. To handle this, we explore a very interesting connection between the theory of martingales and that of laminates, discovered recently by Boros, Shékelyhidi Jr. and Volberg in [8]. This will allow us to show that the constant $C_{p,q}$ in (1.6) and (1.7) is optimal for

$d = 2$. Then we will apply appropriate transference-type arguments to obtain the sharpness for all d.

A few words about the organization of the paper. The next section contains some preliminary material; we analyze there a certain class of differential equations, the solutions to which will be important in our further considerations. Section 3 is devoted to the proof of a certain martingale inequality, which can be regarded as a probabilistic counterpart of (1.5). In Section 4 we exploit the martingale representation of Fourier multipliers to deduce the inequality (1.5) from its stochastic version proved in Section 3. The final paper concerns the sharpness of (1.5). We will prove more: the constant $C_{p,q}$ in (1.6) and (1.7) cannot be improved.

2. A differential equation

Throughout this section, we assume that $1 \leq p < q \leq 2$ are given and fixed. Consider the differential equation

$$p(2-p)h'(x) + p = q(q-1)x^{q-2}h(x)^{2-p}. \tag{2.1}$$

This equation has already appeared in [27], during the study of related class of martingale inequalities. Unfortunately, the results of [27] are too weak for our purposes and do not lead to any form of (1.5). However, as we will see, a deeper investigation into the structure of the solutions to (2.1) gives us the possibility to establish stronger inequalities for martingales. These, in turn, will lead to the estimates for Fourier multipliers announced in Introduction.

We start with the following fact, established in [27] (see Theorem 2.1 and its proof there). See also Figure 1 below for the exemplary case $p = 3/2$, $q = 7/4$.

Theorem 2.1. *There exists a unique nondecreasing, concave solution $h : [0, \infty) \to [0, \infty)$ of (2.1) satisfying $h(0) > 0$ and $h'(t) \to 0$, $h(t) \to \infty$, as $t \to \infty$.*

In all the considerations below, the special solution described in the above theorem will be denoted by h. We will require the following auxiliary fact about this object. Let $F : [0, \infty) \to \mathbb{R}$ be given by

$$F(u) = (h(u) + u)^q - u^q - qu^{q-1}h(u) - (p-1)h(u)^p - \frac{2-p}{2}h(0)^p.$$

Lemma 2.2. *We have $F(u) \geq 0$ for all $u \geq 0$.*

Proof. First we show the estimate for large u. Since $q \leq 2$, an application of the mean value property and then (2.1) gives

$$\begin{aligned}
F(u) &\geq \frac{q(q-1)}{2}(h(u)+u)^{q-2}h(u)^2 - (p-1)h(u)^p - \frac{2-p}{2}h(0)^p \\
&= \frac{1}{2}h(u)^p \left[\left(\frac{h(u)+u}{u} \right)^{q-2} \cdot q(q-1)u^{q-2}h(u)^{2-p} - 2(p-1) \right] - \frac{2-p}{2}h(0)^p \\
&\geq \frac{1}{2}h(u)^p \left[\left(\frac{h(u)+u}{u} \right)^{q-2} p - 2(p-1) \right] - \frac{2-p}{2}h(0)^p \to \infty
\end{aligned}$$

as $u \to \infty$. Next, suppose that F attains a local minimum at some point $u_0 \in (0, \infty)$. We compute that

$$F'(u) = (h'(u)+1)\left[q((h(u)+u)^{q-1} - u^{q-1}) - ph(u)^{p-1} \right], \tag{2.2}$$

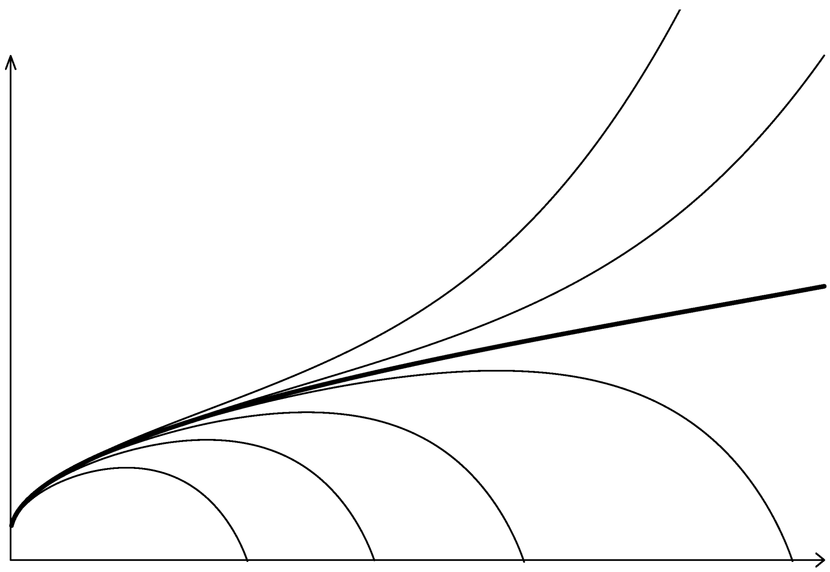

FIGURE 1. The structure of the solutions to (2.1) for $p = 3/2$ and $q = 7/4$. When $h(0)$ is small, the maximal domain of the solution is bounded. On the other hand, if $h(0)$ is large, then the solution is convex for sufficiently large arguments. The bold curve corresponds to the graph of the solution described in Theorem 2.1.

so

$$q((h(u_0)+u_0)^{q-1}-u_0^{q-1})-ph(u_0)^{p-1}=0$$

and in consequence,

$$F(u_0)=(-q+1)u_0^{q-1}h(u_0)+\left(\frac{p}{q}-p+1\right)h(u_0)^p+\frac{p}{q}h(u_0)^{p-1}u_0-\frac{2-p}{2}h(0)^p.$$

We will prove that $F(u_0)\geq 0$; multiplying this estimate by $qh(u_0)^{1-p}$, we get the equivalent form

$$(p-pq+q)h(u_0)+pu_0-q(q-1)u_0^{q-1}h(u_0)^{2-p}-\frac{2-p}{2}qh(0)^p h(u_0)^{1-p}\geq 0,$$

or, combining this with (2.1),

$$(p-pq+q)h(u_0)-p(2-p)u_0h'(u_0)-\frac{2-p}{2}qh(0)^p h(u_0)^{1-p}\geq 0. \tag{2.3}$$

To show this bound, recall that h is a concave function; thus, differentiating both sides of (2.1), we obtain

$$h(u_0)\geq\frac{2-p}{2-q}u_0h'(u_0). \tag{2.4}$$

Furthermore, again by the concavity of h,

$$h(u_0)\geq u_0h'(u_0)+h(0). \tag{2.5}$$

Mutiplying (2.4) by $(2-q)(p-1)$ and (2.5) by $2-p$, and adding the obtained bounds, we get

$$(p-pq+q)h(u_0)-p(2-p)u_0h'(u_0)-(2-p)h(0)\geq 0.$$

This implies (2.3), since $h(u_0)\geq h(0)$ and $q\leq 2$.

Therefore, to complete the proof, we need to show that the inequality $F(0)<0$ cannot hold. Suppose on contrary, that $F(0)$ is negative; then, by the above reasoning, F' does not vanish inside $(0,\infty)$, so $F'(0+)\geq 0$. However, in view of (2.2), this means $qh(0)^{q-1}\geq ph(0)^{p-1}$ or $h(0)^{q-p}\geq p/q$; it remains to observe that

$$F(0)=h(0)^q-\frac{p}{2}h(0)^p=h(0)^p\left[h(0)^{q-p}-\frac{p}{2}\right]\geq 0,$$

a contradiction. □

We conclude this section by introducing another function to be used later: let $H:[h(0),\infty)\to[0,\infty)$ be the inverse to $t\mapsto t+h(t)$. Then, of course,

$$h(H(t))+H(t)=t \qquad \text{and} \qquad h'(H(t))+1=\frac{1}{H'(t)}. \tag{2.6}$$

3. A martingale inequality

The key role in the proof of (1.5) is played by a certain related inequality for differentially subordinated martingales. Let us introduce the necessary background and notation. Assume that $(\Omega,\mathcal{F},\mathbb{P})$ is a complete probability space, equipped with $(\mathcal{F}_t)_{t\geq 0}$, a nondecreasing family of sub-σ-fields of $\mathcal{F}$, such that $\mathcal{F}_0$ contains all the events of probability 0. Let X, Y be two adapted martingales taking values in a certain separable Hilbert space $(\mathcal{H},|\cdot|)$, which may and will be taken to be equal to ℓ_2. As usual, we assume that both processes have right-continuous trajectories which have limits from the left. The symbol $[X,Y]$ will stand for the quadratic covariance process (square bracket) of X and Y. See e.g. Dellacherie and Meyer [16] for details in the case when the processes are real-valued, and extend the definition to the vector setting by $[X,Y]=\sum_{k=0}^\infty[X^k,Y^k]$, where X^k, Y^k are the k-th coordinates of X, Y. Following Bañuelos and Wang [7] and Wang [34], we say that Y is differentially subordinate to X, if the process $([X,X]_t-[Y,Y]_t)_{t\geq 0}$ is nonnegative and nondecreasing as a function of t.

A celebrated theorem of Burkholder [9] compares the L^p-norms of differentially subordinated martingales. We would like to mention that the result was originally formulated in the discrete-time case, and the extension below is due to Wang [34] (see also [9]). We use the notation $||X||_p=\sup_{t\geq 0}||X_t||_p$ for $1\leq p\leq\infty$.

Theorem 3.1. *Assume that X, Y are $\mathcal{H}$-valued martingales such that Y is differentially subordinate to X. Then for $1<p<\infty$ we have*

$$||Y||_p\leq(p^*-1)||X||_p, \tag{3.1}$$

where, as above, $p^*=\max\{p,p/(p-1)\}$. The constant p^*-1 is the best possible.

This result has proved to be very useful in many applications. The literature is too vast to review it here, we refer the interested reader to the papers [3]-[11], [17] and the references therein. Furthermore, the above theorem has been extended in many directions; consult, for instance, [10], [12], [13], [14], [18], [29] and [33].

We will require a certain version of the above estimate, in which the order of the moments of X and Y are different. The main result of this section can be stated as follows.

Theorem 3.2. *Assume that X, Y are $\mathcal{H}$-valued martingales such that Y is differentially subordinate to X. If $1 \leq p < q < 2$, then for any $t \geq 0$ we have*

$$\mathbb{E}(|Y_t|^p - L_{p,q})_+ \leq \mathbb{E}|X_t|^q. \tag{3.2}$$

The constant $L_{p,q}$ is the best possible.

Here by the optimality of $L_{p,q}$ we mean that for any $L < L_{p,q}$, there exists a pair X, Y of martingales such that Y is differentially subordinate to X and $\mathbb{E}(|Y_t|^p - L)_+ > \mathbb{E}|X_t|^q$.

The proof of (3.2) is based on the so-called Burkholder's method. Namely, the validity of this estimate will be shown by constructing certain special functions and exploiting their properties (see [29] for the detailed description of the technique and numerous examples). To construct these special objects, we need an auxiliary function $W_1 : \mathcal{H} \times \mathcal{H} \to \mathbb{R}$, given by the formula

$$W_1(x, y) = \begin{cases} |y|^2 - |x|^2 & \text{if } |x| + |y| \leq 1, \\ 1 - 2|x| & \text{if } |x| + |y| > 1. \end{cases}$$

The crucial property of this function is the following (for the proof, see Wang [34] or Lemma 2.2 in [28]).

Theorem 3.3. *Suppose that X, Y are $\mathcal{H}$-valued martingales such that Y is differentially subordinate to X. Then for any $t \geq 0$ we have*

$$\mathbb{E}W_1(X_t, Y_t) \leq 0.$$

We will also need the following evident property of W_1: for a fixed $x \in \mathcal{H}$,

$$W_1(x, y_1) \leq W(x, y_2) \qquad \text{provided } |y_1| \leq |y_2|. \tag{3.3}$$

We are ready to introduce the special functions corresponding to the martingale inequality (3.2). For $1 \leq p < q < 2$, let h be the solution to (2.1) described in Theorem 2.1. Put

$$w_{p,q}(t) = \frac{p(2-p)}{2} h(H(t))^{p-3} h'(H(t)) H'(t) t^2$$

and define $U_{p,q}$ by

$$U_{p,q}(x, y) = \int_{h(0)}^{\infty} w_{p,q}(t) W_1(x/t, y/t) \mathrm{d}t + \frac{(2-p)h(0)^p}{2}. \tag{3.4}$$

If X, Y are martingales such that Y is differentially subordinate to X, then, obviously, for any $t > 0$ the martingale Y/t is differentially subordinate to X/t. Therefore, by Theorem 3.3 and Fubini's theorem, we obtain

$$\mathbb{E}U_{p,q}(X_t, Y_t) \leq U_{p,q}(0, 0) \qquad \text{for } t \geq 0. \tag{3.5}$$

The function $U_{p,q}$ admits the following explicit formulas (see Lemma 4.1 in [27]).

Lemma 3.4. *We have*

$$U_{p,q}(x, y) = p\frac{|y|^2 - |x|^2}{2h(0)^{2-p}} + \frac{(2-p)h(0)^p}{2} \tag{3.6}$$

if $|x|+|y|\leq h(0)$, *and*

$$\begin{aligned} U_{p,q}(x,y)=&p|y|h(H(|x|+|y|))^{p-1}-(p-1)h(H(|x|+|y|))^p\\ &-H(|x|+|y|)^q-qH(|x|+|y|)^{q-1}\big(|x|-H(|x|+|y|)\big), \end{aligned} \tag{3.7}$$

if $|x|+|y|>h(0)$.

Now we turn to the following majorization property. Let $V_{p,q}:\mathcal{H}\times\mathcal{H}\to\mathbb{R}$ be given by $V_{p,q}(x,y)=\max\{|y|^p,L_{p,q}\}-|x|^q$.

Lemma 3.5. *For any* $(x,y)\in\mathcal{H}\times\mathcal{H}$ *we have* $U_{p,q}(x,y)\geq V_{p,q}(x,y)$.

Proof. Clearly, it suffices to prove the lemma for $\mathcal{H}=\mathbb{R}$, since the dependence of $U_{p,q}$ and $V_{p,q}$ on x, y is only through the norms $|x|$, $|y|$. Furthermore, we will be done if we consider the case x, $y\geq 0$. For the convenience of the reader, the proof is split into a few parts.

Step 1: $y=0$. If $x\geq h(0)$ and we substitute $u=H(x)$, the majorization is equivalent to the assertion of Lemma 2.2. If $x\in(0,h(0))$, we derive that

$$\frac{\partial}{\partial x}\big[U_{p,q}(x,0)-V_{p,q}(x,0)\big]=x\big(-ph(0)^{p-2}+qx^{q-2}\big).$$

Therefore, the derivative is positive for small x and changes sign at most once in the interval $(0,h(0))$. Since $U_{p,q}(0,0)=L_{p,q}=V_{p,q}(0,0)$ and $U_{p,q}(h(0),0)\geq V_{p,q}(h(0),0)$, the majorization follows.

Step 2: $y\in(0,L_{p,q})$. For a fixed x, the function $y\mapsto V_{p,q}(x,y)$ is constant on $[0,L_{p,q}]$, while $y\mapsto U_{p,q}(x,y)$ is nondecreasing on this interval (which follows immediately from (3.3) and (3.4)). Therefore, $U_{p,q}(x,y)-V_{p,q}(x,y)\geq U_{p,q}(x,0)-V_{p,q}(x,0)\geq 0$, by virtue of Step 1.

Step 3: $y\geq L_{p,q}$, $x+y<h(0)$. We easily compute that

$$\frac{\partial}{\partial y}\big[U_{p,q}(x,y)-V_{p,q}(x,y)\big]=py(h(0)^{p-2}-y^{p-2})<0,$$

so $U_{p,q}(x,y)-V_{p,q}(x,y)\geq U_{p,q}(x,h(0)-x)-V_{p,q}(x,h(0)-x)$. Therefore, it suffices to deal with the case $x+y\geq h(0)$, which will be done in Step 4 below.

Step 4: $y\geq L_{p,q}$, $x+y\geq h(0)$. Fix $r\geq h(0)$ and suppose that $|x|+|y|=r$. Denoting $s=|y|$, we see that the inequality $U_{p,q}(x,y)\geq V_{p,q}(x,y)$ is equivalent to $G(s)\geq 0$ for $s\in[L_{p,q},r]$, where

$$\begin{aligned} G(s)=&psh(H(r))^{p-1}-(p-1)h(H(r))^p\\ &-H(r)^q-qH(r)^{q-1}h(H(r))-s^p+(r-s)^q, \qquad s\geq 0. \end{aligned}$$

We have $G(h(H(r)))=G'(h(H(r)))=0$. Furthermore, the second derivative of G, equal to $G''(s)=-p(p-1)s^{p-2}+q(q-1)(r-s)^{q-2}$, is negative on $(0,s_0)$ and positive on (s_0,r) for some $s_0\in(0,r)$. Therefore, to show that $G\geq 0$ on $[L_{p,q},r]$, it suffices to prove that $G(L_{p,q})\geq 0$. But this follows immediately from continuity and Step 2. □

We are ready to establish the main result of this section.

Proof of (3.2). Observe that we may assume that $||X||_q<\infty$, since otherwise there is nothing to prove. The martingale inequality is equivalent to

$$\mathbb{E}\max\{|Y_t|^p,L_{p,q}\}\leq\mathbb{E}|X_t|^q+L_{p,q}, \qquad t\geq 0,$$

i.e., to $\mathbb{E}V_{p,q}(X_t, Y_t) \leq U_{p,q}(0,0)$. But this follows at once from (3.5) and the assertion of Lemma 3.5. The sharpness of (3.2) will be established later, while providing lower bounds for Fourier multipliers, see the beginning of Section 5. □

4. Norm inequalities for Fourier multipliers

We start by recalling the martingale representation of the multipliers from the class (1.1). This is described in full detail in [4] and [5], so we shall be brief. Let m be the multiplier as in (1.1), with the corresponding parameters ϕ, ψ, μ and ν. Assume in addition that $\nu(\mathbb{R}^d)$ is finite and nonzero. Then for any $s < 0$ there is a Lévy process $(X_{s,t})_{t\in[s,0]}$ with $X_{s,s} \equiv 0$, for which Lemmas 4.1 and 4.2 below hold true. To state these, we need some notation. For a given $f \in L^\infty(\mathbb{R}^d)$, define the corresponding parabolic extension $\mathcal{U}_f$ to $(-\infty, 0] \times \mathbb{R}^d$ by

$$\mathcal{U}_f(s,x) = \mathbb{E}f(x + X_{s,0}).$$

Next, fix $x \in \mathbb{R}^d$, $s < 0$ and let f, $\phi \in L^\infty(\mathbb{R}^d)$. We introduce the processes $F = (F_t^{x,s,f})_{s\leq t\leq 0}$ and $G = (G_t^{x,s,f,\phi})_{s\leq t\leq 0}$ by

$$\begin{aligned} F_t &= \mathcal{U}_f(t, x + X_{s,t}), \\ G_t &= \sum_{s<u\leq t} \big[(F_u - F_{u-}) \cdot \phi(X_{s,u} - X_{s,u-})\big] \\ &\quad - \int_s^t \int_{\mathbb{R}^d} \big[\mathcal{U}_f(v, x + X_{s,v-} + z) - \mathcal{U}_f(v, x + X_{s,v-})\big]\phi(z)\nu(\mathrm{d}z)\mathrm{d}v. \end{aligned} \tag{4.1}$$

Finally, fix $s < 0$, a function ϕ on $\mathbb{R}^d$ taking values in the unit ball of $\mathbb{C}$ and define the operator $\mathcal{T} = \mathcal{T}^s$ by the bilinear form

$$\int_{\mathbb{R}^d} \mathcal{T}f(x)g(x)\mathrm{d}x = \int_{\mathbb{R}^d} \mathbb{E}\big[G_0^{x,s,f,\phi} g(x + X_{s,0})\big]\mathrm{d}x, \tag{4.2}$$

where f, $g \in C_0^\infty(\mathbb{R}^d)$. By the results from [4] and [5], the family $\big\{(X_{s,t})_{s\leq t\leq 0}\big\}_{s<0}$ can be chosen so that the following statements are valid.

Lemma 4.1. *For any fixed x, s, f, ϕ as above, the processes $F^{x,s,f}$, $G^{x,s,f,\phi}$ are martingales with respect to $(\mathcal{F}_t)_{s\leq t\leq 0} = (\sigma(X_{s,t} : s \leq t))_{s\leq t\leq 0}$. Furthermore, if $||\phi||_\infty \leq 1$, then $G^{x,s,f,\phi}$ is differentially subordinate to $F^{x,s,f}$.*

Lemma 4.2. *Let $1 < p < \infty$ and $d \geq 2$. The operator $\mathcal{T}^s$ is well defined and extends to a bounded operator on $L^p(\mathbb{R}^d)$, which can be expressed as a Fourier multiplier with the symbol*

$$\begin{aligned} M(\xi) &= M_s(\xi) \\ &= \left[1 - \exp\left(2s\int_{\mathbb{R}^d}(1 - \cos\langle\xi, z\rangle)\nu(dz)\right)\right] \frac{\int_{\mathbb{R}^d}(1 - \cos\langle\xi, z\rangle)\phi(z)\nu(dz)}{\int_{\mathbb{R}^d}(1 - \cos\langle\xi, z\rangle)\nu(dz)} \end{aligned}$$

if $\int_{\mathbb{R}^d}(1 - \cos\langle\xi, z\rangle)\nu(dz) \neq 0$, and $M(\xi) = 0$ otherwise. Furthermore, the identity (4.2) holds if $f \in C_0^\infty(\mathbb{R}^d)$ and $g \in L^q(\mathbb{R}^d)$ for some $1 < q < \infty$.

Equipped with the necessary background, we are ready to establish the main estimate for Fourier multipliers.

Proof of (1.5). Of course, we may assume that $|A| < \infty$. It is convenient to split the reasoning into a few parts.

Step 1. It suffices to deal with the estimate when both p, q lie in $[1,2]$ or both lie in $[2,\infty)$. Indeed, having this done, if we take p, q such that 2 lies between p and q, then

$$\begin{aligned}|A|^{-1/p}||T_m f||_{L^p(A)} &\leq |A|^{-1/2}||T_m f||_{L^2(A)}\\ &\leq C_{2,q}||f||_{L^q(\mathbb{R}^d)}|A|^{-1/q} = C_{p,q}||f||_{L^q(\mathbb{R}^d)}|A|^{-1/q},\end{aligned}$$

as desired.

Step 2. Suppose that $1 \leq p < q \leq 2$. First we show the estimate for the multipliers of the form

$$M_{\phi,\nu}(\xi) = \frac{\int_{\mathbb{R}^d}(1-\cos\langle\xi,z\rangle)\phi(z)\nu(\mathrm{d}z)}{\int_{\mathbb{R}^d}(1-\cos\langle\xi,z\rangle)\nu(\mathrm{d}z)}. \tag{4.3}$$

In addition, we assume that $0 < \nu(\mathbb{R}^d) < \infty$, so that the above approach using Lévy processes is applicable. Fix $s < 0$, a Borel subset A of $\mathbb{R}^d$ satisfying $|A| < \infty$ and a function $f \in C_0^\infty(\mathbb{R}^d)$. We will prove that

$$||\mathcal{T}^s f||_{L^p(A)} \leq \Big[||f||^q_{L^q(\mathbb{R}^d)} + L_{p,q}|A|\Big]^{1/p}. \tag{4.4}$$

To this end, set $g = \chi_A|\mathcal{T}^s f|^{p-2}\mathcal{T}^s f$ (if $\mathcal{T}^s f = 0$, put $g = 0$). This function belongs to L^2; indeed, if $p = 1$, then there is nothing to prove, and for $p > 1$ this follows from Hölder inequality and (1.2) (the function f belongs to $L^r(\mathbb{R}^d)$ for all $1 < r < \infty$). Now, assume that $p = 1$. For a fixed $x \in \mathbb{R}^d$, we have, by (3.2),

$$\begin{aligned}\mathbb{E}|G_0^{x,s,f,\phi}|\cdot|g(x+X_{s,0})| &= \mathbb{E}\chi_{\{x+X_{s,0}\in A\}}|G_0^{x,s,f,\phi}|\\ &\leq \mathbb{E}\chi_{\{x+X_{s,0}\in A\}}\Big[(|G_0^{x,s,f,\phi}| - L_{p,q})_+ + L_{p,q}\Big]\\ &\leq \mathbb{E}(|G_0^{x,s,f,\phi}| - L_{p,q})_+ + L_{p,q}\mathbb{P}(x+X_{s,0}\in A)\\ &\leq \mathbb{E}|F_0^{x,s,f}|^q + L_{p,q}\mathbb{P}(x+X_{s,0}\in A).\end{aligned} \tag{4.5}$$

Thus, by Fubini's theorem,

$$\begin{aligned}\int_A |\mathcal{T}^s f(x)|\mathrm{d}x &= \int_{\mathbb{R}^d}\mathcal{T}^s f(x)g(x)\mathrm{d}x\\ &= \int_{\mathbb{R}^d}\mathbb{E}\left[G_0^{x,s,f,\phi}\,g(x+X_{s,0})\right]\mathrm{d}x\\ &\leq \int_{\mathbb{R}^d}\Big[\mathbb{E}|F_0^{x,s,f}|^q + L_{p,q}\mathbb{P}(x+X_{s,0}\in A)\Big]\mathrm{d}x\\ &= ||f||^q_{L^q(\mathbb{R}^d)} + L_{p,q}|A|,\end{aligned} \tag{4.6}$$

which is (4.4). On the other hand, if $p > 1$, then using Hölder's inequality and Fubini's theorem, we obtain

$$\begin{aligned}
&\int_A |\mathcal{T}^s f(x)|^p \mathrm{d}x \\
&= \int_{\mathbb{R}^d} \mathcal{T}^s f(x) g(x) \mathrm{d}x \\
&= \int_{\mathbb{R}^d} \mathbb{E}\left[G_0^{x,s,f,\phi}\, g(x+X_{s,0})\right] \mathrm{d}x \\
&\leq \left[\int_{\mathbb{R}^d} \mathbb{E}\chi_{\{x+X_{s,0}\in A\}} |G_0^{x,s,f,\phi}|^p \mathrm{d}x\right]^{1/p} \left[\int_{\mathbb{R}^d} \mathbb{E}|g(x+X_{s,0})|^{p'} \mathrm{d}x\right]^{1/p'} \qquad (4.7)\\
&= \left[\int_{\mathbb{R}^d} \mathbb{E}\chi_{\{x+X_{s,0}\in A\}} |G_0^{x,s,f,\phi}|^p \mathrm{d}x\right]^{1/p} ||g||_{L^{p'}(\mathbb{R}^d)} \\
&= \left[\int_{\mathbb{R}^d} \mathbb{E}\chi_{\{x+X_{s,0}\in A\}} |G_0^{x,s,f,\phi}|^p \mathrm{d}x\right]^{1/p} ||\mathcal{T}^s f||_{L^p(\mathbb{R}^d)}^{p/p'}.
\end{aligned}$$

Here, as usual, $p' = p/(p-1)$ denotes the harmonic conjugate to p. A similar argument to that in (4.5) gives that for any x,

$$\mathbb{E}\chi_{\{x+X_{s,0}\in A\}} |G_0^{x,s,f,\phi}|^p \leq \mathbb{E}|F_0^{x,s,f}|^q + L_{p,q}\mathbb{P}(x+X_{s,0}\in A)$$

and therefore the expression in the last square brackets in (4.7) is bounded from above by $\int_{\mathbb{R}^d} |f(x)|^q \mathrm{d}x + L_{p,q}|A|$. It suffices to divide throughout by $||\mathcal{T}^s f||_{L^p(\mathbb{R}^d)}^{p/p'}$, and (4.4) follows.

Next, let us use a homogenization argument: apply (4.4) to λf, divide throughout by λ and optimize over this parameter. We get

$$||\mathcal{T}^s f||_{L^p(A)} \leq C_{p,q} ||f||_{L^q(\mathbb{R}^d)} |A|^{1/p-1/q}.$$

Now if we let $s \to -\infty$, then M_s converges pointwise to the multiplier $M_{\phi,\nu}$ given by (4.3). By Plancherel's theorem, $\mathcal{T}^s f \to T_{M_{\phi,\nu}} f$ in L^2 and hence there is a sequence $(s_n)_{n=1}^\infty$ converging to $-\infty$ such that $\lim_{n\to\infty} \mathcal{T}^{s_n} f \to T_{M_{\phi,\nu}} f$ almost everywhere. Thus Fatou's lemma yields the desired bound for the multiplier $T_{M_{\phi,\nu}}$.

Step 3. Let us still keep p, q between 1 and 2. Now we deduce the result for the general multipliers as in (1.1) (in particular, involving the measure μ) and drop the assumption $0 < \nu(\mathbb{R}^d) < \infty$. For a given $\varepsilon > 0$, define a Lévy measure ν_ε in polar coordinates $(r,\theta) \in (0,\infty) \times \mathbb{S}$ by

$$\nu_\varepsilon(\mathrm{d}r\mathrm{d}\theta) = \varepsilon^{-2}\delta_\varepsilon(\mathrm{d}r)\mu(d\theta),$$

where δ_ε denotes Dirac measure on $\{\varepsilon\}$. Next, consider a multiplier m_ε as in (4.3), in which the Lévy measure is $1_{\{|x|>\varepsilon\}}\nu + \nu_\varepsilon$ and the jump modulator is given by $1_{\{|x|>\varepsilon\}}\phi(x) + 1_{\{|x|=\varepsilon\}}\psi(x/|x|)$. If we let $\varepsilon \to 0$, we see that

$$\begin{aligned}
\int_{\mathbb{R}^d} [1-\cos\langle \xi, x\rangle]\psi(x/|x|)\nu_\varepsilon(\mathrm{d}x) &= \int_{\mathbb{S}} \phi(\theta) \frac{1-\cos\langle \xi, \varepsilon\theta\rangle}{\varepsilon^2} \mu(d\theta) \\
&\to \frac{1}{2}\int_{\mathbb{S}} \langle \xi, \theta\rangle^2 \phi(\theta)\mu(\mathrm{d}\theta).
\end{aligned}$$

This yields the claim by the similar argument as above, using of Plancherel's theorem and the passage to the subsequence which converges almost everywhere.

Step 4. Now, assume that $2 \leq p < q < \infty$. We will use duality and the fact that for a symbol m as in (1.1), its conjugate $\bar{m}$ also belongs to this class. For any f as in (1.5), put $g = |T_m f|^{p-2}\overline{T_m f}$ and write

$$\begin{aligned}
\int_A |T_m f(x)|^p \mathrm{d}x &= \int_{\mathbb{R}^d} T_m f(x) g(x)\chi_A(x)\mathrm{d}x \\
&= \int_{\mathbb{R}^d} m(\xi)\hat{f}(\xi)\widehat{g\chi_A}(\xi)\mathrm{d}\xi \\
&= \int_{\mathbb{R}^d} f(x) T_{\bar{m}}(g\chi_A)(x)\mathrm{d}x \\
&= \int_{\mathbb{R}^d} f(x)\chi_A(x) T_{\bar{m}}(g\chi_A)(x)\mathrm{d}x \\
&\leq ||f||_{L^q(\mathbb{R}^d)} ||T_{\bar{m}}(g\chi_A)||_{L^{q'}(A)} \\
&\leq C_{q',p'} ||f||_{L^q(\mathbb{R}^d)} ||g\chi_A||_{L^{p'}(A)} |A|^{1/q'-1/p'} \\
&= C_{p,q} ||f||_{L^q(\mathbb{R}^d)} \left(\int_A |T_m f(x)|^p \mathrm{d}x \right)^{1/p'} |A|^{1/p-1/q}.
\end{aligned}$$

It remains to divide throughout by $\left(\int_A |T_m f(x)|^p \mathrm{d}x\right)^{1/p'}$ to get the claim. □

Remark 4.3. We have shown above that if f is supported on A, then

$$||T_m f||_{L^p(A)} \leq C_{p,q} ||f||_{L^q(\mathbb{R}^d)} |A|^{1/p-1/q}.$$

A careful inspection of the above proof (Steps 1−3) shows that if $p \leq 2$, then this estimate holds for all $f \in L^q(\mathbb{R}^d)$, that is, the condition $f \equiv 0$ on $\mathbb{R}^d \setminus A$ can be removed. Unfortunately, this is no longer true for $p > 2$ and we do not know the optimal values of $C_{p,q}$ in this case.

In the remainder of this section we discuss the possibility of extending the assertion of Theorem 1.2 to the vector-valued multipliers. For any bounded function $m = (m_1, m_2, \ldots, m_n) : \mathbb{R}^d \to \mathbb{C}^n$, we may define the associated Fourier multiplier acting on complex valued functions on $\mathbb{R}^d$ by the formula $T_m f = (T_{m_1} f, T_{m_2} f, \ldots, T_{m_n} f)$. As we shall see, the reasoning presented above can be easily modified to yield the following statement.

Theorem 4.4. *Let ν, μ be two measures on $\mathbb{R}^d$ and $\mathbb{S}$, respectively, satisfying the assumptions of Theorem 1.2. Assume further that ϕ, ψ are two Borel functions on $\mathbb{R}^d$ taking values in the unit ball of $\mathbb{C}^n$ and let $m : \mathbb{R}^d \to \mathbb{C}^n$ be the associated symbol given by* (1.1). *Then for any Borel subset A of $\mathbb{R}^d$ and any $f \in L^p(\mathbb{R}^d)$ which vanishes outside A we have*

$$||T_m f||_{L^p(A)} \leq C_{p,q} ||f||_{L^q(\mathbb{R}^d)} |A|^{1/p-1/q}.$$

Proof. Suppose first that ν is finite. For a given C^∞ function $f : \mathbb{R}^d \to \mathbb{C}$, we introduce martingales F and $G = (G^1, G^2, \ldots, G^n)$ by the formula (4.1). It is easy to check that G is differentially subordinate to F, arguing as in [4] or [5]. Applying the representation (4.2) to each coordinate of G separately, we obtain the associated multiplier $\mathcal{T} = (\mathcal{T}^1, \mathcal{T}^2, \ldots, \mathcal{T}^n)$, where $\mathcal{T}^j$ has symbol M_{ϕ_j,ν_j} defined in (4.3). Now we repeat the reasoning from (4.5) and (4.7), with a vector valued function $g = \chi_A |\mathcal{T}^s f|^{p-2}\mathcal{T}^s f : \mathbb{R}^d \to \mathbb{C}^n$. An application of (3.2) gives (4.4) and hence, by homogenization, the result follows. □

5. Sharpness

In the final part of the paper we show that the constant $C_{p,q}$ in (1.5) is the best possible. This, of course, will immediately imply that the constant $L_{p,q}$ is optimal in (3.2) (otherwise, its improvement would lead to a smaller constant in (1.5)). As explained in the introductory section, we will be done if we establish the sharpness of (1.6) and (1.7). One easily checks that the multipliers corresponding to the operators $R_j^2 - R_k^2$ and $2R_\ell R_m$ are isometric; i.e., if T_1, T_2 are two such multipliers and m_1, m_2 denote the corresponding symbols, then there is an isometry $I : \mathbb{R}^d \to \mathbb{R}^d$ such that $m_1 \circ I = m_2$. Consequently, in view of Parseval's identity, the optimal constants in (1.6) and (1.7) are the same for all j, k. Hence it is enough to focus on the sharpness of the bound

$$||(R_1^2 - R_2^2)f||_{L^p(A)} \leq C||f||_{L^q(A)}|A|^{1/p-1/q}. \tag{5.1}$$

Our approach will be based on the properties of certain special probability measures, the so-called laminates. For the sake of convenience and clarity, we have decided to split this section into a few separate parts.

5.1. **Necessary definitions.** Let $\mathbb{R}^{m\times n}$ denote the space of all real matrices of dimension $m \times n$ and let $\mathbb{R}^{n\times n}_{sym}$ be the class of all real symmetric $n \times n$ matrices.

Definition 5.1. A function $f : \mathbb{R}^{m\times n} \to \mathbb{R}$ is said to be *rank-one convex*, if $t \mapsto f(A + tB)$ is convex for all $A, B \in \mathbb{R}^{m\times n}$ with rank $B = 1$.

Let $\mathcal{P} = \mathcal{P}(\mathbb{R}^{m\times n})$ stand for the class of all compactly supported probability measures on $\mathbb{R}^{m\times n}$. For $\nu \in \mathcal{P}$, we denote by $\overline{\nu} = \int_{\mathbb{R}^{m\times n}} X d\nu(X)$ the center of mass or *barycenter* of ν.

Definition 5.2. We say that a measure $\nu \in \mathcal{P}$ is a *laminate* (and denote it by $\nu \in \mathcal{L}$), if

$$f(\overline{\nu}) \leq \int_{\mathbb{R}^{m\times n}} f d\nu \tag{5.2}$$

for all rank-one convex functions f. The set of laminates with barycenter 0 is denoted by $\mathcal{L}_0(\mathbb{R}^{m\times n})$.

Laminates arise naturally in several applications of convex integration, where can be used to produce interesting counterexamples, see e.g. [1], [15], [21], [26] and [32]. We will be particularly interested in the case of 2×2 symmetric matrices. The important fact is that laminates can be regarded as probability measures that record the distribution of the gradients of smooth maps, see Corollary 5.6 below and compare it with the discussion in §5.6. Let us briefly explain this; detailed proofs of the statements below can be found for example in [20], [26] and [32].

Definition 5.3. Let $U \subset \mathbb{R}^{2\times 2}$ be a given set. Then $\mathcal{PL}(U)$ denotes the class of *prelaminates* generated in U, i.e., the smallest class of probability measures on U which

(i) contains all measures of the form $\lambda\delta_A + (1-\lambda)\delta_B$ with $\lambda \in [0, 1]$ and satisfying rank$(A - B) = 1$;

(ii) is closed under splitting in the following sense: if $\lambda\delta_A + (1 - \lambda)\tilde{\nu}$ belongs to $\mathcal{PL}(U)$ for some $\tilde{\nu} \in \mathcal{P}(\mathbb{R}^{2\times 2})$ and μ also belongs to $\mathcal{PL}(U)$ with $\overline{\mu} = A$, then also $\lambda\mu + (1 - \lambda)\tilde{\nu}$ belongs to $\mathcal{PL}(U)$.

It follows immediately from the definition that the class $\mathcal{PL}(U)$ contains atomic measures only. Also, by a successive application of Jensen's inequality, we have the inclusion $\mathcal{PL} \subset \mathcal{L}$. Let us state two well-known facts (see [1], [20], [26],[32]).

Lemma 5.4. *Let $\nu = \sum_{i=1}^N \lambda_i \delta_{A_i} \in \mathcal{PL}(\mathbb{R}^{2\times 2}_{sym})$ with $\overline{\nu} = 0$. Moreover, let $0 < r < \frac{1}{2}\min|A_i - A_j|$ and $\delta > 0$. For any bounded domain $\Omega \subset \mathbb{R}^2$ there exists $u \in W_0^{2,\infty}(\Omega)$ such that $\|u\|_{C^1} < \delta$ and for all $i = 1 \dots N$*

$$\left|\{x \in \Omega : \, |D^2u(x) - A_i| < r\}\right| = \lambda_i |\Omega|.$$

Lemma 5.5. *Let $K \subset \mathbb{R}^{2\times 2}_{sym}$ be a compact convex set and $\nu \in \mathcal{L}(\mathbb{R}^{2\times 2}_{sym})$ with $\operatorname{supp}\nu \subset K$. For any relatively open set $U \subset \mathbb{R}^{2\times 2}_{sym}$ with $K \subset\subset U$ there exists a sequence $\nu_j \in \mathcal{PL}(U)$ of prelaminates with $\overline{\nu}_j = \overline{\nu}$ and $\nu_j \overset{*}{\rightharpoonup} \nu$.*

Combining these two lemmas and using a simple mollification, we obtain the following statement, proved by Boros, Shékelyhidi Jr. and Volberg [8]. It links laminates supported on symmetric matrices with second derivatives of functions, and will play a crucial role in our argumentation below. Throughout, $\mathcal{B}$ will denote the unit ball in $\mathbb{R}^2$.

Corollary 5.6. *Let $\nu \in \mathcal{L}_0(\mathbb{R}^{2\times 2}_{sym})$. Then there exists a sequence $u_j \in C_0^\infty(\mathcal{B})$ with uniformly bounded second derivatives, such that*

$$\frac{1}{|\mathcal{B}|}\int_{\mathcal{B}} \phi(D^2u_j(x))\,dx \;\to\; \int_{\mathbb{R}^{2\times 2}_{sym}} \phi\,d\nu$$

for all continuous $\phi : \mathbb{R}^{2\times 2}_{sym} \to \mathbb{R}$.

5.2. **Sharpness in the case $2 \in [p,q]$ and $d = 2$.** We are ready to exploit the above tools; first we study the easier case in which 2 lies between p and q. In what follows, we will often use the notation

$$\operatorname{diag}(x,y) = \begin{bmatrix} x & 0 \\ 0 & y \end{bmatrix} \in \mathbb{R}^{2\times 2}_{sym}.$$

Consider the probability measure $\nu = \frac{1}{2}\delta_{\operatorname{diag}(0,1)} + \frac{1}{2}\delta_{\operatorname{diag}(0,-1)}$ on $\mathbb{R}^{2\times 2}_{sym}$. Directly from the definition, this measure is a prelaminate. Let us introduce the continuous functions

$$\phi_1(A) = |A_{11} - A_{22}|^p \qquad \text{and} \qquad \phi_2(A) = |A_{11} + A_{22}|^q,$$

for which we easily check that

$$\frac{\left(\int_{\mathbb{R}^{2\times 2}_{sym}} \phi_1 \mathrm{d}\nu\right)^{1/p}}{\left(\int_{\mathbb{R}^{2\times 2}_{sym}} \phi_2 \mathrm{d}\nu\right)^{1/q}} = 1.$$

Consequently, if we fix $\varepsilon > 0$, Corollary 5.6 guarantees the existence of $u \in C_0^\infty(\mathcal{B})$ such that

$$\begin{aligned}(1-\varepsilon)|\mathcal{B}|^{1/p-1/q} &\leq \frac{\left(\int_{\mathcal{B}} \phi_1(D^2u(x))\mathrm{d}x\right)^{1/p}}{\left(\int_{\mathcal{B}} \phi_2(D^2u(x))\mathrm{d}x\right)^{1/q}} \\ &= \frac{\left(\int_{\mathcal{B}} |\partial_{11}^2 u(x) - \partial_{22}^2 u(x)|^p \mathrm{d}x\right)^{1/p}}{\left(\int_{\mathcal{B}} |\partial_{11}^2 u(x) + \partial_{22}^2 u(x)|^q \mathrm{d}x\right)^{1/q}}.\end{aligned}$$

Thus, if we put $f = \Delta u$, the inequality becomes

$$||(R_1^2 - R_2^2)f||_{L^p(\mathcal{B})} \geq (1-\varepsilon)||f||_{L^q(\mathcal{B})}|\mathcal{B}|^{1/p-1/q}.$$

Since ε was arbitrary, the sharpness follows.

5.3. **Biconvex functions and a special laminate.** We turn to the much more difficult case when $2 \notin [p,q]$. To study it, we need some additional notation. A function $\zeta : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is said to be biconvex if for any fixed $z \in \mathbb{R}$, the functions $x \mapsto \zeta(x,z)$ and $y \mapsto \zeta(z,y)$ are convex. We start with the following inequality for biconvex functions in the plane. Some heuristic arguments which lead to this particular statement are presented in §5.6 below. Let $1 \leq p < q \leq 2$ be fixed and let h be the solution to (2.1), described in Theorem 2.1.

Lemma 5.7. *Suppose that $\zeta : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is biconvex. Then for any $T > h(0)/2$,*

$$\begin{aligned} &\zeta\left(\frac{h(0)}{2}, \frac{h(0)}{2}\right) \\ &\leq \exp\left[-\int_{h(0)}^{2T} \frac{du}{h(H(u))}\right]\zeta(T,T) \\ &\quad + \int_{h(0)/2}^{T} \frac{\zeta(s-h(H(2s)),s) + \zeta(s,s-h(H(2s)))}{h(H(2s))} \exp\left[-\int_{h(0)}^{2s} \frac{du}{h(H(u))}\right] ds. \end{aligned} \tag{5.3}$$

Proof. By a standard regularization argument, it suffices to show the inequality for $\zeta \in C^1(\mathbb{R}^2)$. Fix $s \geq h(0)/2$. Using biconvexity, we may write, for $\delta < h(0)$,

$$\zeta(s,s) \leq \frac{\delta}{h(H(2s))+\delta}\zeta(s-h(H(2s)),s) + \frac{h(H(2s))}{h(H(2s))+\delta}\zeta(s+\delta,s)$$

and

$$\begin{aligned} \zeta(s+\delta,s) \leq &\frac{h(H(2s+2\delta))-\delta}{h(H(2s+2\delta))}\zeta(s+\delta,s+\delta) \\ &+ \frac{\delta}{h(H(2s+2\delta))}\zeta\big(s+\delta,s+\delta-h(H(2s+2\delta))\big). \end{aligned}$$

Plugging the latter estimate into the former, subtracting $\zeta(s+\delta,s+\delta)$ from both sides and dividing throughout by δ gives

$$\begin{aligned} \frac{\zeta(s,s)-\zeta(s+\delta,s+\delta)}{\delta} \leq &-\frac{h(H(2s))+h(H(2s+2\delta))}{(h(H(2s))+\delta)h(H(2s+2\delta))}\zeta(s+\delta,s+\delta) \\ &+ \frac{\zeta(s-h(H(2s)),s)}{h(H(2s))+\delta} + \frac{\zeta(s+\delta,s+\delta-h(H(2s+2\delta)))}{h(H(2s+2\delta))}. \end{aligned}$$

Letting $\delta \to 0$ yields

$$-\frac{\mathrm{d}}{\mathrm{d}s}\zeta(s,s) \leq -\frac{2}{h(H(2s))}\zeta(s,s) + \frac{\zeta(s-h(H(2s)),s)+\zeta(s,s-h(H(2s)))}{h(H(2s))}.$$

Multiply both sides by $\exp\left[-\int_{h(0)}^{2s}\frac{du}{h(H(u))}\right]$ and work a little bit to obtain

$$\frac{\mathrm{d}}{\mathrm{d}s}\left\{\exp\left[-\int_{h(0)}^{2s}\frac{\mathrm{d}u}{h(H(u))}\right]\zeta(s,s)\right\}$$
$$\geq -\frac{\zeta(s-h(H(2s)),s)+\zeta(s,s-h(H(2s)))}{h(H(2s))}\exp\left[-\int_{h(0)}^{2s}\frac{\mathrm{d}u}{h(H(u))}\right].$$

It suffices to integrate this inequality over s from $h(0)/2$ to T to get the claim. □

Let $\mu=\mu_T\in\mathcal{P}(\mathbb{R}^{2\times 2})$ be defined by the right-hand side of (5.3); that is, for any $f\in C(\mathbb{R}^{2\times 2})$, let

$$\int f\,d\mu_T$$
$$:=\exp\left[-\int_{h(0)}^{2T}\frac{\mathrm{d}u}{h(H(u))}\right]f(\mathrm{diag}(T,T))+\int_{h(0)/2}^{T}L(s)\exp\left[-\int_{h(0)}^{2s}\frac{\mathrm{d}u}{h(H(u))}\right]\mathrm{d}s,$$

where

$$L(s)=\frac{f(\mathrm{diag}(s-h(H(2s)),s))+f(\mathrm{diag}(s,s-h(H(2s))))}{h(H(2s))}.$$

Then μ_T is a probability with barycenter $\overline{\mu}_T=\mathrm{diag}(h(0)/2,h(0)/2)$. Moreover, observe that if f is rank-one convex, then $(x,y)\mapsto f(\mathrm{diag}(x,y))$ is biconvex. Therefore, using Lemma 5.7 we see that μ_T is a laminate. Consequently, the measure $\tilde{\mu}_T$, defined by the identity $\tilde{\mu}_T(A)=\mu_T(-A)$, is also a laminate, and has barycenter $\mathrm{diag}(-h(0)/2,-h(0)/2)$. Introduce another probability measure ν_T on $\mathbb{R}^{2\times 2}$ by

$$\nu_T:=\frac{1}{4}\mu_T+\frac{1}{4}\tilde{\mu}_T+\frac{1}{4}\delta_{\mathrm{diag}(-h(0)/2,h(0)/2)}+\frac{1}{4}\delta_{\mathrm{diag}(h(0)/2,-h(0)/2)}.$$

Obviously, the barycenter of ν_T equals 0. Furthermore, ν_T is a laminate: indeed, μ_T, $\tilde{\mu}_T$ have this property, so if f is a rank-one convex function on $\mathbb{R}^{2\times 2}$, then

$$\int_{\mathbb{R}^{2\times 2}}f\mathrm{d}\nu_T\geq\frac{1}{4}\bigg[f\left(\mathrm{diag}\left(\frac{h(0)}{2},\frac{h(0)}{2}\right)\right)+f\left(\mathrm{diag}\left(-\frac{h(0)}{2},-\frac{h(0)}{2}\right)\right)$$
$$+f\left(\mathrm{diag}\left(-\frac{h(0)}{2},\frac{h(0)}{2}\right)\right)+f\left(\mathrm{diag}\left(\frac{h(0)}{2},-\frac{h(0)}{2}\right)\right)\bigg]$$
$$\geq f(\mathrm{diag}(0,0))=f(\bar{\nu}_T).$$

Here the latter estimate follows directly from rank-one convexity of f. Next, consider the function $\phi:\mathbb{R}^{2\times 2}_{sym}\to\mathbb{R}$ given by

$$\phi(A)=|A_{11}-A_{22}|^p-|A_{11}+A_{22}|^q-L_{p,q}. \tag{5.4}$$

We have

$$\begin{aligned}\int_{\mathbb{R}^{2\times 2}_{sym}} \phi \mathrm{d}\mu_T &= \exp\left[-\int_{h(0)}^{2T} \frac{\mathrm{d}u}{h(H(u))}\right](-(2T)^q - L_{p,q}) \\ &\quad + \int_{h(0)/2}^{T} \frac{2((h(H(2s)))^p - H(2s)^q - L_{p,q})}{h(H(2s))} \exp\left[-\int_{h(0)}^{2s} \frac{\mathrm{d}u}{h(H(u))}\right]\mathrm{d}s \\ &= \exp\left[-\int_{h(0)}^{2T} \frac{\mathrm{d}u}{h(H(u))}\right](-(2T)^q - L_{p,q}) \\ &\quad + \int_{h(0)}^{2T} \frac{(h(H(s)))^p - H(s)^q - L_{p,q}}{h(H(s))} \exp\left[-\int_{h(0)}^{s} \frac{\mathrm{d}u}{h(H(u))}\right]\mathrm{d}s.\end{aligned}$$

Now we let T go to ∞. Using the substitution $r = H(u)$, we get that

$$\begin{aligned}\exp\left[-\int_{h(0)}^{2T} \frac{\mathrm{d}u}{h(H(u))}\right] &= \exp\left[-\int_{0}^{H(2T)} \frac{h'(r)+1}{h(r)}\mathrm{d}r\right] \\ &= \frac{h(0)}{h(H(2T))}\exp\left[-\int_0^{H(2T)} \frac{\mathrm{d}r}{h(r)}\right].\end{aligned} \tag{5.5}$$

Furthermore, by (2.1) and the concavity of h, we have that if $r > 1$, then

$$q(q-1)r^{q-2}h(r)^{2-p} \le p + p(2-p)h'(1),$$

that is, $h(r) \le cr^{(2-q)/(2-p)}$, where $c^{2-p} = (p + p(2-p)h'(1))/(q(q-1))$. This implies that for large T,

$$\int_0^{H(2T)} \frac{\mathrm{d}r}{h(r)} \ge \frac{1}{c}\int_1^{H(2T)} r^{(q-2)/(2-p)}\mathrm{d}r = O(T^{(q-p)/(2-p)}),$$

since

$$\lim_{T\to\infty} \frac{H(T)}{T} = \lim_{T\to\infty} \frac{T}{h(T)+T} = 1.$$

Therefore,

$$\lim_{T\to\infty} \exp\left[-\int_{h(0)}^{2T} \frac{\mathrm{d}u}{h(H(u))}\right](-(2T)^q - L_{p,q}) = 0.$$

In addition, using the calculations from (5.5), we may write

$$\begin{aligned}&\int_{h(0)}^{\infty} \frac{(h(H(s)))^p - H(s)^q - L_{p,q}}{h(H(s))} \exp\left[-\int_{h(0)}^{s} \frac{\mathrm{d}u}{h(H(u))}\right]\mathrm{d}s \\ &= h(0)\int_{h(0)}^{\infty} \frac{(h(H(s)))^p - H(s)^q - L_{p,q}}{(h(H(s)))^2} \exp\left[-\int_0^{H(s)} \frac{\mathrm{d}r}{h(r)}\right]\mathrm{d}s \\ &= h(0)\int_0^{\infty} \frac{h(u)^p - u^q - L_{p,q}}{h(u)^2} \exp\left[-\int_0^{u} \frac{\mathrm{d}r}{h(r)}\right](h'(u)+1)\mathrm{d}u,\end{aligned}$$

where the latter passage follows from the substitution $u = H(s)$. Now, since

$$\frac{\mathrm{d}}{\mathrm{d}u}\left(\frac{1}{h(u)}\exp\left[-\int_0^u \frac{\mathrm{d}r}{h(r)}\right]\right) = -\frac{h'(u)+1}{h(u)^2}\exp\left[-\int_0^u \frac{\mathrm{d}r}{h(r)}\right],$$

the integration by parts gives

$$\begin{aligned} h(0)\int_0^\infty \frac{-u^q-L_{p,q}}{h(u)^2}\exp\left[-\int_0^u \frac{\mathrm{d}r}{h(r)}\right](h'(u)+1)\mathrm{d}u \\ &= -L_{p,q}-h(0)q\int_0^\infty \frac{u^{q-1}}{h(u)}\exp\left[-\int_0^u\frac{\mathrm{d}r}{h(r)}\right]\mathrm{d}u \\ &= -L_{p,q}-h(0)q(q-1)\int_0^\infty u^{q-2}\exp\left[-\int_0^u \frac{\mathrm{d}r}{h(r)}\right]\mathrm{d}u. \end{aligned}$$

By (2.1), we have $q(q-1)u^{q-2}=ph(u)^{p-2}+p(2-p)h(u)^{p-2}h'(u)$ and therefore,

$$\begin{aligned} & h(0)\int_0^\infty \frac{h(u)^p-u^q-L_{p,q}}{h(u)^2}\exp\left[-\int_0^u\frac{\mathrm{d}r}{h(r)}\right](h'(u)+1)\mathrm{d}u \\ &= -L_{p,q}+h(0)\int_0^\infty h(u)^{p-2}\exp\left[-\int_0^u\frac{\mathrm{d}r}{h(r)}\right]\big[h'(u)+1-p-p(2-p)h'(u)\big]\mathrm{d}u \\ &= -L_{p,q}+h(0)(p-1)\int_0^\infty\left(h(u)^{p-1}\exp\left[-\int_0^u\frac{\mathrm{d}r}{h(r)}\right]\right)'\mathrm{d}u \\ &= -L_{p,q}-(p-1)h(0)^p=-\frac{p}{2}h(0)^p. \end{aligned}$$

Summarizing, we have proved that

$$\lim_{T\to\infty}\int_{\mathbb{R}^{2\times 2}_{sym}}\phi\mathrm{d}\mu_T=-\frac{p}{2}h(0)^p$$

and since $\phi(-A)=\phi(A)$ for any $A\in\mathbb{R}^{2\times 2}$, we also have

$$\lim_{T\to\infty}\int_{\mathbb{R}^{2\times 2}_{sym}}\phi\mathrm{d}\tilde{\mu}_T=-\frac{p}{2}h(0)^p.$$

Consequently,

$$\int_{\mathbb{R}^{2\times 2}_{sym}}\phi\mathrm{d}\nu_T=\frac12\int_{\mathbb{R}^{2\times 2}_{sym}}\phi\mathrm{d}\mu+\frac12(h(0)^p-L_{p,q})\xrightarrow{T\to\infty}\frac{2-p}{2}h(0)^p-L_{p,q}=0.$$

5.4. **Sharpness for** $2\notin[p,q]$ **and** $d=2$**.** By duality, it suffices to show that $C_{p,q}$ is optimal in the case $1\le p<q<2$. By the above reasoning, if $\varepsilon>0$ is a given number, then we can pick $T>0$ such that $\int_{\mathbb{R}^{2\times 2}_{sym}}\phi\mathrm{d}\nu_T>-\varepsilon$. Therefore, an application of Corollary 5.6 yields the existence of a C^∞ function u, supported on $\mathcal{B}$, such that $\int_{\mathcal{B}}\phi(D^2u(x))\mathrm{d}x>-2\varepsilon|\mathcal{B}|$ or, by the definition of ϕ,

$$\int_{\mathcal{B}}|\partial^2_{11}u(x)-\partial^2_{22}u(x)|^p\mathrm{d}x\ge\int_{\mathcal{B}}|\Delta u(x)|^q\mathrm{d}x+(L_{p,q}-2\varepsilon)\,|\mathcal{B}|.$$

Therefore, if we put $f=\Delta u$, we obtain the bound

$$\int_{\mathcal{B}}|(R_1^2-R_2^2)f(x)|^p\mathrm{d}x\ge\int_{\mathcal{B}}|f(x)|^q\mathrm{d}x+(L_{p,q}-2\varepsilon)\,|\mathcal{B}|. \tag{5.6}$$

Now suppose that C is a constant such that

$$||(R_1^2-R_2^2)f||_{L^p(\mathcal{B})}\le C||f||_{L^q(\mathcal{B})}|\mathcal{B}|^{1/p-1/q}$$

for all integrable f which vanish outside $\mathcal{B}$. Then, by Young's inequality,

$$\int_{\mathcal{B}}|(R_1^2-R_2^2)f(x)|^p\mathrm{d}x\le\int_{\mathcal{B}}|f(x)|^q\mathrm{d}x+\frac{q-p}{q}\left(\frac{p}{q}\right)^{p/(q-p)}C^{pq/(q-p)}|\mathcal{B}|.$$

Combining this with (5.6) and the fact that ε was arbitrary, we see that

$$\frac{q-p}{q}\left(\frac{p}{q}\right)^{p/(q-p)} C^{pq/(q-p)} \geq L_{p,q},$$

which is equivalent to $C \geq C_{p,q}$. This proves the desired sharpness of (1.6) and (1.7) in the case $d = 2$.

5.5. **The case $d \geq 3$.** Suppose that for fixed $1 \leq p < q < \infty$ and some positive constant C we have

$$\left(\int_A |(R_1^2 - R_2^2)f(x)|^p \mathrm{d}x\right)^{1/p} \leq C\left(\int_A |f(x)|^p \mathrm{d}x\right)^{1/p} |A|^{1/p-1/q} \tag{5.7}$$

for all Borel subsets A of $\mathbb{R}^d$ and all Borel functions $f : \mathbb{R}^d \to \mathbb{R}$ supported on A. For $t > 0$, define the dilation operator δ_t as follows: for any function $g : \mathbb{R}^2 \times \mathbb{R}^{d-2} \to \mathbb{R}$, we let $\delta_t g(\xi,\zeta) = g(\xi, t\zeta)$; for any $A \subset \mathbb{R}^2 \times \mathbb{R}^{d-2}$, let $\delta_t A = \{(\xi, t\zeta) : (\xi,\zeta) \in A\}$. If f is supported on A, then $\delta_t f$ is supported on $\delta_t^{-1}A$ and hence, by (5.7), the operator $T_t := \delta_t^{-1} \circ (R_1^2 - R_2^2) \circ \delta_t$ satisfies

$$\begin{aligned}\left(\int_A |T_t f(x)|^p \mathrm{d}x\right)^{1/p} &= \left(t^{d-2}\int_{\delta_t^{-1}A} |(R_1^2 - R_2^2)\circ\delta_t f(x)|^p \mathrm{d}x\right)^{1/p}\\ &\leq C\left(t^{d-2}\int_{\delta_t^{-1}A} |\delta_t f(x)|^q \mathrm{d}x\right)^{1/q} \left(t^{d-2}|\delta_t^{-1}A|\right)^{1/p-1/q}\\ &= C\left(\int_A |f(x)|^q \mathrm{d}x\right)^{1/q} |A|^{1/p-1/q}.\end{aligned} \tag{5.8}$$

Now fix $f \in L^q(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$. It is straightforward to check that the Fourier transform $\mathcal{F}$ satisfies the identity $\mathcal{F} = t^{d-2}\delta_t \circ \mathcal{F} \circ \delta_t$, so the operator T_t has the property that

$$\widehat{T_t f}(\xi,\zeta) = -\frac{\xi_1^2 - \xi_2^2}{|\xi|^2 + t^2|\zeta|^2}\widehat{f}(\xi,\zeta), \qquad (\xi,\zeta) \in \mathbb{R}^2 \times \mathbb{R}^{d-2}.$$

By Lebesgue's dominated convergence theorem, we have

$$\lim_{t\to 0}\widehat{T_t f}(\xi,\zeta) = \widehat{T_0 f}(\xi,\zeta)$$

in $L^2(\mathbb{R}^d)$, where $\widehat{T_0 f}(\xi,\zeta) = (\xi_2^2 - \xi_1^2)\widehat{f}(\xi,\zeta)/|\xi|^2$. By Plancherel's theorem and Fatou's lemma, we see that (5.8) implies

$$\left(\int_A |T_0 f(x)|^p \mathrm{d}x\right)^{1/p} \leq C\left(\int_A |f(x)|^q \mathrm{d}x\right)^{1/q} |A|^{1/p-1/q}. \tag{5.9}$$

Now pick an arbitrary function $g : \mathbb{R}^2 \to \mathbb{R}$ supported on the unit ball $\mathcal{B}$ and define $f : \mathbb{R}^2 \times \mathbb{R}^{d-2} \to \mathbb{R}$ by $f(\xi,\zeta) = g(\xi)1_{[0,1]^{d-2}}(\zeta)$. Denoting by $\mathcal{R}_1$ and $\mathcal{R}_2$ the *planar* Riesz transforms, we have $T_0 f(\xi,\zeta) = (\mathcal{R}_1^2 - \mathcal{R}_2^2)g(\xi)1_{[0,1]^{d-2}}(\zeta)$, because of the identity

$$\widehat{T_0 f}(\xi,\zeta) = -\frac{\xi_1^2 - \xi_2^2}{|\xi|^2}\,\widehat{g}(\xi)\widehat{1_{[0,1]^{d-2}}}(\zeta).$$

Plug this into (5.9) with the choice $A = \mathcal{B} \times [0,1]^{d-2}$ to obtain

$$\left(\int_{\mathcal{B}} |(\mathcal{R}_1^2 - \mathcal{R}_2^2) g(\xi)|^p \mathrm{d}\xi\right)^{1/p} \leq C \left(\int_{\mathcal{B}} |g(\xi)|^q \mathrm{d}\xi\right)^{1/q} |\mathcal{B}|^{1/p-1/q}.$$

As we have computed in the previous subsections, this implies $C \geq C_{p,q}$. The proof is complete.

5.6. **On the search of an appropriate laminate.** The inequality which appears in the statement of Lemma 5.7 is strictly related to the extremal example in (3.2). Suppose that $d = 2$ and let us look at the inequality (5.1) in the non-homogeneous form (which follows easily from Young's inequality)

$$\int_A |(R_1^2 - R_2^2) f(x)|^p \mathrm{d}x \leq \int_A |f(x)|^p \mathrm{d}x + L_{p,q} \cdot |A|,$$

or, since $R_1^2 + R_2^2 = Id$,

$$\frac{1}{|A|} \int_A |(R_1^2 - R_2^2) f(x)|^p \mathrm{d}x - L_{p,q} \leq \frac{1}{|A|} \int_A |(R_1^2 + R_2^2) f(x)|^p \mathrm{d}x.$$

On the other hand, a slightly weaker form of the inequality (3.2) can be rewritten in the form $\mathbb{E}|F_t - G_t|^p - L_{p,q} \leq \mathbb{E}|F_t + G_t|^q$, or

$$\mathbb{E}\phi(\mathrm{diag}(F_t, G_t)) \leq 0, \tag{5.10}$$

where ϕ is given by (5.4) and the martingale $F - G$ is differentially subordinate to $F + G$. Thus Corollary 5.6 suggests the following approach: find the extremal martingale pair (F, G) (for which the equality in (5.10) is attained, or almost attained); then the distribution of the random variable $\mathrm{diag}(F_t, G_t)$ is the desired laminate.

The paper [27] contains the description of the extremal pairs of martingales $(F - G, F + G)$ such that $\mathbb{E}|F_t - G_t|^p - \mathbb{E}|F_t + G_t|^p - L_{p,q}$ is almost 0 for large t and such that $F - G$ is differentially subordinate to $F + G$. We recall here the construction and express it in terms of the pair (F, G) (which is more convenient to us, in the light of the above remarks). Namely, fix $\delta > 0$, $T > h(0)/2$ and consider the discrete-time Markov martingale (f, g) whose transition function is uniquely determined by the following conditions:

(i) (f, g) starts from $(0, 0)$.

(ii) The state $(0, 0)$ leads to $(0, h(0)/2)$ or $(0, -h(0)/2)$.

(iii) for $\varepsilon \in \{-1, 1\}$, the state $(0, \varepsilon h(0)/2)$ leads to $(-\varepsilon h(0)/2, \varepsilon h(0)/2)$ or to $(\varepsilon h(0)/2, \varepsilon h(0)/2)$.

(iv) for $\varepsilon \in \{-1, 1\}$ and $h(0)/2 \leq s \leq T$, the state $(\varepsilon s, \varepsilon s)$ leads to $(\varepsilon(s - h(H(2s))), \varepsilon s)$ or to $(\varepsilon(s + \delta), \varepsilon s)$.

(v) for $h(0)/2 \leq s \leq T$, the state $(\varepsilon(s + \delta), \varepsilon s)$ leads to $(\varepsilon(s + \delta), \varepsilon(s + \delta))$ or to $(\varepsilon(s + \delta), \varepsilon(s + \delta - h(H(2s + 2\delta))))$.

(vi) All the remaining states are absorbing.

It is not difficult to check that if we let $\delta \to 0$, then the distributions of the pointwise limits $\mathrm{diag}(f_\infty, g_\infty)$ converge weakly to the laminate ν_T exploited in §5.4. This explains the use of this particular probability measure. See also [8] for a similar discussion.

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