SHARP MAXIMAL INEQUALITIES FOR THE MARTINGALE SQUARE BRACKET

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ABSTRACT. We determine the smallest number β for which the following holds. If X is a continuous-path martingale and $[X,X]$ denotes its square bracket, then

$$||\sup_t X_t||_1 \leq \beta ||[X,X]^{1/2}||_1.$$

Then we extend this inequality to (i) the class of martingales with no positive jumps and (ii) the class of discrete-time conditionally symmetric martingales. We also study some generalizations of the inequality involving an extra term depending on the initial value X_0.

1. Introduction

Square function inequalities play an important role in harmonic analysis, potential theory and both classical and noncommutative probability, see e.g. [5], [10], [18], [20], [23], ... The objective of the present paper is to determine the best constants in some maximal estimates for the martingale square function, both in the discrete- and continuous-time setting.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, filtered by a nondecreasing family $(\mathcal{F}_n)_{n=0}^\infty$ of sub-σ fields of $\mathcal{F}$. Assume that $f = (f_n)_{n=0}^\infty$ is an adapted real-valued martingale and let $df = (df_n)_{n=0}^\infty$ denote its difference sequence, defined by the equations

$$f_n = \sum_{k=0}^n df_k, \qquad n = 0,\, 1,\, 2,\, \ldots.$$

Then $S(f)$, the *square function* of f, and f^*, the *one-sided maximal function* of f, are given by

$$S(f) = \left(\sum_{k=0}^\infty |df_k|^2 \right)^{1/2} \qquad \text{and} \qquad f^* = \sup_k f_k.$$

We will also use the notation

$$S_n(f) = \left(\sum_{k=0}^n |df_k|^2 \right)^{1/2} \qquad \text{and} \qquad f_n^* = \sup_{k \leq n} f_k,$$

for $n = 0,\, 1,\, 2,\, \ldots$.

In the literature, there has been an interest in comparison of the moments of a martingale and its square function; see [2]–[9], [19] and references therein. Also, consult [15], [16] for some more recent progress on the subject. It was shown by

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Burkholder in [2], that there are finite positive c_p, C_p, $1 < p < \infty$, such that if f is a martingale, then

$$c_p||S(f)||_p \leq ||f||_p \leq C_p||S(f)||_p, \tag{1.1}$$

where $||f||_p = \sup_n ||f_n||_p$. Then Burkholder refined his proof in [4] and showed (1.1) with $c_p^{-1} = C_p = \max\{p-1, (p-1)^{-1}\}$. Moreover, this choice of c_p the best possible for $1 < p \leq 2$ and the choice of C_p is optimal for $2 \leq p < \infty$. See also [19] for a different approach. In the remaining cases, the best values of c_p and C_p are not known. Furthermore, in the limit case $p = 1$, the left inequality does not hold with any $c_p > 0$ and, as shown by the author in [15], we have that the optimal choice for C_1 is 2. For $p < 1$, neither of the inequalities holds without extra assumptions on the martingale f.

Let us now turn to the related maximal estimate

$$c_p^*||S(f)||_p \leq ||\sup_n |f_n|\,||_p \leq C_p^*||S(f)||_p, \qquad 1 \leq p < \infty. \tag{1.2}$$

Its validity for $p > 1$ follows from (1.1) and Doob's maximal inequality. The case $p = 1$ was studied by Davis [8], who established the double estimate using a clever decomposition of a martingale. What about the optimal values of the constants c_p^* and C_p^*? This is easy when $p = 2$: then $||S(f)||_2 = ||f||_2$, so (1.2) is sharp with $c_2^* = 1$ and $C_2^* = 2$. If $p \neq 2$, then, to the best of our knowledge, there are two results in this direction, both due to Burkholder: if $p > 2$, then the smallest possible C_p^* equals p (see [5]); furthermore, the optimal value of c_1^* is $1/\sqrt{3}$ (see [6]).

The inequalities (1.1) and (1.2) have also been studied for some special classes of martingales. A martingale f is *conditionally symmetric*, if for any $n \geq 1$, the laws of df_n and $-df_n$ given $\mathcal{F}_{n-1}$ coincide. For example, this holds for the important class of *dyadic martingales* (or *Haar martingales*, **or Paley-Walsh martingales**). Recall that a martingale on the **probability space** $([0,1], \mathcal{B}([0,1]), |\cdot|)$ is dyadic if for all $n \geq 0$ its n-th term and the absolute value of the $(n+1)$-st term of its difference sequence are both constant on $[(k-1)/2^n, k/2^n)$, $k = 1, 2, \ldots, 2^n$. Inequalities (1.1) and (1.2) in the dyadic case were investigated by Khintchine [11], Littlewood [12], Marcinkiewicz [13], Marcinkiewicz and Zygmund [14] and Paley [17] (without using the concept of a martingale, the results were formulated in terms of the partial sums of Rademacher and Haar series). Wang [24] studied (1.1) for conditionally symmetric martingales and showed that the following estimates are sharp, even in the dyadic case:

$$\begin{aligned} ||f||_p &\leq \nu_p||S(f)||_p, &\qquad 0 < p \leq 2,\\ ||f||_p &\leq \mu_p||S(f)||_p, &\qquad p \geq 3,\\ \nu_p||S(f)||_p &\leq ||f||_p, &\qquad p \geq 2. \end{aligned}$$

Here ν_p is the smallest positive zero of the confluent hypergeometric function and μ_p is the largest positive zero of the parabolic cylinder function of parameter p. For the remaining values of parameter p, the best constants in (1.1) are not known. The problem of determining optimal c_p^* and C_p^* in (1.2) for conditionally symmetric martingales is also open (except for the trivial case $p = 2$).

One of the main results of the present paper concerns the inequality for one-sided maximal function of a conditionally symmetric martingale f. Let $s_0 = -0.8745\ldots$

be the unique negative solution to the equation

$$\int_0^{s_0} \exp\left(\frac{u^2}{2}\right) du + 1 = 0 \tag{1.3}$$

and set

$$\beta = \exp\left(\frac{s_0^2}{2}\right) = 1.4658\ldots.$$

We will prove the following fact.

Theorem 1.1. *If f is a conditionally symmetric martingale, then*

$$||f^*||_1 \leq \beta ||S(f)||_1. \tag{1.4}$$

The constant is the best possible. It is already the best possible for dyadic martingales.

The inequalities (1.1) and (1.2) have been also been investigated for continuous-time martingales. Assume that the probability space is complete and suppose it is equipped with a continuous-time filtration $(\mathcal{F}_t)_{t\in[0,\infty)}$ such that $\mathcal{F}_0$ contains all the events of probability 0. Let X be a right-continuous martingale with limits from the left and let $[X,X]$ stand for its square bracket (which is continuous analogue of the square function. See [10] for details). Finally, denote $X^* = \sup_s X_s$ and $X_t^* = \sup_{s\leq t} X_s$.

The optimal constants in (1.1) for continuous-path martingales were found by Davis [9]: he showed that the best choice for c_p is μ_p for $1 < p \leq 2$ and ν_p when $2 \leq p < \infty$; on the other hand, the best possible value of C_p is ν_p if $0 < p \leq 2$ and μ_p for $2 \leq p < \infty$. However, the optimal constants in (1.2) are not known (again, except for the trivial case $p = 2$).

Our contribution in this direction is described in the theorem below.

Theorem 1.2. *If X has no positive jumps, then*

$$||X^*||_1 \leq \beta ||[X,X]^{1/2}||_1 \tag{1.5}$$

and the constant is the best possible. It is already the best possible if X is assumed to have continuous paths.

The inequalities (1.4) and (1.5) can be strengthened if one allows an extra term $\mathbb{E}f_0$ or $\mathbb{E}X_0$ on the right (see Section 4 below). In particular, it will be shown that if the martingales are assumed to start from 0, then the best constant in (1.4) and (1.5) decreases to $\nu_1 = 1,30693\ldots$.

The paper is organized as follows. The proof of the announced results exploits some properties of the confluent hypergeometric function of parameter 1. In the next section we introduce this function and establish some technical estimates to be needed later. Then in Section 3 we provide the proofs of Theorems 1.1 and 1.2. The final part of the paper is devoted to the extensions of the inequalities (1.4) and (1.5) mentioned above. **Furthermore, as an application, we present there some inequalities for a stopped local time of Brownian motion and three-dimensional Bessel process**.

2. Auxiliary technical facts

The central role in the paper is played by the function $\phi:\mathbb{R}\to\mathbb{R}$ given by

$$\phi(s)=-\exp\left(\frac{s^2}{2}\right)+s\int_0^s\exp\left(\frac{u^2}{2}\right)\mathrm{d}u.$$

We have that $-\phi$ is a confluent hypergeometric function of parameter 1, see [1]. We easily check that ϕ satisfies

$$\phi'(s)=\int_0^s\exp\left(\frac{u^2}{2}\right)du,\ \ \phi''(s)=\exp\left(\frac{s^2}{2}\right) \tag{2.1}$$

and the differential equation

$$\phi''(s)-s\phi'(s)+\phi(s)=0. \tag{2.2}$$

Throughout the paper, the number $\nu_1=1,30693\ldots$ will stand for the unique positive root of ϕ.

In the two lemmas below we establish key inequalities for the function ϕ.

Lemma 2.1. *Let $x>0$, $d\geq 0$ and $y\leq 0$. Then*

$$\sqrt{x^2+d^2}\left[\phi\left(\frac{y+d}{\sqrt{x^2+d^2}}\wedge 0\right)+\phi\left(\frac{y-d}{\sqrt{x^2+d^2}}\right)\right]\leq 2x\phi\left(\frac{y}{x}\right). \tag{2.3}$$

Proof. We will show that

$$\sqrt{x^2+d^2}\left[\phi\left(\frac{y+d}{\sqrt{x^2+d^2}}\right)+\phi\left(\frac{y-d}{\sqrt{x^2+d^2}}\right)\right]\leq 2x\phi\left(\frac{y}{x}\right), \tag{2.4}$$

which is stronger than (2.3), since $\phi(s)\geq\phi(0)$ for $s\geq 0$ (see (2.1)). If $d=0$, then both sides of (2.4) are equal; hence it suffices to prove that the left-hand side, as a function of d, has nonpositive derivative. This is equivalent to

$$\frac{d}{\sqrt{x^2+d^2}}\left[\phi(s_1)+\phi(s_2)\right]+\phi'(s_1)\frac{x^2-yd}{x^2+d^2}-\phi'(s_2)\frac{x^2+yd}{x^2+d^2}\leq 0,$$

where

$$s_1=\frac{y+d}{\sqrt{x^2+d^2}}\qquad\text{and}\qquad s_2=\frac{y-d}{\sqrt{x^2+d^2}}.$$

In view of (2.2), $\phi(s_1)=s_1\phi'(s_1)-\phi''(s_1)$ and $\phi(s_2)=s_2\phi'(s_2)-\phi''(s_2)$. Therefore the inequality can be written in the form

$$\phi'(s_1)-\phi'(s_2)-\frac{s_1-s_2}{2}(\phi''(s_1)+\phi''(s_2))\leq 0,$$

or

$$\int_{s_2}^{s_1}\phi''(u)\mathrm{d}u\leq\frac{s_1-s_2}{2}(\phi''(s_1)+\phi''(s_2)).$$

It suffices to use the fact that $\phi''(s)=\exp(s^2/2)$ is positive and convex. $\square$

Lemma 2.2. *Let $x>0$ and y, $d\leq 0$. Then*

$$\sqrt{x^2+d^2}\phi\left(\frac{y+d}{\sqrt{x^2+d^2}}\right)\leq x\phi\left(\frac{y}{x}\right)+\phi'\left(\frac{y}{x}\right)d. \tag{2.5}$$

Proof. If $d = 0$, then both sides are equal; hence we may assume that $d < 0$. The inequality (2.5) is equivalent to

$$-\sqrt{x^2+d^2}\exp\left(\frac{(y+d)^2}{2(x^2+d^2)}\right)+x\exp\left(\frac{y^2}{2x^2}\right)\leq (y+d)\int_{(y+d)/\sqrt{x^2+d^2}}^{y/x}\exp\left(\frac{u^2}{2}\right)\mathrm{d}u$$

and after substitution $X = x/\sqrt{x^2+d^2}$, $Y = y/\sqrt{x^2+d^2}$, $D = d/\sqrt{x^2+d^2}$ (note that $X^2 + D^2 = 1$ and $Y + D < 0$) becomes

$$\frac{1}{Y+D}\left[-\exp\left(\frac{(Y+D)^2}{2}\right)+X\exp\left(\frac{Y^2}{2X^2}\right)\right]-\int_{Y+D}^{Y/X}\exp\left(\frac{u^2}{2}\right)\mathrm{d}u\geq 0. \tag{2.6}$$

Fix D and denote the left-hand side by $F(Y)$. We have

$$F'(Y)=\frac{1}{(Y+D)^2}\left[-\frac{YD+1}{X}\exp\left(\frac{Y^2}{2X^2}\right)+\exp\left(\frac{(Y+D)^2}{2}\right)\right],$$

which, as we will prove now, is nonpositive. This is equivalent to (recall that $X = \sqrt{1-D^2}$)

$$G(Y)=\log\left(\frac{YD+1}{\sqrt{1-D^2}}\right)-\frac{(Y+D)^2}{2}+\frac{Y^2}{2(1-D^2)}\geq 0. \tag{2.7}$$

But, since $\log(1-x)+x \leq 0$ for all $x < 1$, we have

$$G(0)=-\frac{1}{2}(\log(1-D^2)+D^2)\geq 0$$

and, furthermore,

$$G'(Y)=-D^2Y\left(\frac{1}{YD+1}-\frac{1}{1-D^2}\right)\leq 0.$$

Thus (2.7) follows and F is nonincreasing. In consequence, to establish (2.6), we need to show that $F(0) \geq 0$. That is,

$$H(D)=D^{-1}\left[-\exp\left(\frac{D^2}{2}\right)+\sqrt{1-D^2}\right]+\int_0^D\exp\left(\frac{u^2}{2}\right)\mathrm{d}u\geq 0.$$

However, we have $\lim_{D\to 0-}H(D) = 0$ and

$$H'(D)=\frac{1}{D^2}\left[-\frac{1}{\sqrt{1-D^2}}+\exp\left(\frac{D^2}{2}\right)\right]\leq 0,$$

where the latter estimate follows from the elementary bound $e^s \leq (1-s)^{-1}$, applied to $s = D^2$. Thus $H \geq 0$, and the proof is complete. □

3. Proofs of Theorems 1.1 and 1.2

Introduce the function $U : (0,\infty)\times\mathbb{R}^2 \to \mathbb{R}$ by

$$U(x,y,z)=y+\sqrt{x}\phi\left(\frac{y-z}{\sqrt{x}}\right). \tag{3.1}$$

Let us list some of the properties of U, which will be used later.

Lemma 3.1. *(i) The function U satisfies the partial differential equations*

$$U_x + \frac{1}{2}U_{yy} = 0 \qquad \text{on } (0,\infty)\times\mathbb{R}^2, \tag{3.2}$$

and

$$U_z(x,y,y) = 0 \qquad \text{for all } x>0,\ y\in\mathbb{R}. \tag{3.3}$$

(ii) For any $x>0$ and $y\leq z$, $d\leq 0$ we have

$$U(x^2+d^2, y+d, z) + U(x^2+d^2, y-d, z) \leq 2U(x^2,y,z) \tag{3.4}$$

and

$$U(x^2+d^2, y+d, z) \leq U(x^2,y,z) + U_y(x^2,y,z)d. \tag{3.5}$$

(iii) For any $x>0$ and $y\leq z$ we have

$$U(x,y,z) \geq z - \beta\sqrt{x}. \tag{3.6}$$

Proof. (i) The equation (3.2) follows from (2.2), while (3.3) is a consequence of $\phi'(0)=0$.

(ii) A direct computation shows that (3.4) reduces to (2.3) and that (3.5) follows from (2.5).

(iii) Divide throughout by $\sqrt{x}$ and substitute $s=(y-z)/\sqrt{x}\leq 0$. The inequality becomes

$$\phi(s)+s \geq -\beta.$$

The derivative of the left-hand side is equal to

$$\int_0^s \exp\left(\frac{u^2}{2}\right)\mathrm{d}u + 1,$$

and hence the function $s\mapsto \phi(s)+s$, $s\leq 0$, attains its minimum at s_0 given by (1.3). As one easily checks, the minimum equals $-\beta$. □

Now we shall provide the proofs of the announced estimates.

Proof of (1.4). We may and do assume that $||S(f)||_1<\infty$. Let $\varepsilon>0$ be fixed and set $h_n=(\varepsilon+S_n^2(f), f_n, f_n^*)$, $n=0,1,2,\ldots$. We will show that the sequence $(U(h_n))$ is an $(\mathcal{F}_n)$-supermartingale. To this end, note that for any n,

$$\frac{f_n^*-f_n}{\sqrt{\varepsilon+S_n^2(f)}} \leq \frac{2\sum_{k=0}^n |df_k|}{\sqrt{\varepsilon+\max_{k\leq n}|df_k|}} \leq 2n+2,$$

so $U(h_n)\leq |f_n| + \sqrt{\varepsilon+S_n^2(f)}\cdot\phi(2n+2)$ and, in particular, $U(h_n)$ is integrable. By the conditional symmetry of f and (3.4) we have, for any $n\geq 1$,

$$\begin{aligned}
\mathbb{E}[U(h_n)|\mathcal{F}_{n-1}] &= \mathbb{E}[U(\varepsilon+S_{n-1}^2(f)+df_n^2, f_{n-1}+df_n, f_{n-1}^*\vee(f_{n-1}+df_n)|\mathcal{F}_{n-1}]\\
&= \frac{1}{2}\mathbb{E}[U(\varepsilon+S_{n-1}^2(f)+df_n^2, f_{n-1}+df_n, f_{n-1}^*\vee(f_{n-1}+df_n)|\mathcal{F}_{n-1}]\\
&\quad + \frac{1}{2}\mathbb{E}[U(\varepsilon+S_{n-1}^2(f)+df_n^2, f_{n-1}-df_n, f_{n-1}^*\vee(f_{n-1}-df_n)|\mathcal{F}_{n-1}]\\
&\leq U(h_{n-1}).
\end{aligned}$$

Therefore, using (3.6), we have that for any n,

$$\begin{aligned}
||f_n^*||_1 - \beta||\sqrt{\varepsilon+S_n^2(f)}||_1 &\leq \mathbb{E}U(h_n) \leq \mathbb{E}U(h_0) = \mathbb{E}U(\varepsilon+f_0^2, f_0, f_0)\\
&= f_0 - \sqrt{\varepsilon+f_0^2} \leq 0.
\end{aligned}$$

It suffices to let $\varepsilon \to 0$ and then $n \to \infty$ to obtain (1.4), in view of Lebesgue's dominated and monotone convergence theorems. □

Now we turn to the continuous-time setting. For any martingale X there exists a unique continuous part X^c of X satisfying

$$[X,X]_t = |X_0|^2 + [X^c,X^c]_t + \sum_{0<s\leq t} |\triangle X_s|^2 \tag{3.7}$$

for all $t \geq 0$. Here $\triangle X_s = X_s - X_{s-}$ is the jump of X at time s. Furthermore, we have that $[X^c,X^c] = [X,X]^c$, the pathwise continuous part of $[X,X]$.

Proof of (1.5). With no loss of generality we may assume that $||[X,X]^{1/2}||_1 < \infty$. Fix $\varepsilon > 0$ and a positive integer N. Consider a stopping time

$$\sigma_N = \inf\{t > 0 : X_t^* - X_t \geq N\sqrt{\varepsilon + [X,X]_t}\}$$

and a martingale

$$X^N = (X_{\sigma_N \wedge t}). \tag{3.8}$$

Clearly, it suffices to show (1.5) for X^N; then the passage $N \to \infty$ together with Lebesgue's monotone convergence theorem yield the estimate for the initial process X. Therefore, from now on, we may and do assume that $X = X^N$. The advantage is that the process

$$Z_t = (\varepsilon + [X,X]_t, X_t, X_t^*), \qquad t \geq 0, \tag{3.9}$$

satisfies $\mathbb{E}U(Z_t) < \infty$ for all t. Indeed, we have that

$$\frac{X_t^* - X_t}{\sqrt{\varepsilon + [X,X]_t}} \leq \frac{X_{t-}^* - X_{t-}}{\sqrt{\varepsilon + [X,X]_{t-}}} + \frac{|\Delta X_t|}{\sqrt{\varepsilon + [X,X]_t}} \leq N + 1, \tag{3.10}$$

so $U(Z_t) \leq X_t + \sqrt{\varepsilon + [X,X]_t}\phi(N+1)$ and the integrability follows.

The function U is of class C^2, so we may apply Itô's formula to (Z_t) and obtain

$$U(Z_t) = U(Z_0) + I_1 + I_2 + I_3 + I_4,$$

where

$$I_1 = \int_{0+}^t U_x(Z_{s-})\mathrm{d}[X,X]_s + \frac{1}{2}\int_0^t U_{yy}(Z_{s-})\mathrm{d}[X^c,X^c]_s,$$

$$I_2 = \int_{0+}^t U_y(Z_{s-})\mathrm{d}X_s,$$

$$I_3 = \int_{0+}^t U_z(Z_{s-})\mathrm{d}X_s^*,$$

$$I_4 = \sum_{0<s\leq t} \Big[U(Z_s) - U(Z_{s-}) - U_x(Z_{s-})|\Delta X_s|^2 - U_y(Z_{s-})\Delta X_s - U_z(Z_{s-})\Delta X_s^*\Big].$$

As we have already seen in the previous proof, $U(Z_0) \leq 0$. Furthermore, using (3.7) and then (3.2), we may write

$$\begin{aligned} I_1 &= \int_0^t (U_x(Z_{s-}) + \frac{1}{2}U_{yy}(Z_{s-}))\mathrm{d}[X^c,X^c]_s + \sum_{0<s\leq t} U_x(Z_{s-})|\triangle X_s|^2 \\ &= \sum_{0<s\leq t} U_x(Z_{s-})|\triangle X_s|^2. \end{aligned}$$

The assumption that X does not have positive jumps, has two consequences. First, the process X^* has continuous paths. Second, the inequality (3.5) implies that for any $s \in (0,t]$,

$$U(Z_s) - U(Z_{s-}) - U_y(Z_{s-})\Delta X_s \leq 0.$$

This gives $I_1 + I_4 \leq 0$. Furthermore, I_2 has zero expectation, by the properties of stochastic integrals. Finally, we have $I_3 = 0$, due to (3.3): indeed, since X^* has continuous trajectories, $\mathrm{d}X^*$ is supported on the set $\{s : X_s = X_s^*\}$, and we have $U_z(Z_{s-}) = 0$ there. Thus we have established the inequality $\mathbb{E}U(Z_t) \leq 0$, so, in virtue of (3.6),

$$\mathbb{E}X_t^* \leq \beta \mathbb{E}\sqrt{\varepsilon + [X,X]_t}. \tag{3.11}$$

It suffices to let $\varepsilon \to 0$ and then $t \to \infty$ to obtain (1.5). □

Now we will show that the constant β is optimal in (1.4) and (1.5). First let us deal with the continuous-time setting.

Sharpness of (1.5) *for continuous-path martingales.* We will provide an appropriate example. First let us recall the result of Shepp [22]. Suppose that $B^0 = (B_t^0)$ is a Brownian motion starting from 0 and consider a stopping time $T = T_a$ defined by

$$T_a = \inf\{t > 0 : |B_t^0| = a\sqrt{1+t}\}.$$

The result of Shepp we will need is that if $a < \nu_1$, then the stopping time T_a belongs to $L^{1/2}$.

Now let $B^1 = (B_t^1)$ be a Brownian motion starting from 1 and let

$$\tau = \inf\{t > 0 : (B_t^1)^* - B_t^1 = -s_0\sqrt{1+t}\}. \tag{3.12}$$

In view of Levy's theorem, the distributions of τ and T_{-s_0} coincide, so in particular we have $\mathbb{E}\tau^{1/2} < \infty$ (since $-s_0 < \nu_1$). Let $X = (B^1_{\tau\wedge t})$ and consider the process $Y = (U([X,X]_t, X_t, X_t^*))$. By (3.2), (3.3) and Itô's formula, Y is a martingale which converges in L^1: indeed, arguing as in (3.10), we can show that

$$|Y_t| \leq X^* - \phi(s_0)\sqrt{[X,X]}, \qquad t \geq 0,$$

and the variable on the right is integrable by (1.5) and the equality $[X,X] = 1+\tau$. It suffices to note that $\mathbb{E}Y_0 = 0$ and $Y_t \to X^* - \beta[X,X]^{1/2}$ almost surely as $t \to \infty$. This shows that

$$||X^*||_1 = \beta||[X,X]^{1/2}||_1 \tag{3.13}$$

and we are done. □

Sharpness of (1.4). We will prove that the optimal constant in (1.4) for the dyadic case is not smaller than the one in (1.5) for continuous-path martingales. This will be done using an extension of Burkholder's argument (see Chapter 10 in [5]). Suppose that for any dyadic martingale f we have $||f^*||_1 \leq \gamma||S(f)||_1$. Introduce the function $W : [0,\infty) \times \mathbb{R}^2 \to \mathbb{R}$ by

$$W(x^2, y, z) = \sup\left\{\mathbb{E}(f^* \vee z) - \gamma\mathbb{E}\sqrt{x^2 - y^2 + S^2(f)}\right\},$$

where the supremum is taken over the class of all simple dyadic martingales f starting from y (recall that a martingale f is called simple if for any n the random variable f_n takes only a finite number of values and there is deterministic N such that $f_N = f_{N+1} = f_{N+2} = \ldots$ with probability 1). Arguing as in [5], one can prove that W satisfies the following three conditions:

1° For any $x \in \mathbb{R}$, we have $W(x^2, x, x) \leq 0$.
2° For any x, y, $z \in \mathbb{R}$ we have $W(x^2, y, z) \geq V(x^2, y, z) = y \vee z - \gamma|x|$.
3° For any x, $d \geq 0$, and y, $z \in \mathbb{R}$, we have

$$W(x^2 + d^2, y + d, (y + d) \vee y \vee z) + W(x^2 + d^2, y - d, y \vee z) \leq 2W(x^2, y, y \vee z).$$

For a fixed $\delta \in (0, 1)$, let $g^\delta : \mathbb{R}^3 \to [0, \infty)$ be a function of class C^∞, supported on the ball of center 0 and radius δ, and satisfying $\int_{\mathbb{R}^3} g^\delta = 1$. Introduce W^δ, V^δ : $[\delta, \infty) \times \mathbb{R}^2 \to \mathbb{R}$ by the convolutions

$$W^\delta(x, y, z) = \int_{[-\delta,\delta]^3} W(x - u, y - v, z - w) g^\delta(u, v, w) \mathrm{d}x\mathrm{d}y\mathrm{d}w$$

and

$$V^\delta(x, y, z) = \int_{[-\delta,\delta]^3} V(x - u, y - v, z - w) g^\delta(u, v, w) \mathrm{d}x\mathrm{d}y\mathrm{d}w.$$

Clearly, the condition 3° is still valid for the function W^δ (provided $x^2 \geq \delta$); moreover, since W^δ is of class C^∞, we see that letting $d \to 0$ yields

$$W^\delta_x(x, y, z) + \frac{1}{2} W^\delta_{yy}(x, y, z) \leq 0 \qquad \text{if } x > \delta \text{ and } y < z, \tag{3.14}$$

and, if one takes $y = z$,

$$W^\delta_z(x, y, y) \leq 0 \qquad \text{for } x > \delta \text{ and any } y. \tag{3.15}$$

Let $B = B^1$ be a Brownian motion starting from 1 and suppose that τ is a stopping time defined by (3.12). Fix $\varepsilon \geq 2\delta$, an integer $N \geq 2$ and let $\eta_N = \tau \wedge \inf\{t > 0 : |B_t| \geq N\}$. By Itô's formula, the inequalities (3.14) and (3.15) guarantee that the process $(W^\delta(1 + \varepsilon + \eta_N \wedge t, B_{\eta_N \wedge t}, B^*_{\eta_N \wedge t}))$ is a supermartingale. Furthermore, by 2°, we have $V^\delta \leq W^\delta$, so for any $t \geq 0$,

$$\begin{aligned} \mathbb{E}V^\delta(1 + \varepsilon + \eta_N \wedge t, B_{\eta_N \wedge t}, B^*_{\eta_N \wedge t}) &\leq \mathbb{E}W^\delta(1 + \varepsilon + \eta_N \wedge t, B_{\eta_N \wedge t}, B^*_{\eta_N \wedge t}) \\ &\leq \mathbb{E}W^\delta(1 + \varepsilon, B_0, B_0) = W^\delta(1 + \varepsilon, 1, 1). \end{aligned}$$

Now it is clear from the definition of W and 1° that for $(u, v, w) \in [-\delta, \delta]^3$,

$$\begin{aligned} W(1 + \varepsilon - u, 1 - v, 1 - w) &\leq W(\sqrt{1 - v}, 1 - v, 1 - w) \\ &\leq W(\sqrt{1 - v}, 1 - v, 1 - v) + |w - v| \\ &\leq 2\delta, \end{aligned}$$

so $W^\delta(1+\varepsilon, 1, 1) \leq 2\delta$. Thus, letting $\delta \to 0$ and using the fact that V is a continuous function (so $V^\delta \to V$), yields

$$\mathbb{E}B^*_{\eta_N \wedge t} - \gamma \mathbb{E}\sqrt{1 + \varepsilon + \eta_N \wedge t} \leq 0.$$

It suffices to let $\varepsilon \to 0$ and $t \to \infty$, $N \to \infty$ to obtain, by Lebesgue's theorems,

$$||B^*_\tau||_1 \leq \gamma||\sqrt{1 + \tau}||_1,$$

which implies $\gamma \geq \beta$, in view of (3.13). □

Remark: In fact, the above argumentation can be used to provide an alternative proof of the inequality (1.5) for continuous-path martingales. Indeed, the validity of (1.4) implies the existence of the special function W, defined as above (with γ replaced by β), and an application of the smoothing argument and Itô's formula yields (1.5). It is evident that this method can be used as a general tool to transfer the inequalities from

the conditionally symmetric (or dyadic) setting to the continuous time continuous-path case. However, two remarks are in order. First, observe that the continuous-time versions may, but need not remain sharp: see the results of Wang [24] and Davis [9] mentioned in the introduction. Second, the above argumentation does not seem to work if we want to obtain the estimates for the wider class which consists of martingales with no positive jumps. This is why we have taken a different approach and exploited the special function U.

4. A class of stronger estimates and applications

In this section we present certain extensions of the inequalities (1.4) and (1.5). First we need to introduce some parameters. Let

$$b_* = 1 - \nu_1 \exp\left(-\frac{\nu_1^2}{2}\right)$$

and, for $b \leq b_*$, let $s_0(b)$ be the unique negative number satisfying

$$\int_0^{s_0(b)} \exp\left(\frac{u^2}{2}\right) \mathrm{d}u = -\frac{1}{1-b}. \tag{4.1}$$

Note that

$$s_0(\cdot) \text{ is continuous, decreasing, } \lim_{b\to-\infty} s_0(b) = 0 \text{ and } s_0(b_*) = -\nu_1. \tag{4.2}$$

Furthermore, for $b \in \mathbb{R}$, set

$$C(b) = \begin{cases} (1-b)\exp\left(\frac{s_0^2(b)}{2}\right) & \text{if } b < b_*, \\ \nu_1 & \text{if } b \in [b_*, 2-b_*], \\ (b-1)\exp\left(\frac{s_0^2(2-b)}{2}\right) & \text{if } b > 2-b_*. \end{cases} \tag{4.3}$$

It can be easily verified that the function C is of class C^1. Furthermore, it satisfies the symmetry condition $C(b) = C(2-b)$ for all b.

The main result in this section can be stated as follows.

Theorem 4.1. *Let $b \in \mathbb{R}$.*

(i) For any conditionally symmetric martingale f we have

$$||f^*||_1 \leq b\mathbb{E}f_0 + C(b)||S(f)||_1. \tag{4.4}$$

The constant $C(b)$ is the best possible. It is already the best possible for dyadic martingales.

(ii) For any martingale X with no positive jumps we have

$$||X^*||_1 \leq b\mathbb{E}X_0 + C(b)||[X,X]^{1/2}||_1. \tag{4.5}$$

The constant $C(b)$ is the best possible. It is already the best possible for continuous-path martingales.

Remark 4.2. *The result above generalizes Theorems 1.1 and 1.2: they correspond to the choice $b = 0$.*

Proof of (4.4) *and* (4.5). We will only give the detailed argumentation leading to (4.5); it will be clear how to modify the proof so that it works for (4.4). We will consider the cases $b \leq b_*$, $b \in (b_*, 2-b_*)$, $b \geq 2-b_*$ separately.

The case $b \leq b_*$ Define $U_b : (0,\infty)\times\mathbb{R}^2 \to \mathbb{R}$ by

$$U_b(x,y,z) = by + (1-b)U(x,y,z),$$

where U is given by (3.1). First note that we have the majorization

$$U_b(x,y,z) \geq z - (1-b)\exp\left(\frac{s_0^2(b)}{2})\right)\sqrt{x} \tag{4.6}$$

for $x > 0$ and $y \leq z$. Indeed, after substitution $s = (y-z)/\sqrt{x} \leq 0$, this is equivalent to

$$s + (1-b)\phi(s) + C(b) \geq 0,$$

and, as one easily verifies, the left-hand side, as a function of $s \leq 0$, attains its minimum 0 at $s = s_0(b)$.

Now fix $\varepsilon > 0$, $N > 0$ and let X be a martingale with no positive jumps. We may and do assume that $X = X^N$, where X^N is given by (3.8). If we define (Z_t) by (3.9), then, as we have shown in the proof of (1.5), we have $\mathbb{E}U(Z_t) \leq 0$ for any t. Consequently,

$$||X_t^*||_1 - C(b)||\sqrt{\varepsilon + [X,X]_t}||_1 \leq \mathbb{E}U_b(Z_t) \leq b\mathbb{E}X_0 \tag{4.7}$$

and it suffices to let $\varepsilon \to 0$ and $t \to \infty$.

The case $b \in (b_*, 2-b^*]$. By (4.7), applied to $b := b_*$ and a trivial bound $\mathbb{E}X_0 \leq \mathbb{E}|X_0|$, we get

$$||X_t^*||_1 - \nu_1||\sqrt{\varepsilon + [X,X]_t}||_1 \leq \mathbb{E}U_{b_*}(Z_t) = \mathbb{E}X_0 - (1-b_*)\mathbb{E}|X_0| \leq b_*\mathbb{E}X_0 \leq b\mathbb{E}X_0.$$

Since ε and t are arbitrary, the inequality follows.

The case $b > 2-b_*$. Applying (4.7) to $b := 2-b$ yields

$$||X_t^*||_1 - C(2-b)||\sqrt{\varepsilon + [X,X]_t}||_1 \leq \mathbb{E}U_{2-b}(Z_t) = \mathbb{E}X_0 - (b-1)\mathbb{E}|X_0| \leq b\mathbb{E}X_0.$$

Let $\varepsilon \to 0$ and $t \to \infty$ to complete the proof. □

Sharpness. First we will show that (4.5) is sharp for continuous-path martingales. The arguments are essentially the same as in the proof of the sharpness of (1.5), so we will only sketch the proof. We consider three cases: $b < b_*$, $b \in [b_*, 2-b_*]$, $b > 2-b_*$.

The case $b < b_*$. Let $B = B^1$ be a Brownian motion starting from 1 and

$$\tau_b = \inf\{t > 0 : B_t^* - B_t = -s_0(b)\sqrt{1+t}\}. \tag{4.8}$$

By (4.2), $-s_0(b) < \nu_1$, so the result of Shepp [22] gives $\mathbb{E}\tau_b^{1/2} < \infty$. Since U_b satisfies (3.2) and (3.3), the process $(U_b(1+\tau_b\wedge t, B_{\tau_b\wedge t}, B^*_{\tau_b\wedge t}))$ is a martingale convergent in L^1 to $U_b(1+\tau_b, B_{\tau_b}, B^*_{\tau_b}) = B^*_{\tau_b} - C(b)\sqrt{1+\tau_b}$. Thus if $X = (B_{\tau_b\wedge t})$, then

$$b\mathbb{E}X_0 = \mathbb{E}U_b([X,X]_0, X_0, X_0^*) = ||X^*||_1 - C(b)||[X,X]^{1/2}||_1. \tag{4.9}$$

The case $b \in [b_*, 2-b_*]$. Let $B = B^0$ be a Brownian motion starting from 0 and let $b' < b_*$. Repeating the argumentation from the previous case, we have that if $X = (B_{\tau_{b'}\wedge t})$, then, by (4.9),

$$0 = ||X^*||_1 - C(b')||[X,X]^{1/2}||_1. \tag{4.10}$$

Now if $b' \uparrow b_*$, then $C(b') \to \nu_1 = C(b)$, so the constant $C(b)$ cannot be replaced in (4.5) by a smaller one.

The case $b > 2 - b_*$. Take a Brownian motion $B = B^{-1}$ starting from -1 and repeat the argumentation from the case $b < b_*$, with b replaced by $2 - b$. Let $X = (B_{\tau_{2-b}\wedge t})$ and note that since $U_b([X,X]_0, X_0, X_0^*) = (2-b)X_0 + (b-1)(X_0 + X_0) = bX_0$, (4.9) becomes

$$b\mathbb{E}X_0 = ||X^*||_1 - C(2-b)||[X,X]^{1/2}||_1 = ||X^*||_1 - C(b)||[X,X]^{1/2}||_1.$$

This establishes the sharpness of (4.5). To show that the constant $C(b)$ is also optimal in the dyadic case, we repeat the arguments used in the proof of the sharpness of (1.4). One needs to consider a function $W : [0,\infty) \times \mathbb{R}^2 \to \mathbb{R}$ given by

$$W(x^2, y, z) = \sup \left\{ \mathbb{E}(f^* \vee z) - b\mathbb{E}f_0 - \gamma \mathbb{E}\sqrt{x^2 - y^2 + S^2(f)} \right\},$$

the supremum being taken over all simple dyadic martingales starting from y. Here γ is the optimal value of the constant $C(b)$ in (4.4). Then one shows the versions of the conditions 1°, 2° and 3°, and uses the convolution argument, in order to transfer the problem to the continuous-time setting. The details are omitted and left to the reader. □

Remark 4.3. *Let* $b < b_*$. *Comparing* (4.9) *and the equality*

$$||(B^1_{\tau_b})^*||_1 = \mathbb{E}B^1_0 - s_0(b)||\sqrt{1+\tau_b}||_1,$$

which is a direct consequence of (4.8)*, we derive that*

$$||\sqrt{1+\tau_b}||_1 = \frac{1}{\phi(s_0(b))}. \tag{4.11}$$

See also [22] *for related formulas.*

One can try to choose the optimal b in (4.4) and (4.5) to obtain the best upper bound for $||f^*||_1$ and $||X^*||_1$. This will be done below.

Corollary 4.4. *(i) Let f be a conditionally symmetric martingale.*

If $\mathbb{E}f_0 \geq 0$, *then the optimal choice of* b *in* (4.4) *is given by*

$$b = 1 + \left(\int_0^s \exp\left(\frac{u^2}{2}\right) du \right)^{-1}, \quad \textit{where } s \leq 0 \textit{ satisfies } \phi(s) = \frac{\mathbb{E}f_0}{||S(f)||_1}. \tag{4.12}$$

In particular, if $\mathbb{E}f_0 = 0$, *then*

$$||f^*||_1 \leq \nu_1 ||S(f)||_1.$$

If $\mathbb{E}f_0 < 0$, *then the optimal choice of* b *in* (4.4) *is given by*

$$b = 1 - \left(\int_0^s \exp\left(\frac{u^2}{2}\right) du \right)^{-1}, \quad \textit{where } s \leq 0 \textit{ satisfies } \phi(s) = -\frac{\mathbb{E}f_0}{||S(f)||_1}. \tag{4.13}$$

The obtained inequalities are sharp even in the dyadic case.

(ii) Let X be a martingale with no positive jumps.

If $\mathbb{E}X_0 \geq 0$, *then the optimal choice of* b *in* (4.4) *is given by*

$$b = 1 + \left(\int_0^s \exp\left(\frac{u^2}{2}\right) du \right)^{-1}, \quad \textit{where } s \leq 0 \textit{ satisfies } \phi(s) = \frac{\mathbb{E}X_0}{||[X,X]^{1/2}||_1}. \tag{4.14}$$

In particular, if $\mathbb{E}X_0 = 0$, *then*

$$||X^*||_1 \leq \nu_1 ||[X,X]^{1/2}||_1.$$

If $\mathbb{E}X_0 < 0$*, then the optimal choice of* b *in* (4.4) *is given by*

(4.15)
$$b = 1 - \left(\int_0^s \exp\left(\frac{u^2}{2}\right) du\right)^{-1}, \quad \textit{where } s \leq 0 \textit{ satisfies } \phi(s) = -\frac{\mathbb{E}X_0}{||[X,X]^{1/2}||_1}.$$

The obtained inequalities are sharp even for continuous-path martingales.

Proof. This follows from a straightforward analysis of the right-hand sides of (4.4) and (4.5). We will focus only on the continuous-time setting. The right-hand of (4.5), as a function of b, has derivative $\mathbb{E}X_0 + C'(b)\mathbb{E}[X,X]^{1/2}$; furthermore,

$$C'(b) = \begin{cases} \phi(s_0(b)) & \text{if } b < b_*, \\ 0 & \text{if } b \in [b_*, 2 - b_*], \\ -\phi(s_0(2-b)) & \text{if } b > 2 - b_*. \end{cases}$$

Therefore, if $\mathbb{E}X_0 \geq 0$, the right-hand side of (4.4) attatins its minimum for b satisfying $\phi(s_0(b)) = -\mathbb{E}X_0/\mathbb{E}[X,X]^{1/2}$; such b exists, since $-\mathbb{E}X_0/\mathbb{E}[X,X]^{1/2} \in [-1, 0]$, which is precisely the range of $b \mapsto \phi(s_0(b))$, $b \leq b_*$. Using (4.1), we obtain the formula (4.14) for b. Analogously, if $\mathbb{E}X_0 < 0$, then the optimal b is the solution to the equation $\phi(s_0(2-b)) = -\mathbb{E}X_0/\mathbb{E}[X,X]^{1/2}$, or, in view of (4.1), is given by (4.15).

The examples showing the optimality of the constants are those used in the sharpness of (4.5). Indeed, if $b < b_*$ (respectively, $b > 2 - b_*$), then we have equality in (4.5) and in (4.14) (respectively, in (4.5) and in (4.15)), in view of (4.11). The case $\mathbb{E}X_0 = 0$ is covered by (4.10). ☐

As an application of the results presented above, we will establish the following two facts.

(i) *An inequality for a stopped local time.* Suppose that B is a standard one-dimensional Brownian motion and let $L = (L_t)$ be the local time in 0 of B: that is, for any $t \geq 0$,

$$|B_t| = \int_0^t \mathbf{sgn}(B_s) dB_s + L_t.$$

It is well-known consequence of Skorokhod's lemma (see e.g. page 239 in Revuz and Yor [21]) that L can be written in the form $L_t = \sup_{s \leq t} D_s \vee 0 = \sup_{s \leq t} D_s$, where D is an adapted standard Brownian motion. Thus, by Theorem 1.2 and Corollary 4.4, we get the following result.

Theorem 4.5. *For any stopping time* τ *of* B*, we have*

$$||1 + L_\tau||_1 \leq \beta||\sqrt{1+\tau}||_1$$

and

$$||L_\tau||_1 \leq \nu_1 ||\tau^{1/2}||_1.$$

Both inequalities are sharp.

(ii) *An inequality for a stopped three-dimensional Bessel process.* Let $\rho = (\rho_t)$ be a three-dimensional Bessel process started at 0: for $t \geq 0$, $\rho_t = |B_t|$, where B is a Brownian motion in $\mathbb{R}^3$ such that $B_0 = 0$. By Pitman's theorem, we have $\rho_t = 2\sup_{s \leq t} D_s - D_t$, where D is an adapted standard one-dimensional Brownian motion; therefore, Theorem 1.2 and Corollary 4.4 yield the following.

Theorem 4.6. *For any stopping time τ of ρ,*

$$||2+\rho_\tau||_1 \leq 2\beta||\sqrt{1+\tau}||_1$$

and

$$||\rho_\tau||_1 \leq 2\nu_1||\tau^{1/2}||_1.$$

Both estimates are sharp.

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