SHARP $L^{p,\infty} \to L^q$ ESTIMATES FOR THE DYADIC-LIKE MAXIMAL OPERATORS

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ABSTRACT. For each $1 \le q < p < \infty$ we study the sharp versions of the $L^{p,\infty} \to L^q$ estimates for the dyadic maximal operator on $\mathbb{R}^n$. Actually, this is done in the more general setting of maximal operators associated with a tree-like structure. The proof rests on a novel combination of the Bellman function technique and optimization arguments.

1. Introduction

The motivation for the results of this paper comes from a natural question about sharp versions of certain inequalities for the dyadic maximal operator on $\mathbb{R}^n$. Recall that this operator is given by the formula

$$\mathcal{M}_d\phi(x) = \sup\left\{\frac{1}{|Q|}\int_Q |\phi(u)|\mathrm{d}u : x \in Q,\, Q \subset \mathbb{R}^n \text{ is a dyadic cube}\right\},$$

where ϕ is a locally integrable function on $\mathbb{R}^n$ and the dyadic cubes are those formed by the grids $2^{-N}\mathbb{Z}^n$, $N = 0, 1, 2, \ldots$. The maximal operator plays an important role in analysis and PDEs, and from the viewpoint of applications it is often of interest to have optimal, or at least good bounds for its norms. For example, $\mathcal{M}_d$ satisfies the weak-type $(1,1)$ inequality

$$\lambda\Big|\{x \in \mathbb{R}^n : \mathcal{M}_d\phi(x) \ge \lambda\}\Big| \le \int_{\{\mathcal{M}_d\phi \ge \lambda\}} |\phi(u)|\mathrm{d}u \tag{1.1}$$

for any $\phi \in L^1(\mathbb{R}^n)$ and any $\lambda > 0$. This bound is sharp: it is easy to construct an exemplary non-zero ϕ for which both sides are equal. Integrating the above estimate, we obtain the L^p estimate

$$||\mathcal{M}_d\phi||_{L^p(\mathbb{R}^n)} \le \frac{p}{p-1}||\phi||_{L^p(\mathbb{R}^n)}, \qquad 1 < p \le \infty, \tag{1.2}$$

in which the constant $p/(p-1)$ is also the best possible. These two statements are absolutely classical, and form a starting point for various extensions and numerous applications. It is impossible for us to review all these results here, and we will only mention some statements which are closely related to the subject of this paper. First, both (1.1) and (1.2) hold in the more general setting of maximal operators $\mathcal{M}_\mathcal{T}$ associated with tree-like structure $\mathcal{T}$. To define the necessary notions, assume that (X, μ) is a nonatomic probability space. Two measurable subsets A, B of X are said to be almost disjoint if $\mu(A \cap B) = 0$.

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Definition 1.1. A set $\mathcal{T}$ of measurable subsets of X will be called a tree if the following conditions are satisfied:

(i) $X \in \mathcal{T}$ and for every $I \in \mathcal{T}$ we have $\mu(I) > 0$.

(ii) For every $I \in \mathcal{T}$ there is a finite subset $C(I) \subset \mathcal{T}$ containing at least two elements such that

  (a) the elements of $C(I)$ are pairwise almost disjoint subsets of I,

  (b) $I = \bigcup C(I)$.

(iii) $T = \bigcup_{m \geq 0} \mathcal{T}^m$, where $\mathcal{T}^0 = \{X\}$ and $T^{m+1} = \bigcup_{I \in \mathcal{T}^m} C(I)$.

(iv) We have $\lim_{m\to\infty} \sup_{I \in \mathcal{T}^m} \mu(I) = 0$.

Any probability space equipped with a tree gives rise to the corresponding maximal operator $\mathcal{M}_{\mathcal{T}}$, given by

$$\mathcal{M}_{\mathcal{T}}\phi(x) = \sup\left\{ \frac{1}{\mu(I)} \int_I |\phi(u)| \mathrm{d}\mu(u) : x \in I, I \in \mathcal{T} \right\}.$$

Let us briefly describe the interplay between the tree setting and its dyadic counterpart. Observe that in the dyadic case, it is enough to study (1.1) and (1.2) for functions supported on $[0,1]^n$; the passage to general locally integrable ϕ's follows immediately from straightforward dilation arguments. Next, the class of dyadic cubes contained in $[0,1]^n$ forms a tree, and the associated maximal operator coincides with the dyadic maximal operator (restricted to the functions supported on $[0,1]^n$). Thus the setting of trees is indeed more general; it is also closely related to the theory of martingales (for the explanation, see [6]).

We turn our attention to other estimates for the operators $\mathcal{M}_{\mathcal{T}}$. It is well-known that if $p = 1$, then the inequality

$$||\mathcal{M}_{\mathcal{T}}\phi||_{L^p(X)} \leq C_p ||\phi||_{L^p(X)}$$

does not hold with any finite constant C_p, even in the dyadic case. This leads to the question about an appropriate substitute of this bound. Motivated by the classical results of Zygmund, Melas [7] proposed an answer to this question in terms of sharp LlogL-type estimates. The subsequent work [8] of Melas concerns another extension of (1.2): the action of $\mathcal{M}_{\mathcal{T}}$, considered as an operator from $L^p(X)$ to $L^q(X)$ (for $1 \leq q < p$), is studied there. Specifically, among other things, Melas determined the best constant $C_{p,q}$ in the following local inequality: for any $E \in \mathcal{T}$,

$$\left(\int_E (\mathcal{M}_{\mathcal{T}}\phi)^q \mathrm{d}\mu \right)^{1/q} \leq C_{p,q} \left(\int_X |\phi|^p \mathrm{d}\mu \right)^{1/p} \mu(E)^{1/q - 1/p}.$$

The paper [10] by Melas and Nikolidakis continues the research in this direction and treats the following sharp version of Kolmogorov's inequality: for any $0 < q < 1$ and any $E \in \mathcal{T}$,

$$\left(\int_E |\mathcal{M}_{\mathcal{T}}\phi|^q \mathrm{d}\mu \right)^{1/q} \leq \left(\frac{1}{1-q} \right)^{1/q} \left(\int_X |\phi| \mathrm{d}\mu \right) \mu(E)^{1/q-1}.$$

Finally, let us mention here three papers devoted to weak-type estimates. First, Melas and Nikolidakis [9] investigated various sharp extensions of the inequality

$$||\mathcal{M}_{\mathcal{T}}\phi||_{L^{p,\infty}(X)} \leq ||\phi||_{L^p(X)}, \qquad 1 \leq p < \infty,$$

where $||\phi||_{L^{p,\infty}(X)} = \sup_{\lambda>0} \lambda\left[\mu(\{x \in X : |\phi(x)| \geq \lambda\})\right]^{1/p}$ is the usual weak p-th norm. Next, the work [12] of Nikolidakis gives the sharp comparison of weak norms:

$$||\mathcal{M}_{\mathcal{T}}\phi||_{L^{p,\infty}(X)} \leq \frac{p}{p-1}||\phi||_{L^{p,\infty}(X)}, \qquad 1 < p < \infty.$$

Consult also the recent Nikolidakis' paper [13] for the further development in this direction.

We should point out here that in the works cited above, much more is proved. Namely, the papers actually contain the derivation of the so-called Bellman functions associated with the estimates. This provides much more information about the action of maximal operators on the corresponding spaces: for the necessary definitions and the explanation of this fact, see Section 2 below.

In this paper, we continue this line of research. We will be interested in the explicit formula for the norm of $\mathcal{M}_{\mathcal{T}}$ as an operator from $L^{p,\infty}(X)$ to $L^q(X)$, $1 \leq q < p < \infty$. One of our main results can be stated as follows.

Theorem 1.2. *Suppose that $1 \leq q < p < \infty$ are fixed parameters. Then for any locally μ-integrable function ϕ on X,*

$$||\mathcal{M}_{\mathcal{T}}\phi||_{L^q(X)} \leq \left(\frac{p}{p-q}\right)^{1/q} \frac{p}{p-1}||\phi||_{L^{p,\infty}(X)} \tag{1.3}$$

and the constant on the right-hand side is the best possible.

There is a probabilistic analogue of this result, which can be expressed in the language of martingales, and which follows from Theorem 1.2 by straightforward approximation. Though we will not go any further in this direction, we find the version worth stating as a separate theorem. For the necessary definitions and related results, we refer the reader to the classical monograph of Doob [5].

Theorem 1.3. *Suppose that $f = (f_n)_{n\geq 0}$ is a martingale on a certain probability space (with respect to its natural filtration). Then for any $1 \leq q < p < \infty$ and any $n \geq 0$ we have*

$$\left|\left|\sup_{0\leq k\leq n} |f_k|\right|\right|_q \leq \left(\frac{p}{p-q}\right)^{1/q} \frac{p}{p-1}||f_n||_{p,\infty}$$

and the constant on the right-hand side is the best possible.

Let us come back to Theorem 1.2. Actually, in analogy with the papers cited above, we will prove much more: we will identify the explicit formula for the Bellman function corresponding to (1.3). It should be pointed out that our proof will not be just a mere repetition of the arguments appearing in [6]-[13]. More specifically, the reasoning will be based on a novel unification of Monge-Ampére argument and combinatorial/optimization techniques. This approach has a lot flexibility and, as we hope, can be applied in other results of this type.

We have organized the paper as follows. The next section contains the description of Bellman function technique and explains the methodology which leads to our main results. In Section 3 we apply the method and, in particular, obtain the estimate (1.3). Section 4 is devoted to the sharpness: we construct appropriate extremal examples there. The final part of the paper presents some steps which have led us to the discovery of special functions of Section 3.

2. On the method of proof

There are several powerful techniques which can be used to obtain the estimates for maximal operators (covering theorems, Calderón-Zygmund-type decompositions, etc.). As we have already mentioned above, in this paper we will be particularly interested in the so-called Bellman function method. Roughly speaking, this approach relates the problem of proving a given inequality for the maximal operator to the existence of a special function which possesses certain concavity and majorization properties. This technique, if used appropriately, allows to determine optimal constants involved in the estimate under investigation. Furthermore, the special function often provides some additional insight into the structure and the behavior of the maximal operator. As an illustration, consider the following function, introduced by Nazarov and Treil in [11] during the study of (1.2) in the dyadic setting:

$$\begin{aligned}&\mathfrak{B}_p(f,F,L)\\&=\sup\left\{\frac{1}{|Q|}\int_Q(\mathcal{M}_d\phi)^p:\frac{1}{|Q|}\int_Q\phi=f,\ \frac{1}{|Q|}\int_Q\phi^p=F,\ \sup_{R:Q\subseteq R}\frac{1}{|R|}\int_R\phi=L\right\}.\end{aligned}$$

Here Q is a fixed dyadic cube, the variables f, F, L satisfy $0\leq f\leq L$, $f^p\leq F$ and the supremum is taken over all nonnegative functions $\phi\in L^p(Q)$ and all dyadic cubes R containing Q. Alternatively, the formula above can be rewritten as

$$\mathfrak{B}_p(f,F,L)=\sup\left\{\frac{1}{|Q|}\int_Q\max\{\mathcal{M}_d\phi,L\}^p:\frac{1}{|Q|}\int_Q\phi=f,\ \frac{1}{|Q|}\int_Q\phi^p=F\right\}.$$

Since $\max\{\mathcal{M}_d\phi,L\}\geq L$, the above definition implies

$$\mathfrak{B}_p(f,F,L)\geq L^p. \tag{2.1}$$

The interplay between $\mathfrak{B}_p$ and (1.2) is evident: once one proves the majorization

$$\mathfrak{B}_p(f,F,L)\leq\left(\frac{p}{p-1}\right)^pF, \tag{2.2}$$

the L^p-estimate follows. However, it is clear that the function $\mathfrak{B}_p$ codifies much more information on the action of $\mathcal{M}$ on L^p than (1.2), and thus it is of interest to derive its explicit formula. To study the structure of $\mathfrak{B}_p$, Nazarov and Treil established the so-called "main inequality", which describes the martingale-like dynamics of this function. Namely, for any $f_\pm$, $F_\pm$, L satisfying $f_\pm^p\leq F_\pm$, we have

$$\mathfrak{B}_p\left(\frac{f_-+f_+}{2},\frac{F_-+F_+}{2},L\right)\geq\frac{\mathfrak{B}_p(f_-,F_-,\max\{f_-,L\})+\mathfrak{B}_p(f_+,F_+,\max\{f_+,L\})}{2}.$$

A beautiful observation in [11] is that if one constructs *any function* which satisfies (2.1), (2.2) and the above concavity (in the literature, such an object is commonly called a supersolution), then (1.2) is established. Nazarov and Treil constructed such a function, thus reproving the maximal L^p-bound.

The discovery of the explicit formula for $\mathfrak{B}_p$ is due to Melas [6], actually in the above more general setting of trees. Namely, the function

$$\mathfrak{B}_p(f,F,L)=\sup\left\{\int_X\max\{\mathcal{M}\phi,L\}^p\mathrm{d}\mu:\int_X\phi\,\mathrm{d}\mu=f,\ \int_X\phi^p\mathrm{d}\mu=F\right\}$$

satisfies

$$\mathfrak{B}_p(f,F,L)=\begin{cases} Fw_p\left(\frac{pL^{p-1}f-(p-1)L^p}{F}\right)^p & \text{if } L<\frac{p}{p-1}f,\\ L^p+\left(\frac{p}{p-1}\right)^p(F-f^p) & \text{if } L\geq \frac{p}{p-1}f. \end{cases} \tag{2.3}$$

Here $w_p:[0,1]\to[1,p/(p-1)]$ is the inverse function of $z\mapsto -(p-1)z^p+pz^{p-1}$. Melas' approach is combinatorial in nature; the key step is to narrow down the class of functions among which the optimizers of the underlying extremal problem are found. Roughly speaking, in this line of reasoning one finds the Bellman function as the appropriate integral of the optimizer. This approach does not use the martingale dynamics of the problem (i.e., the "main inequality") and is specific to the discrete maximal operator. In particular, it does not directly apply to other dyadic operators, nor does it seem to work for other maximal functions. This technique should be contrasted with a general PDE- and geometry-based method first used by Slavin, Stokolos and Vasyunin [15]. There, the "main inequality" of Nazarov and Treil was turned into a Monge-Ampère PDE on a plane domain, whose solution turned out to be Melas' function. The optimizers were then built along the straight-line characteristics of the PDE. Recently, Monge-Ampère equations have found many applications in Bellman-function problems [1, 14, 15, 16, 17, 18, 19, 20]. Typically, they arise in settings with integral norms, such as L^p; and those where the main inequality can be interpreted as a convexity/concavity statement. However, one can also get a differential equation in other cases, as long as the main inequality is infinitesimally non-trivial. This approach has its roots in the works of Burkholder [2, 3, 4].

Coming back to the results of this paper, our main contribution is the identification of the explicit formula for the Bellman function

$$\mathfrak{B}_{p,q}(f,F,L)=\sup\left\{\int_X \max\{\mathcal{M}_{\mathcal{T}}\phi,L\}^q\,\mathrm{d}\mu:\int_X\phi\,\mathrm{d}\mu=f,\ ||\phi||^p_{L^{p,\infty}(X)}\leq F\right\},$$

with the methods developed in [6] and [15]. The natural domain of this function consists of all (f,F,L) satisfying $f\leq L$ and $f\leq\frac{p}{p-1}F^{1/p}$. Indeed, we have $\mathcal{M}_{\mathcal{T}}\phi\geq\int_X\phi\,\mathrm{d}\mu=f$, so there is no point in considering $L<f$ (the above formula gives $\mathfrak{B}_{p,q}(f,F,L)=\mathfrak{B}_{p,q}(f,F,f)$ for such L). Furthermore, integrating by parts we get

$$\begin{aligned} f=\int_0^\infty\mu(\phi\geq t)\mathrm{d}t&=\int_0^{||\phi||_{L^{p,\infty}(X)}}\mu(\phi\geq t)\mathrm{d}t+\int_{||\phi||_{L^{p,\infty}(X)}}^\infty\mu(\phi\geq t)\mathrm{d}t\\ &\leq ||\phi||_{L^{p,\infty}(X)}+\int_{||\phi||_{L^{p,\infty}(X)}}^\infty ||\phi||^p_{L^{p,\infty}(X)}t^{-p}\mathrm{d}t\\ &=\frac{p}{p-1}||\phi||_{L^{p,\infty}(X)}=\frac{p}{p-1}F^{1/p}, \end{aligned} \tag{2.4}$$

and actually, one easily shows that for $f\leq\frac{p}{p-1}F^{1/p}$, the class of all functions $\phi:X\to[0,\infty)$ satisfying $\int_X\phi\,\mathrm{d}\mu=f$, $||\phi||^p_{L^{p,\infty}(X)}=F$ is nonempty (see e.g. [9]).

How can we find the formula for $\mathfrak{B}_{p,q}$ (where p, q are fixed parameters as above)? At the first glance, the Monge-Ampére approach is not applicable, since the weak norms are not integral. In other words, the function $\mathfrak{B}_{p,q}$ is not governed by any version of "main inequality" of Nazarov and Treil. To overcome this difficulty, we propose the following novel two step procedure. Fix a function $\Phi:[0,\infty)\to[0,\infty)$

and consider

$$\mathfrak{B}_\Phi(f,F,L)=\sup\left\{\int_X \max\{\mathcal{M}\phi,L\}^q\,\mathrm{d}\mu:\int_X\phi\,\mathrm{d}\mu=f,\int_X\Phi(\phi)\,\mathrm{d}\mu\leq F\right\}. \tag{2.5}$$

Of course, the Monge-Ampére approach works here, at least if Φ is sufficiently regular (e.g., convex, nondecreasing and of class C^1). Observe that if Φ satisfies the additional condition

$$\int_X \Phi(\phi)\mathrm{d}\mu\leq ||\phi||^p_{L^{p,\infty}(X)} \tag{2.6}$$

for any $\phi\geq 0$, then we have $\mathfrak{B}_{p,q}\leq\mathfrak{B}_\Phi$. The key point is the following: if there is a function ϕ for which both sides of (2.6) are equal and which is an optimizer in (2.5), then in fact we have $\mathfrak{B}_{p,q}=\mathfrak{B}_\Phi$ and we are done. Thus, we have reduced the problem to that of finding an appropriate Φ (and then identifying the corresponding $\mathfrak{B}_\Phi$). As we shall see in Section 5 below, the first step of the above procedure exploits certain optimization arguments, while the second part involves the methods of [15].

Now it is high time to formulate the solution to the above problem. For any $1<q<p<\infty$, consider the function $B_{p,q}$, defined for all (f,F,L) satisfying $f\leq L$ and $f\leq\frac{p}{p-1}F^{1/p}$, given by the following formula: if $L\leq\left[F\left(\frac{p}{p-1}\right)^p/f\right]^{1/(p-1)}$, then

$$\begin{aligned}&B_{p,q}(f,F,L)\\&=\frac{(p-1)q}{(p-q)(q-1)}\left(F\left(\frac{p}{p-1}\right)^p\right)^{(q-1)/(p-1)}f^{(p-q)/(p-1)}+L^q-\frac{q}{q-1}fL^{q-1}.\end{aligned}$$

On the other hand, if $L>\left[F\left(\frac{p}{p-1}\right)^p/f\right]^{1/(p-1)}$, then let

$$B_{p,q}(f,F,L)=\frac{q}{p-q}\left(\frac{p}{p-1}\right)^pFL^{q-p}+L^q.$$

Furthermore, let $B_{p,1}(f,F,L)=\lim_{q\downarrow 1}B_{p,q}(f,F,L)$ for any $p>1$ and any (f,F,L) as above. That is,

$$B_{p,1}(f,F,L)=\frac{f}{p-1}\left\{1+\ln\left[F\left(\frac{p}{p-1}\right)^p/(fL^{p-1})\right]\right\}+L$$

if $L\leq\left[F\left(\frac{p}{p-1}\right)^p/f\right]^{1/(p-1)}$, and

$$B_{p,1}(f,F,L)=\frac{1}{p-1}\left(\frac{p}{p-1}\right)^pFL^{1-p}+L$$

otherwise. Here is the main result of this paper.

Theorem 2.1. *For $1\leq q<p<\infty$, the functions $\mathfrak{B}_{p,q}$ and $B_{p,q}$ coincide.*

We will establish this statement in the next two sections.

3. Proof of the inequality $\mathfrak{B}_{p,q} \leq B_{p,q}$

Let c be a fixed positive parameter and let $1 < q < p < \infty$. Define the function $\Phi_c : [0,\infty) \to [0,\infty)$ by

$$\Phi_c(x) = \begin{cases} 0 & \text{if } x \leq \frac{p-1}{p}c, \\ \frac{p}{q-1}\left(\frac{p}{p-1}\right)^{q-1} x^q - \frac{pq}{q-1}c^{q-1}x + (p-1)c^q & \text{if } x > \frac{p-1}{p}c. \end{cases} \tag{3.1}$$

It is easy to check that the function Φ_c is convex and of class C^1 on $(0,\infty)$. Next, introduce special functions $B_c : \{(x,y) : 0 \leq x \leq y\} \to \mathbb{R}$ by the formula

$$B_c(x,y) = \begin{cases} y^q + \frac{q}{q-1}(c^{q-1} - y^{q-1})x & \text{if } y < c, \\ y^q - \Phi_c(x) & \text{if } y \geq c,\ x \leq \frac{p-1}{p}y, \\ p\left(y^q - \frac{q}{q-1}y^{q-1}x + \frac{q}{q-1}c^{q-1}x\right) - (p-1)c^q & \text{if } y \geq c,\ x \geq \frac{p-1}{p}y. \end{cases} \tag{3.2}$$

It the lemma below, we study two properties of B_c which, in a sense, can be regarded as appropriate versions of (2.2) and the "main inequality".

Lemma 3.1. *(i) For any $c > 0$ and $0 \leq x \leq y$ we have the majorization*

$$B_c(x,y) \geq y^q - \Phi_c(x). \tag{3.3}$$

(ii) For any $0 \leq x \leq y$ and any $d \geq -x$ we have

$$B_c(x+d, y \vee (x+d)) \leq B_c(x,y) + B_{cx}(x,y)d. \tag{3.4}$$

Proof. (i) Fix $y \geq 0$ and consider the function $\xi(x) = B_c(x,y) - y^q + \Phi_c(x)$, $x \in [0,y]$. If $y < c$, then we have $\xi(0) = 0$ and the function ξ is increasing (since both $x \mapsto B_c(x,y)$ and $x \mapsto \Phi_c(x)$ have this property). This implies (3.3). On the other hand, if $y \geq c$, we easily verify that ξ is convex and vanishes, along with its derivative, at the point $x = (p-1)y/p$. This establishes the majorization.

(ii) The inequality is clear when $x + d \leq y$, since the function $x \mapsto B_c(x,y)$ is concave on $[0,y]$. Suppose then that $x + d > y$ and consider the function $\zeta(s) = B_c(s,s)$. We have $\zeta'(s) = B_{cx}(s,s) + B_{cy}(s,s) = B_{cx}(s,s)$, which combined with the aforementioned concavity of $x \mapsto B_c(x,y)$ gives

$$\begin{aligned} B_c(x,y) + B_{cx}(x,y)d &\geq B_c(y,y) + B_{cx}(x,y)(d-x) \\ &\geq B_c(y,y) + B_{cx}(y,y)(d-x) \\ &= \zeta(y) + \zeta'(y). \end{aligned}$$

Thus, we will be done if we show that ζ is concave. But this is evident: ζ is of class C^1 on $(0,\infty)$ and admits the formula

$$\zeta(y) = \begin{cases} -\frac{y^q}{q-1} + \frac{q}{q-1}c^{q-1}y & \text{if } y < c, \\ -\frac{p}{q-1}y^q + \frac{pq}{q-1}c^{q-1}y - (p-1)c^q & \text{if } y \geq c. \end{cases}$$

The proof is complete. $\square$

We are ready to establish the first half of Theorem 2.1.

Theorem 3.2. *For any $1 \leq q < p < \infty$ we have $\mathfrak{B}_{p,q} \leq B_{p,q}$.*

Proof. Clearly, it suffices to establish the inequality for q strictly larger than 1; the case $q=1$ follows immediately by passing to the limit. It is convenient to split the reasoning into two parts.

Step 1. First we will use B_c to establish the estimate

$$\int_X \max\{\mathcal{M}_{\mathcal{T}}\phi, L\}^q \mathrm{d}\mu \leq \int_X \Phi_c(\phi)\mathrm{d}\mu + \mathfrak{B}_c\left(\int_X \phi\,\mathrm{d}\mu, \max\left\{\int_X \phi\,\mathrm{d}\mu, L\right\}\right) \tag{3.5}$$

for any nonnegative $\phi \in L^q(X)$. To this end, fix such a ϕ and consider the associated sequence $(\phi_n)_{n\geq 0}$ of conditional expectations of ϕ with respect to $(\mathcal{T}^n)_{n\geq 0}$. That is, for any $x \in X$ and any nonnegative integer n, put

$$\phi_n(x) = \frac{1}{\mu(E)}\int_E \phi\,\mathrm{d}\mu, \tag{3.6}$$

where E is the element of $\mathcal{T}^n$ which contains x (since the elements of $\mathcal{T}^n$ are pairwise almost disjoint, such a set E is determined uniquely for μ-almost all x). We will also use the notation

$$\mathcal{M}_{\mathcal{T}^n}\phi(x) = \sup\left\{\frac{1}{\mu(I)}\int_I |\phi(u)|\mathrm{d}\mu(u) : x \in I, I \in \mathcal{T}^k \text{ for some } k \leq n\right\}.$$

Next, pick an integer $n \geq 0$, $E \in \mathcal{T}^n$ and let E_1, E_2, $\ldots$, E_m be the elements of $\mathcal{T}^{n+1}$ whose union is E. We will prove that

$$\int_E B_c\left(\phi_{n+1}, \max\{\mathcal{M}_{\mathcal{T}^{n+1}}\phi, L\}\right)\mathrm{d}\mu \leq \int_E B_c\left(\phi_n, \max\{\mathcal{M}_{\mathcal{T}^n}\phi, L\}\right)\mathrm{d}\mu. \tag{3.7}$$

To do this, note that both ϕ_n and $\max\{\mathcal{M}_{\mathcal{T}^n}\phi, L\}$ are constant on E: denote the values of these functions by x and y, respectively. On the other hand, we have the equality $\max\{\mathcal{M}_{\mathcal{T}^{n+1}}\phi, L\} = \max\{\mathcal{M}_{\mathcal{T}^n}\phi, L, \phi_{n+1}\}$ and the function ϕ_{n+1} is constant on E_1, E_2, $\ldots$, E_m. Letting $d_j = (\phi_{n+1}-\phi_n)|_{E_j} = \phi_{n+1}|_{E_j} - x$, it follows directly from (3.6) that

$$d_j \geq x \quad \text{and} \quad \sum_{j=1}^m |E_j| d_j = 0. \tag{3.8}$$

Now, apply (3.4) to x, y and $d = d_j$, and multiply both sides by $|E_j|$, $j = 1, 2, \ldots, m$. If we sum up the obtained inequalities, we get

$$\sum_{j=1}^m |E_j| B_c(\phi_{n+1}|_{E_j}, \max\{\mathcal{M}_{\mathcal{T}^{n+1}}\phi, L\}|_{E_j}) \leq |E| B_c(x,y),$$

which is precisely (3.7). Summing these estimates over all $E \in \mathcal{T}_n$, we get (3.7) and hence, by induction, we obtain

$$\int_X B_c\left(\phi_n, \max\{\mathcal{M}_{\mathcal{T}^n}\phi, L\}\right)\mathrm{d}\mu \leq \int_X B_c\left(\phi_0, \max\{\mathcal{M}_{\mathcal{T}^0}\phi, L\}\right)\mathrm{d}\mu.$$

However, we have $\phi_0 = \mathcal{M}_{\mathcal{T}^0}\phi = \int_X \phi\mathrm{d}\mu$ and hence the right hand side is equal to $B_c\left(\int_X \phi\mathrm{d}\mu, \max\left\{\int_X \phi\mathrm{d}\mu, L\right\}\right)$. To deal with the left-hand side, we make use of the majorization (3.3) and, as the result, obtain the bound

$$\begin{aligned}\int_X \max\{\mathcal{M}_{\mathcal{T}^n}\phi, L\}^q \mathrm{d}\mu &\leq \int_X \Phi_c(\phi_n)\,\mathrm{d}\mu + B_c\left(\int_X \phi\,\mathrm{d}\mu, \max\left\{\int_X \phi\,\mathrm{d}\mu, L\right\}\right)\\ &\leq \int_X \Phi_c(\phi)\,\mathrm{d}\mu + B_c\left(\int_X \phi\,\mathrm{d}\mu, \max\left\{\int_X \phi\,\mathrm{d}\mu, L\right\}\right).\end{aligned}$$

Here the latter estimate follows from Jensen inequality: Φ_c is a convex function. It remains to observe that if we let $n \to \infty$, then $\mathcal{M}_{\mathcal{T}^n}\phi$ increases to $\mathcal{M}_{\mathcal{T}}\phi$; therefore, (3.5) follows from Lebesgue's monotone convergence theorem.

Step 2. Now we turn to the bound $\mathfrak{B}_{p,q} \leq B_{p,q}$. Let $\phi \geq 0$ be a μ-integrable function satisfying $\int_X \phi \, \mathrm{d}\mu = f$ and $||\phi||^p_{L^{p,\infty}(X)} \leq F$, and let $L \geq f$ be a fixed number. We rewrite (3.5) in the form

$$\begin{aligned}\int_X \max\{\mathcal{M}_{\mathcal{T}}\phi, L\}^q \mathrm{d}\mu &\leq \int_0^\infty \Phi_c'(t)\mu(\phi \geq t)\mathrm{d}t + B_c\left(f, \max\{f, L\}\right) \\ &= \int_{(p-1)c/p}^\infty \frac{pq}{q-1}\left[\left(\frac{p}{p-1}\right)^{q-1} t^{q-1} - c^{q-1}\right]\mu(\phi \geq t)\mathrm{d}t \\ &\quad + B_c\left(f, L\right).\end{aligned}$$

The expression in the square brackets above is nonnegative when $t \geq (p-1)c/p$; furthermore, directly from the definition of weak norm, we have $\mu(\phi \geq t) \leq ||\phi||^p_{L^{p,\infty}(X)} t^{-p} \leq F t^{-p}$ for all $t \geq 0$. Thus,

$$\begin{aligned}\int_X \max\{\mathcal{M}_{\mathcal{T}}\phi, L\}^q \mathrm{d}\mu &\leq F \int_{(p-1)c/p}^\infty \frac{pq}{q-1}\left[\left(\frac{p}{p-1}\right)^{q-1} t^{q-1} - c^{q-1}\right] t^{-p}\mathrm{d}t \\ &\quad + B_c\left(f, L\right) \\ &= \frac{qc^{q-p}}{p-q}\left(\frac{p}{p-1}\right)^p F + B_c\left(f, L\right).\end{aligned}$$

Now it is time to specify c; we consider two cases. If $L > \left[F\left(\frac{p}{p-1}\right)^p / f\right]^{1/(p-1)}$, then we take $c = L$. Since $B_L(f, L) = L^q$, we obtain

$$\int_X \max\{\mathcal{M}_{\mathcal{T}}\phi, L\}^q \mathrm{d}\mu \leq \frac{qF}{p-q}\left(\frac{p}{p-1}\right)^p L^{q-p} + L^q,$$

as claimed. On the other hand, if $L \leq \left[F\left(\frac{p}{p-1}\right)^p / f\right]^{1/(p-1)}$, then we take c equal to the right-hand side of this estimate. We have

$$B_c(f, L) = L^q + \frac{q}{q-1} f\left(\left[F\left(\frac{p}{p-1}\right)^p / f\right]^{(q-1)/(p-1)} - L^{q-1}\right),$$

and hence, after some manipulations, we get the inequality

$$\begin{aligned}\int_X \max\{\mathcal{M}_{\mathcal{T}}\phi, L\}^q \mathrm{d}\mu \leq& \frac{(p-1)q}{(p-q)(q-1)}\left(F\left(\frac{p}{p-1}\right)^p\right)^{(q-1)/(p-1)} f^{(p-q)/(p-1)} \\ &+ L^q - \frac{q}{q-1} f L^{q-1}.\end{aligned}$$

This is precisely the desired bound. The proof is complete. $\square$

As an application, we will easily establish the inequality of Theorem 1.2, formulated in the introductory section.

Proof of (1.3). All we need is to show that

$$\mathfrak{B}_{p,q}(f, F, f) \leq \frac{p}{p-q}\left(\frac{p}{p-1}\right)^q F^{q/p}. \tag{3.9}$$

To do this, observe that $f \leq \left[F\left(\frac{p}{p-1}\right)^p/f\right]^{1/(p-1)}$ (see (2.4)). Therefore,

$$B_{p,q}(f,F,f) = \frac{(p-1)q}{(p-q)(q-1)}\left(F\left(\frac{p}{p-1}\right)^p\right)^{(q-1)/(p-1)} f^{(p-q)/(p-1)} - \frac{f^q}{q-1}.$$

Now we optimize over f. It is easy to see, using the direct differentiation, that the right-hand side above attains its maximal value for $f = pF^{1/p}/(p-1)$. Plugging this extremal f, we obtain (3.9). This finishes the proof. □

4. Proof of the inequality $\mathfrak{B}_{p,q} \geq B_{p,q}$

For the sake of clarity, we have decided to split this section into three parts. First, we deal with the construction of appropriate extremal functions. Then, in the next two subsections we present the calculations in the case $q > 1$ and $q = 1$, respectively.

4.1. **An example.** We start with the following lemma, which can be found in [6].

Lemma 4.1. *For every $I \in \mathcal{T}$ and every $\alpha \in (0,1)$ there is a subfamily $F(I) \subset \mathcal{T}$ consisting of pairwise almost disjoint subsets of I such that*

$$\mu\left(\bigcup_{J\in F(I)} J\right) = \sum_{J\in F(I)} \mu(J) = \alpha\mu(I).$$

Now, let f, F and L be fixed numbers satisfying $f \leq \frac{p}{p-1}F^{1/p}$ and $f \leq L$. In the boundary case $f = \frac{p}{p-1}F$, any function ϕ satisfying $\int_X \phi\,\mathrm{d}\mu = f$ and $||\phi||^p_{L^{p,\infty}(X)} \leq F$ has the distribution $\mu(\phi \geq \lambda) = \min\{F\lambda^{-p}, 1\}$ for all $\lambda > 0$ (see (2.4)). Now it is a matter of easy calculations to prove that $\mathfrak{B}_{p,q}(f,F,L) \geq B_{p,q}(f,F,L)$. Thus, in our further considerations, we assume that f is strictly smaller than $\frac{p}{p-1}F^{1/p}$. Fix $q \in [1,p)$ and let $r > p$ be a fixed parameter (eventually, we will let r go down to p). Define

$$c = \left[F\left(\frac{r}{r-1}\right)^p/f\right]^{1/(p-1)}. \tag{4.1}$$

The next step is to pick a large positive integer N and put $\delta = (c-f)/N$ (recall that $f < \frac{p}{p-1}F^{1/p}$, or $c > f$: hence $\delta > 0$). We have the following fact.

Lemma 4.2. *If δ is sufficiently small (that is, N is large enough), then for any $x \geq 0$ we have*

$$\left(1+\frac{x}{c}\right)^{p/r} \leq 1 + \frac{x}{c+r\delta}.$$

Proof. This is straightforward. The function $\xi(x) = (1+x/c)^{p/r} - 1 - x/(c+r\delta)$ satisfies $\xi(0) = 0$ and

$$\xi'(x) = \frac{p}{rc}\left(1+\frac{x}{c}\right)^{p/r-1} - \frac{1}{c+r\delta} \leq \frac{p}{rc} - \frac{1}{c+r\delta}$$

is negative provided δ is small enough. □

We are ready for the construction. By the inductive use of Lemma 4.1, there is a sequence $\{X\} = A_0 \supset A_1 \supset A_2 \supset \ldots$ which enjoys the following properties.

(i) For each k, A_k is a union of certain pairwise disjoint subsets from $\mathcal{T}$: we have $A_k = \bigcup F_k$ for some $F_k \subset \mathcal{T}$.

(ii) For each $k = 1, 2, \ldots, N$ and any $I \in F_{k-1}$ we have

$$\frac{\mu(A_k \cap I)}{\mu(I)} = \frac{f + (k-1)\delta}{f + k\delta}.$$

(iii) For each $k = N+1, N+2, \ldots$ and any $I \in F_{k-1}$ we have

$$\frac{\mu(A_k \cap I)}{\mu(I)} = \frac{f + (k-1)\delta}{f + (k-1)\delta + r\delta}.$$

Observe that in particular, (ii) and (iii) imply that

(ii') For each $k = 1, 2, \ldots, N$ we have

$$\frac{\mu(A_k)}{\mu(A_{k-1})} = \frac{f + (k-1)\delta}{f + k\delta}.$$

(iii') For each $k = N+1, N+2, \ldots$ and any $I \in F_k$ we have

$$\frac{\mu(A_k)}{\mu(A_{k-1})} = \frac{f + (k-1)\delta}{f + (k-1)\delta + r\delta}.$$

Introduce the function ϕ by

$$\phi = \begin{cases} 0 & \text{on } A_0 \setminus A_N, \\ (f + (k-1)\delta)(r-1)/r & \text{on } A_{k-1} \setminus A_k, \quad k = N+1, N+2, \ldots. \end{cases}$$

We will show below that $\mu(A_k) \to 0$ as $k \to \infty$. Thus we can treat ϕ as a well-defined function on X (as it is actually given on a subset of full measure).

Let us study the properties of the above objects.

Lemma 4.3. *(a) For any $k \geq 0$ we have*

$$\mu(A_{N+k}) \leq \left[(c + k\delta)\frac{r-1}{r} \right]^{-p} F. \tag{4.2}$$

(b) We have $||\phi||^p_{L^{p,\infty}(X)} \leq F$.

Proof. Write down (ii') for $k = 1, 2, \ldots, N$ and multiply the equations to get

$$\mu(A_N) = \frac{\mu(A_N)}{\mu(A_0)} = \frac{f}{f + N\delta} = \frac{f}{c} = \left(\frac{r-1}{r}c\right)^{-p} F.$$

Consequently, both sides of (4.2) are equal for $k = 0$. If $k \geq 1$, a similar use of (iii') gives

$$\begin{aligned}
\mu(A_{N+k}) &= \mu(A_N)\prod_{j=1}^{k}\frac{f+(N+j-1)\delta}{f+(N+j-1)\delta+r\delta}\\
&= \frac{f}{c}\prod_{j=1}^{k}\frac{c+(j-1)\delta}{c+(j-1)\delta+r\delta}\\
&= \frac{f}{c}\prod_{j=1}^{k}\left(1-\frac{r\delta}{c+(j-1)\delta+r\delta}\right) \qquad (4.3)\\
&\leq \frac{f}{c}\exp\left(-r\delta\sum_{j=1}^{k}\frac{1}{c+(j-1)\delta+r\delta}\right)\\
&\leq \frac{f}{c}\exp\left(-r\delta\int_{c+r\delta}^{c+k\delta+r\delta}\frac{1}{s}\mathrm{d}s\right) = \frac{f}{c}\left(1+\frac{k\delta}{c+r\delta}\right)^{-r}.
\end{aligned}$$

However, by Lemma 4.2, this does not exceed

$$\frac{f}{c}\left(1+\frac{k\delta}{c}\right)^{-p} = \left[(c+k\delta)\frac{r-1}{r}\right]^{-p}F.$$

(b) We must prove that $\mu(\phi \geq \lambda) \leq F\lambda^{-p}$ for all $\lambda > 0$. The function ϕ vanishes on $X \setminus A_N$ and is at least $c(r-1)/r$ on A_N. Therefore, using (a), we see that for $\lambda \leq c(r-1)/r$ we have

$$\mu(\phi \geq \lambda) = \mu(A_N) \leq \left(c\frac{r-1}{r}\right)^{-p}F \leq F\lambda^{-p}.$$

On the other hand, if $\lambda > c(r-1)/r$, then let k be the unique positive integer satisfying

$$\frac{(c+(k-1)\delta)(r-1)}{r} < \lambda \leq \frac{(c+k\delta)(r-1)}{r}.$$

Then $\phi < \lambda$ on $X \setminus A_{N+k}$ and $\phi \geq \lambda$ on A_{N+k}, so by (a),

$$\mu(\phi \geq \lambda) = \mu(A_{N+k}) \leq \left[(c+k\delta)\frac{r-1}{r}\right]^{-p}F \leq F\lambda^{-p}.$$

This completes the proof. □

Lemma 4.4. *For any $k \geq 0$ we have*

$$\frac{1}{\mu(A_k)}\int_{A_k}\phi\, d\mu = f + k\delta.$$

In particular, $\int_X \phi\, d\mu = f$.

Remark 4.5. This implies a slightly stronger statement. Let $k \geq 0$ and let F_k be the subset of $\mathcal{T}$ whose union is A_k. Then for any $I \in F_k$ we have

$$\frac{1}{\mu(I)}\int_I \phi\, \mathrm{d}\mu = f + k\delta.$$

Indeed, by (ii), (iii) and the definition of ϕ, the conditional distribution of ϕ is the same on each $I \in F_k$ (i.e., $\mu(\{x \in I : \phi(x) \geq \lambda\})/\mu(I)$ does not depend on I, but only on the "level" k to which I belongs).

Proof of Lemma 4.4. It is enough to show that the sequence $\alpha_k = \int_{A_k} \phi \,\mathrm{d}\mu - \mu(A_k)(f + k\delta)$, $k = 0, 1, 2, \ldots$ is constant (indeed, by the preceding lemma, this sequence converges to 0). Now, if $0 \leq k \leq N$, we have $\int_{A_k} \phi \,\mathrm{d}\mu = \int_{A_N} \phi \,\mathrm{d}\mu$, directly from the definition of ϕ; furthermore, by (ii'),

$$\mu(A_k)(f + k\delta) = \mu(A_{k+1})(f + (k+1)\delta) = \ldots = \mu(A_N)(f + N\delta)$$

and hence $\alpha_k = \alpha_N$. Now, if $k \geq N + 1$, then

$$\begin{aligned}\alpha_k - \alpha_{k-1} &= -\int_{A_{k-1}\setminus A_k} \phi \,\mathrm{d}\mu - \mu(A_k)(f + k\delta) + \mu(A_{k-1})(f + (k-1)\delta) \\ &= \frac{1}{r}\Big[\mu(A_{k-1})(f + (k-1)\delta) - \mu(A_k)(f + (k-1)\delta + \delta r)\Big] = 0,\end{aligned}$$

where the last equality is due to (iii'). □

The final observation of this subsection concerns the function $\mathcal{M}_{\mathcal{T}}\phi$ and follows directly from Remark 4.5. Namely, for any $k \geq 0$ we have

$$\mathcal{M}_{\mathcal{T}}\phi \geq f + k\delta \qquad \text{on } A_k. \tag{4.4}$$

Equipped with the above facts, we are ready to prove the bound $\mathfrak{B}_{p,q} \geq B_{p,q}$.

4.2. **The case $q > 1$.** Assume first that

$$L > \left[F\left(\frac{p}{p-1}\right)^p / f\right]^{1/(p-1)}.$$

Then, for r sufficiently close to p we also have $L > c$ and hence, if M denotes the unique positive integer satisfying $f + (M-1)\delta < L \leq f + M\delta$, then $M > N$. Using (4.4), we write

$$\int_X \max\{\mathcal{M}_{\mathcal{T}}\phi, L\}^q \mathrm{d}\mu \geq L^q(1 - \mu(A_M)) + \sum_{k=M}^{\infty} (f + k\delta)^q \mu(A_k \setminus A_{k+1}).$$

Now we will let $\delta \to 0$. Repeating the reasoning from (4.3), we see that for $k \geq N$,

$$\mu(A_k) = \frac{f}{c}\left(1 + \frac{k\delta}{c}\right)^{-r} + o(\delta).$$

Since $M - N$ is of order $(L - c)/\delta$, we get $\mu(A_M) = fc^{r-1}L^{-r} + o(\delta)$ and

$$\lim_{\delta \to 0} L^q(1 - \mu(A_M)) = L^q - fc^{r-1}L^{q-r}.$$

Furthermore, by (iii'), if $k \geq N$, then

$$\mu(A_k \setminus A_{k+1}) = \frac{r\delta\mu(A_{k+1})}{f + k\delta},$$

and therefore

$$\begin{aligned}\lim_{\delta\to 0}\sum_{k=M}^{\infty}(f+k\delta)^q\mu(A_k\setminus A_{k+1}) &= \lim_{\delta\to 0} r\delta\sum_{k=M}^{\infty}(f+k\delta)^{q-1}\mu(A_{k+1})\\ &= r\int_L^{\infty} s^{q-1}\cdot\frac{f}{c}\left(\frac{s}{c}\right)^{-r}\mathrm{d}s\\ &= \frac{frc^{r-1}L^{q-r}}{r-q}.\end{aligned}$$

Combining the above two facts and the definition (4.1) of c, we obtain

$$\mathfrak{B}_{p,q}(f,F,L)\geq L^q+\frac{qfL^{q-r}}{r-q}\left[F\left(\frac{r}{r-1}\right)^p/f\right]^{(r-1)/(p-1)}.$$

However, if $r\downarrow p$, then the right-hand side above converges to $B_{p,q}(f,F,L)$. This gives the desired lower bound.

Now we turn our attention to the case when

$$L<\left[F\left(\frac{p}{p-1}\right)^p/f\right]^{1/(p-1)}.$$

We define M in the same way as previously: as the unique integer satisfying $f+(M-1)\delta<L\leq f+M\delta$. Then $M\leq N$ for sufficiently small δ. We have

$$\int_X \max\{\mathcal{M}_{\mathcal{T}}\phi,L\}^q\mathrm{d}\mu\geq L^q(1-\mu(A_M))+\sum_{k=M}^{\infty}(f+k\delta)^q\mu(A_k\setminus A_{k+1}). \tag{4.5}$$

By (ii'), we have $\mu(A_k)=f/(f+k\delta)$ for $k\leq N$, and therefore

$$\lim_{\delta\to 0}L^q(1-\mu(A_M))=L^q-L^{q-1}f. \tag{4.6}$$

Furthermore,

$$\begin{aligned}\lim_{\delta\to 0}\sum_{k=M}^{N}(f+k\delta)^q\mu(A_k\setminus A_{k+1}) &= \lim_{\delta\to 0} f\delta\sum_{k=M}^{N}(f+k\delta)^{q-1}(f+(k+1)\delta)^{-1}\\ &= f\int_L^c s^{q-2}\mathrm{d}s=\frac{f(c^{q-1}-L^{q-1})}{q-1}\end{aligned} \tag{4.7}$$

and, repeating the calculations from the preceding case,

$$\lim_{\delta\to 0}\sum_{k=N}^{\infty}(f+k\delta)^q\mu(A_k\setminus A_{k+1})=\frac{frc^{q-1}}{r-q}. \tag{4.8}$$

Combining all the above facts and the equation (4.1), we get

$$\mathfrak{B}_{p,q}(f,F,L)\geq L^q-\frac{qfL^{q-1}}{q-1}+\frac{(p-1)qf}{(p-q)(q-1)}\left[F\left(\frac{r}{r-1}\right)^p/f\right]^{(q-1)/(p-1)}.$$

It suffices to let $r\downarrow p$ to get the desired bound $\mathfrak{B}_{p,q}(f,F,L)\geq B_{p,q}(f,F,L)$.

Finally, we deal with the case

$$L=\left[F\left(\frac{p}{p-1}\right)^p/f\right]^{1/(p-1)}.$$

Then $f < L$, since, as we have assumed, $f \leq \frac{p}{p-1}F^{1/p}$. Pick an arbitrary $L' \in (f, L)$. Then by the very definition of $\mathfrak{B}_{p,q}$ and the above reasoning,

$$\mathfrak{B}_{p,q}(f,F,L) \geq \mathfrak{B}_{p,q}(f,F,L') \geq B_{p,q}(f,F,L').$$

It remains to note that the function $B_{p,q}$ is continuous. Thus letting $L' \uparrow L$ yields the claim.

4.3. **The case** $q = 1$**.** Essentially, all the calculations in this case are the same as previously. Indeed, for

$$L > \left[F\left(\frac{p}{p-1}\right)^p/f\right]^{1/(p-1)}$$

we repeat the reasoning word-by-word and obtain

$$\mathfrak{B}_{p,1}(f,F,L) \geq L + \frac{fL^{1-r}}{r-1}\left[F\left(\frac{r}{r-1}\right)^p/f\right]^{(r-1)/(p-1)},$$

and letting $r \downarrow p$ gives $\mathfrak{B}_{p,q}(f,F,L) \geq B_{p,q}(f,F,L)$. If

$$L < \left[F\left(\frac{p}{p-1}\right)^p/f\right]^{1/(p-1)},$$

then we write down (4.5), (4.6) and (4.8) with $q = 1$; furthermore, letting $q \to 1$ in (4.7) gives

$$\lim_{\delta\to 0}\sum_{k=M}^{N}(f+k\delta)^q\mu(A_k\setminus A_{k+1}) = f\ln(c/L).$$

Thus we obtain

$$\mathfrak{B}_{p,1}(f,F,L) \geq L - f + f\ln(c/L) + \frac{fr}{r-1} = L + \frac{f}{r-1} + f\ln(c/L).$$

It remains to use (4.1) and let $r \downarrow p$ to obtain $\mathfrak{B}_{p,1}(f,F,L) \geq B_{p,1}(f,F,L)$.

5. On the search for special parameters

In the final part of the paper, we sketch some steps which have led us to the discovery of the functions Φ_c and B_c defined in (3.1) and (3.2). We would like to stress that the reasoning presented below is informal and serves as intuitive guideline in the search for these special objects. In particular, we will feel free to impose some additional assumptions on Φ_c and B_c (e.g., smoothness), which greatly help with the search but do not follow directly from the definitions.

As we have explained in Section 2, we are interested in the Bellman function

$$\mathfrak{B}_\Phi(f,F,L) = \sup\left\{\int_X \max\{\mathcal{M}_\mathcal{T}\phi, L\}^q\,\mathrm{d}\mu : \int_X \phi\,\mathrm{d}\mu = f, \int_X \Phi(\phi)\,\mathrm{d}\mu \leq F\right\} \tag{5.1}$$

for an appropriate Φ (see below). To study this object, let us study a related, simpler function of two variables:

$$\mathfrak{B}(f,L) = \sup\left\{\int_X \max\{\mathcal{M}_\mathcal{T}\phi, L\}^q - \Phi(\phi)\,\mathrm{d}\mu\ :\ \int_X \phi\mathrm{d}\mu = f\right\}, \tag{5.2}$$

where $0 \leq f \leq L$. To put it into a general framework, we rewrite the definition of $\mathfrak{B}$ in the form

$$\mathfrak{B}(f,L) = \sup\left\{\int_X G\big(\phi, \max\{\mathcal{M}_\mathcal{T}\phi, L\}\big)\,\mathrm{d}\mu\ :\ \int_X \phi\mathrm{d}\mu = f\right\},$$

where the payoff function G is given by $G(f,L)=L^q-\Phi(f)$.

The reasoning splits naturally into two parts.

5.1. **Looking for optimizers ϕ.** As we have explained in Section 2, we search for $\Phi\geq 0$ satisfying

$$\int_X \Phi(\phi)\mathrm{d}\mu \leq ||\phi||^p_{L^{p,\infty}(X)} \qquad \text{for all } \phi\geq 0 \tag{5.3}$$

and the following further condition: there is a function $\phi\geq 0$ which realizes equality in (5.2) and for which both sides of (5.3) are equal to F. Clearly, such a function is also an optimizer in (5.1) and $\mathfrak{B}_\Phi(f,F,L)=\mathfrak{B}(f,L)+F$.

First take a look at (5.3). If Φ is continuous and nondecreasing, we rewrite this condition in the form

$$\int_0^\infty \Phi'(\lambda)\mu(\phi\geq\lambda)\mathrm{d}\lambda \leq ||\phi||^p_{L^{p,\infty}(X)}, \qquad \text{for all } \phi\geq 0. \tag{5.4}$$

Let us restrict ourselves to those ϕ, which have fixed weak norm: $||\phi||^p_{L^{p,\infty}(X)}=F$. Then, at the first glance, there is only one extremal function ϕ in the above inequality (i.e., there is only one function for which both sides are equal). Indeed, if $\Phi'>0$ on $(0,\infty)$, then the choice $\mu(\phi\geq\lambda)=\min\{F\lambda^{-p},1\}$ maximizes the left-hand side, and the use of any other distribution gives a smaller quantity on the left. But this class of optimizers is not sufficient for our purposes, since such a ϕ satisfies

$$f=\int_X\phi=\frac{p}{p-1}||\phi||_{L^{p,\infty}(X)}=\frac{p}{p-1}F^{1/p}.$$

In particular, this class does not contain elements for f, F satisfying $f<\frac{p}{p-1}F^{1/p}$. To overcome this difficulty, we allow Φ to be zero on a certain interval $[0,d]$ (and assume that Φ is a *convex* increasing function on (d,∞)). This enlarges the class of optimizers, since any function which satisfies $\mu(\phi\geq\lambda)\leq\min\{F\lambda^{-p},1\}$ for $\lambda\geq d$ gives equality in (5.4) (there are no restrictions on the behavior of $\mu(\phi\geq\lambda)$ for $\lambda<d$, except for those coming from the weak norm condition). Actually, a little thought and experimentation suggests considering ϕ's satisfying $\mu(\phi\geq\lambda)=\mu(\phi\geq d)$ for $\lambda\in(0,d)$: in other words, is seems plausible to work with those ϕ, which do not take values in the set $(0,d)$. It turns out that this class of functions is large enough to satisfy the following: for any f and F, there is ϕ such that $\int_X\phi\,\mathrm{d}\mu=f$ and $||\phi||^p_{L^{p,\infty}(X)}=F$. Namely, the distribution of such ϕ is given by

$$\mu(\phi\geq\lambda)=\begin{cases} Fd^{-p} & \text{if } 0<\lambda<d,\\ F\lambda^{-p} & \text{if } \lambda\geq d,\end{cases} \tag{5.5}$$

where $d=\big[pF/((p-1)f)\big]^{1/(p-1)}$.

5.2. **Construction of $\mathfrak{B}$ and Φ.** Now we will use some abstract facts concerning the function $\mathfrak{B}$ defined in (5.2). For the detailed explanation, we refer the reader to [11]; a very interesting discussion about the martingale interpretation of the subject can be found in Burkholder's papers [3] and [4]. From now on, we assume that $X=[0,1]$, μ is the Lebesgue measure and the tree $\mathcal{T}$ consists of dyadic intervals of X (the functions we will obtain work for other probability spaces and trees as well). The key fact is the following.

Theorem 5.1. *The function $\mathfrak{B}$ is the least function satisfying the conditions*

1° *We have* $B(f,L) \geq L^q - \Phi(f)$ *for all* $f \leq L$.

2° *For any* $f \leq L$ *and any* $f_\pm \geq 0$ *such that* $(f_- + f_+)/2 = f$, *we have*

$$\mathfrak{B}(f,L) \geq \Big[\mathfrak{B}\big(f_-, \max\{f_-, L\}\big) + \mathfrak{B}\big(f_+, \max\{f_+, L\}\big)\Big]/2.$$

We see that the conditions 1° and 2° are just the appropriate versions of (2.1) and the "main inequality" of Nazarov and Treil. From now on, we will work under additional assumption that $\mathfrak{B}$ is of class C^2. The infinitesimal version of 2° reads

$$\mathfrak{B}_{xx}(f,L) \leq 0 \qquad \text{and} \qquad \mathfrak{B}_y(f,f) \leq 0. \tag{5.6}$$

Actually, it is not difficult to gather some intuition about $\mathfrak{B}$. Since for any fixed L the function $f \mapsto L^q - \Phi(f)$ is concave (we have assumed above that Φ is convex), the first idea is just to put $\mathfrak{B}(f,L) = L^q - \Phi(f)$. However, this function does not work: the second condition in (5.6) is violated. Therefore, $\mathfrak{B}$ must be raised a little bit. But how much? A standard step in the search (cf. [15]) is to assume that in (5.6) both sides are equal, at least at a large part of the domain. Roughly speaking, this reflects the general phenomenon that a Bellman function should realize equality in the underlying PDEs on some open subset C of the domain, and on the compliment of C, it should be equal to the corresponding payoff function G. Keeping this fact in mind and working a little bit with the geometry of $\mathfrak{B}$, we come upon the following idea. For each L there should be a "threshold" $\gamma(L) \in [0,L]$ such that

$$\mathfrak{B}(f,L) = \begin{cases} L^q - \Phi(f) & \text{for } f \in [0, \gamma(L)], \\ a_L f + b_L & \text{for } f \in [\gamma(L), L], \end{cases} \tag{5.7}$$

where a_L, b_L are some coefficients. If $\gamma(L) > 0$, then a_L and b_L can be identified from the smoothness property of $\mathfrak{B}$: $a_L = \Phi'(\gamma(L))$ and $b_L = L^q - \Phi(\gamma(L)) - \Phi'(\gamma(L))\gamma(L)$. On the other hand, if $\gamma(L) = 0$, then we must necessarily have $b_L = L^q - \Phi(\gamma(L)) = L^q$ and $a_L \geq 0$: the latter inequality follows from the assumption $\Phi(t) = 0$ for $t \in (0,d)$. This condition on Φ implies further that the function $\gamma(L)$ can be assumed to take values in $\{0\} \cup [d,\infty)$. Indeed, if $\gamma(L) \in (0,d)$, then automatically $\mathfrak{B}(f,L) = L^q$ for all $f \in [0,L]$; hence we may change the value of $\gamma(L)$ to 0, and this will produce the same formula. Finally, a little experimentation suggests the existence of $L_0 \geq 0$ such that $\gamma(L) = 0$ for $L < L_0$ and $\gamma(L) \geq d$ for $L \geq L_0$.

To determine γ (and a_L and b_L for $L < L_0$), we take a look at the second condition in (5.6) and assume equality there. Combining this additional condition with (5.7) and the above facts implies the following: for $L < L_0$ we have $\alpha_L = -\frac{q}{q-1}L^{q-1} + \beta$ for some constant β; furthermore, for $L > L_0$, the threshold $\gamma(L)$ satisfies the differential equation

$$\Phi''(\gamma(L))\gamma'(L) = qL^{q-2}.$$

Thus we obtain that

$$\mathfrak{B}(f,L) = \begin{cases} L^q - \frac{q}{q-1}L^{q-1}f + \beta f & \text{if } L < L_0, \\ L^q - \Phi(f) & \text{if } L \geq L_0,\ f \leq \gamma(L), \\ L^q - \Phi(\gamma(L)) + \Phi'(\gamma(L))(f - \gamma(L)) & \text{if } L \geq L_0,\ f \geq \gamma(L). \end{cases}$$

Since $\mathfrak{B}$ is of class C^1, a look at the partial derivative $\mathfrak{B}_x(0,L)$ for $L < L_0$ and $L > L_0$ gives $\beta = \frac{q}{q-1}L_0^{q-1}$.

The final part of the analysis is to determine the function Φ. To achieve this, we take a look at the above explicit formula, and construct optimizers corresponding to $\mathfrak{B}(f, L)$. As explained in [15], these extremal functions can be built along the straight-line characteristics of the PDEs (5.6). Some tedious and lengthy calculations allow to express the distribution of such ϕ's in terms of Φ; on the other hand, these optimizers have distribution as in (5.5). We will not present the calculations here, and only say that they lead to $\gamma(L) = (p-1)L/p$, $L_0 = pd/(p-1)$ and the differential equation

$$\Phi''(s) = pq \left(\frac{p}{p-1}\right)^{q-1} s^{q-2}, \qquad s > d.$$

Solving this equation and using the fact that $\mathfrak{B}$ is smooth, we obtain that

$$\Phi(s) = \frac{p}{q-1}\left(\frac{p}{p-1}\right)^{q-1} s^q - \frac{pq}{q-1}\left(\frac{p}{p-1}\right)^{q-1} d^{q-1}s + p\left(\frac{p}{p-1}\right)^{q-1} d^q$$

for $s \geq d$. Thus, we get the functions Φ, $\mathfrak{B}$ which coincide with those given by (3.1) and (3.2), with $c = pd/(p-1)$.

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