A SPLITTING PROCEDURE FOR BELLMAN FUNCTIONS AND THE ACTION OF DYADIC MAXIMAL OPERATORS ON L^p

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Abstract. The purpose of the paper is to introduce a novel "splitting" procedure which can be helpful in the derivation of explicit formulas for various Bellman functions. As an illustration, we study the action of the dyadic maximal operator on L^p. The associated Bellman function $\mathfrak{B}_p$, introduced by Nazarov and Treil, was found explicitly by Melas with the use of combinatorial properties of the maximal operator, and was later re-discovered by Slavin, Stokolos and Vasyunin with the use of the corresponding Monge–Ampère PDE. Our new argument enables an alternative simple derivation of $\mathfrak{B}_p$.

1. Introduction

Bellman function method is a powerful tool in proving various types of inequalities of harmonic analysis. The technique has its origins in the theory of stochastic optimal control, and its fruitful connection with other areas of mathematics was firstly observed by Burkholder in [1], during the study of certain sharp inequalities for martingale transforms. The method has developed rapidly after the appearance of the fundamental paper [9] by Nazarov and Treil (inspired by the preprint version of [10]) and has been applied in various settings: see e.g. [2, 3, 4, 5, 6, 7, 8, 11, 16, 17, 18, 19, 20, 21, 22] and references therein.

Roughly speaking, the technique relates the validity of a given inequality to the existence of a certain special function, which possesses appropriate majorization and concavity-type properties. Actually, this object carries all the information on the underlying estimate: sharp constants, the extremal functions or sequences, and in many cases it provides a further insight into the structure of the problem. However, the discovery of the Bellman function is in general a very difficult task, and the methods leading to the solution involve the exploitation of combinatorial aspects of the problem or the analysis of complicated intrinsic Monge–Ampère PDE's, and any novel tool here is of considerable interest. One of the principal goals of this paper is to introduce a general procedure which simplifies the technicalities arising in the study of such problems. More precisely, we will show how to split the search of a given Bellman function into two parts. The first step is to consider *a family* of simpler, less dimensional Bellman functions, while the second involves an optimization argument which yields the desired object.

As it is rather impossible to put the aforementioned procedure into an abstract framework, we have decided to describe the approach by working on a specific example, associated with the action of the dyadic maximal operator on $L^p(\mathbb{R}^n)$,

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$1 < p < \infty$. Recall that this operator is given by the formula

$$\mathcal{M}_d\phi(x) = \sup\left\{\frac{1}{|Q|}\int_Q |\phi(u)|\mathrm{d}u : x \in Q,\, Q \subset \mathbb{R}^n \text{ is a dyadic cube}\right\},$$

where ϕ is a locally integrable function on $\mathbb{R}^n$ and the dyadic cubes are those formed by the grids $2^{-N}\mathbb{Z}^n$, $N = 0, 1, 2, \ldots$. Following Nazarov and Treil [9], define the associated Bellman function

$$\begin{aligned}&\mathfrak{B}_p(f, F, L)\\ &= \sup\left\{\frac{1}{|Q|}\int_Q (\mathcal{M}_d\phi)^p : \frac{1}{|Q|}\int_Q \phi = f,\, \frac{1}{|Q|}\int_Q \phi^p = F,\, \sup_{R: Q\subseteq R}\frac{1}{|R|}\int_R \phi = L\right\}.\end{aligned}$$

Here Q is a fixed dyadic cube, the variables f, F, L satisfy $0 < f \leq L$, $f^p \leq F$ and the supremum is taken over all nonnegative functions $\phi \in L^p(Q)$ and all dyadic cubes R containing Q. Alternatively, the formula above can be rewritten as

$$\mathfrak{B}_p(f, F, L) = \sup\left\{\frac{1}{|Q|}\int_Q \max\{\mathcal{M}_d\phi, L\}^p : \frac{1}{|Q|}\int_Q \phi = f,\, \frac{1}{|Q|}\int_Q \phi^p = F\right\}.$$

By a standard dilation argument, we see that $\mathfrak{B}_p$ is independent of Q. The connection between this function and the L^p-boundedness of $\mathcal{M}_d$ is evident: the identification of the explicit formula for $\mathfrak{B}_p$ provides a sharp refinement of the Hardy-Littlewood-Doob maximal inequality

$$||\mathcal{M}_d\phi||_p \leq \frac{p}{p-1}||\phi||_p. \tag{1.1}$$

It is shown in [9] that $\mathfrak{B}_p(f, F, L) \leq q^pF - pqfL^{p-1} + pL^p$, which implies (1.1). The discovery of the explicit formula for $\mathfrak{B}_p$ is due to Melas: the proof in [4] exploits deep combinatorial properties of the operator $\mathcal{M}_d$. An alternative approach, based on the solution of the underlying Monge–Ampère PDE, can be found in the paper [16] by Slavin, Stokolos and Vasyunin. Actually, Melas works in the more general setting of maximal operators $\mathcal{M}_\mathcal{T}$ associated with tree-like structure $\mathcal{T}$. To introduce the necessary notions, assume that (X, μ) is a nonatomic probability space. Two measurable subsets A, B of X are said to be almost disjoint if $\mu(A \cap B) = 0$.

Definition 1.1. A set $\mathcal{T}$ of measurable subsets of X will be called a tree if the following conditions are satisfied:

- (i) $X \in \mathcal{T}$ and for every $I \in \mathcal{T}$ we have $\mu(I) > 0$.
- (ii) For every $I \in \mathcal{T}$ there is a finite subset $C(I) \subset \mathcal{T}$ containing at least two elements such that
  - (a) the elements of $C(I)$ are pairwise almost disjoint subsets of I,
  - (b) $I = \bigcup C(I)$.
- (iii) $T = \bigcup_{m\geq 0}\mathcal{T}^m$, where $\mathcal{T}^0 = \{X\}$ and $T^{m+1} = \bigcup_{I\in\mathcal{T}^m} C(I)$.
- (iv) We have $\lim_{m\to\infty}\sup_{I\in\mathcal{T}^m}\mu(I) = 0$.

Any probability space equipped with a tree gives rise to the corresponding maximal operator $\mathcal{M}_\mathcal{T}$, given by

$$\mathcal{M}_\mathcal{T}\phi(x) = \sup\left\{\frac{1}{\mu(I)}\int_I |\phi(u)|\mathrm{d}\mu(u) : x \in I, I \in \mathcal{T}\right\}.$$

Now one extends the definition of $\mathcal{B}_p$ as follows:

$$\mathfrak{B}_p(f,F,L)=\sup\left\{\int_X \max\{\mathcal{M}_\mathcal{T}\phi,L\}^p\mathrm{d}\mu:\int_X\phi\,\mathrm{d}\mu=f,\ \int_X\phi^p\mathrm{d}\mu=F\right\},$$

defined, as previously, for f,F,L satisfying $0<f\leq L$ and $f^p\leq F$. It is clear that this generalizes the dyadic setting studied previously: simply take X to be equal to a certain dyadic cube Q, let μ denote the normalized Lebesgue's measure and take $\mathcal{T}$ to be the class of all dyadic sub-cubes of Q.

One of the main results of [4] asserts that

$$\mathfrak{B}_p(f,F,L)=\begin{cases} Fw_p\left(\frac{pL^{p-1}f-(p-1)L^p}{F}\right)^p & \text{if } L<\frac{p}{p-1}f,\\ L^p+\left(\frac{p}{p-1}\right)^p(F-f^p) & \text{if } L\geq\frac{p}{p-1}f,\end{cases} \tag{1.2}$$

where $w_p:[0,1]\to[1,p/(p-1)]$ is the inverse to $H_p(z)=-(p-1)z^p+pz^{p-1}$. This formula does not depend on $\mathcal{T}$, so in particular it holds true in the above dyadic setting as well.

We will show how the aforementioned splitting argument leads to a yet another proof of (1.2). The idea is as follows: instead of studying the complicated, three-dimensional $\mathfrak{B}_p$, we will search for a certain family $(\mathbb{B}_{c,p})_{c\geq 0}$ of simpler functions depending on two variables only. This family, roughly speaking, arises from moving the assumption $\int_X\phi^p\mathrm{d}\mu=F$ appearing in the definition of $\mathfrak{B}_p(f,F,L)$ into the optimized expression (cf. Section 2 below). The second step of the analysis is to optimize over the parameter c to obtain $\mathfrak{B}_p$: see Section 3 for the details.

We would like to point out here that the argument works fine also in other settings as well. For instance, we have successfully applied it to obtain Burkholder's function corresponding to L^p estimates for martingale transforms (see [13]). The range of potential applications is much wider. See Section 4 for more information.

2. A family of related Bellman functions

For any $c\geq 0$, introduce the function

$$\mathbb{B}_{c,p}(f,L)=\sup\left\{\int_X\Big[\max\{\mathcal{M}_\mathcal{T}\phi,L\}^p-c^p\phi^p\Big]\mathrm{d}\mu:\phi\in L^p_+(X),\int_X\phi\,\mathrm{d}\mu=f\right\},$$

given for $0\leq f\leq L$. Comparing this to the definition of $\mathfrak{B}_p(f,F,L)$, we see that we have removed the assumption $\int_X\phi^p\mathrm{d}\mu=F$; instead, we have plugged $-c^p\int_X\phi^p\mathrm{d}\mu$ into the optimized expression. As the result, we have got rid of the dependence on F: the function above is truly two-dimensional.

In the statement below, we provide the explicit formulas for $\mathbb{B}_{c,p}$. For the sketch of some steps leading to the discovery of these, see Remark 2.4 below.

Theorem 2.1. *If $c<p/(p-1)$, then*

$$\mathbb{B}_{c,p}(f,L)=\infty,\qquad \textit{for all } f,\ L. \tag{2.1}$$

If $c\geq p/(p-1)$, then

$$\mathbb{B}_{c,p}(f,L)=\begin{cases}\frac{\gamma}{\gamma-1}\left[L^p-\frac{p}{p-1}L^{p-1}f\right] & \textit{if } L\leq\gamma f,\\ L^p-c^pf^p & \textit{if } L>\gamma f,\end{cases} \tag{2.2}$$

where γ is the unique number from the interval $(1,p/(p-1)]$ satisfying

$$(p-1)(\gamma-1)=\left(\frac{\gamma}{c}\right)^p. \tag{2.3}$$

For any f, $L \geq 0$, denote by $B_{c,p}(f,L)$ the right-hand side of (2.1) or (2.2), depending on whether $c < p/(p-1)$ or $c \geq p/(p-1)$. In the proof of the above theorem, we will require the following properties of $B_{c,p}$.

Lemma 2.2. *Suppose that $c \geq p/(p-1)$.*

(i) The function $B_{c,p}$ is continuous on the first quadrant and of class C^1 in its interior.

(ii) For any x, $z \geq 0$ we have the majorization

$$B_{c,p}(x,z) \geq z^p - c^p x^p.$$

(iii) For any $x, y \geq 0$ and $z \geq x$ we have

$$B_{c,p}(y, \max\{y,z\}) \leq B_{c,p}(x,z) + \frac{\partial B_{c,p}(x,z)}{\partial x}(y-x). \tag{2.4}$$

Proof. (i) This is straightforward: the details are left to the reader.

(ii) Clearly, we only need to handle the case $z \leq \gamma x$. By (2.3), we have $c^p = \gamma^p (p-1)^{-1}(\gamma-1)^{-1}$, and the majorization can be rewritten in the form

$$\frac{(\gamma x)^p}{p} + \frac{(p-1)z^p}{p} \geq z^{p-1} \cdot (\gamma x).$$

This bound follows directly from Young's inequality.

(iii) For a fixed z, the left-hand side of (2.4), considered as a function of $y \in [0,z]$, is concave, while the right hand side is linear; furthermore, both expressions agree, along with their derivatives, at $y = x$. This establishes the inequality for $y \leq z$; suppose then, that y is larger than z. Using the aforementioned concavity of $B_{c,p}(\cdot, z)$ on $[0,z]$ (which implies that the right-hand side of (2.4) is a nonincreasing function of x), we see that it suffices to show the estimate for $x = z$. Then the bound is equivalent to $y^p - z^p \geq pz^{p-1}(y-z)$, which follows at once from the mean-value theorem. □

We are ready for the proof of Theorem 2.1. We start with the upper bound for the function $\mathbb{B}_{c,p}$.

Proof of the inequality $\mathbb{B}_{c,p} \leq B_{c,p}$. Of course, it suffices to establish the claim for $c \geq p/(p-1)$. The proof is a slight modification of the arguments presented in [9]. Let ϕ be an arbitrary p-integrable function on X with mean f, and introduce the sequences $(\phi_n)_{n\geq 0}$, $(\mathcal{M}^n_{\mathcal{T}}\phi)_{n\geq 0}$ of measurable functions on X as follows. Given an integer n, an element E of $\mathcal{T}_n$ and a point $x \in E$, set

$$\phi_n(x) = \frac{1}{\mu(E)} \int_E \phi(t) \mathrm{d}\mu(t),$$

$$\mathcal{M}^n_{\mathcal{T}}\phi(x) = \sup\left\{ \frac{1}{\mu(I)} \int_I |\phi(u)| \mathrm{d}\mu(u) : x \in I, I \in \mathcal{T}_k \text{ for some } k \leq n \right\}.$$

In other words, $\mathcal{M}^n_{\mathcal{T}}\phi$ describes the action on ϕ of a maximal operator associated with the truncated tree $(\mathcal{T}_0, \mathcal{T}_1, \ldots, \mathcal{T}_{n-1}, \mathcal{T}_n, \mathcal{T}_n, \mathcal{T}_n, \ldots)$. Observe that if n, E are as above, and E_1, E_2, $\ldots$, E_m are the elements of $\mathcal{T}_{n+1}$ whose union is E, then

$$\frac{1}{\mu(E)} \int_E \phi_n(t) \mathrm{d}\mu(t) = \sum_{i=1}^m \frac{\mu(E_i)}{\mu(E)} \cdot \frac{1}{\mu(E_i)} \int_{E_i} \phi_{n+1}(t) \mathrm{d}\mu(t). \tag{2.5}$$

Consequently, for any $L \geq f$ we have

$$\begin{aligned}\int_E B_{c,p}(\phi_n(t), \max\{\mathcal{M}^n_{\mathcal{T}}\phi(t), L\})\mathrm{d}\mu(t) \\ \geq \int_E B_{c,p}(\phi_{n+1}(t), \max\left\{\mathcal{M}^{n+1}_{\mathcal{T}}\phi(t), L\right\})\mathrm{d}\mu(t).\end{aligned} \tag{2.6}$$

Indeed, we have $\mathcal{M}^{n+1}_{\mathcal{T}}\phi = \max\{\mathcal{M}^n_{\mathcal{T}}\phi, \phi_{n+1}\}$, so by (2.4), applied to $x = \phi_n(t)$, $y = \phi_{n+1}(t)$ and $z = \max\{\mathcal{M}^n_{\mathcal{T}}\phi(t), L\}$ (for some $t \in E$), we obtain

$$\begin{aligned}&B_{c,p}(\phi_{n+1}, \max\left\{\mathcal{M}^{n+1}_{\mathcal{T}}\phi, L\right\}) \\ &\quad\leq B_{c,p}(\phi_n, \max\{\mathcal{M}^n_{\mathcal{T}}\phi, L\}) + \frac{\partial B_{c,p}(\phi_n, \max\{\mathcal{M}^n_{\mathcal{T}}\phi, L\})}{\partial x}(\phi_{n+1} - \phi_n)\end{aligned} \tag{2.7}$$

on E. It suffices to integrate both sides over E and use (2.5) to get (2.6). Summing over all $E \in \mathcal{T}_n$, we see that

$$\int_X B_{c,p}(\phi_n, \max\{\mathcal{M}^n_{\mathcal{T}}\phi, L\})\mathrm{d}\mu \geq \int_X B_{c,p}(\phi_{n+1}, \max\left\{\mathcal{M}^{n+1}_{\mathcal{T}}\phi, L\right\})\mathrm{d}\mu$$

and therefore, by induction,

$$\int_X B_{c,p}(\phi_0, \max\left\{\mathcal{M}^0_{\mathcal{T}}\phi, L\right\})\mathrm{d}\mu \geq \int_X B_{c,p}(\phi_n, \max\{\mathcal{M}^n_{\mathcal{T}}\phi, L\})\mathrm{d}\mu.$$

However, we have $\mathcal{M}^0_{\mathcal{T}}\phi = \phi_0 \equiv \int_X \phi\mathrm{d}\mu = f$. Thus, the left-hand side equals $B_{c,p}(f, \max\{f, L\}) = B_{c,p}(f, L)$, while the right can be bounded from below with the use of Lemma 2.2 (ii). As the result, we obtain

$$\int_X \Big[\max\{\mathcal{M}^n_{\mathcal{T}}\phi, L\}\Big]^p \mathrm{d}\mu \leq c^p \int_X \phi_n^p \mathrm{d}\mu + B_{c,p}(f, L) \leq c^p \int_X \phi^p \mathrm{d}\mu + B_{c,p}(f, L),$$

where in the last line we have exploited Jensen's inequality. Letting $n \to \infty$ and using Fatou's lemma, we obtain

$$\int_X \Big[\max\{\mathcal{M}_{\mathcal{T}}\phi, L\}\Big]^p \mathrm{d}\mu - c^p \int_X \phi^p \mathrm{d}\mu \leq B_{c,p}(f, L),$$

Taking the supremum over all ϕ yields the desired bound $\mathbb{B}_{c,p}(f, L) \leq B_{c,p}(f, L)$. $\square$

The proof of the reverse bound for $\mathbb{B}_{c,p}$ rests on the construction of appropriate examples. We will need a lemma, which can be found in [4].

Lemma 2.3. *For every $I \in \mathcal{T}$ and every $\alpha \in (0, 1)$ there is a subfamily $F(I) \subset \mathcal{T}$ consisting of pairwise almost disjoint subsets of I such that*

$$\mu\left(\bigcup_{J \in F(I)} J\right) = \sum_{J \in F(I)} \mu(J) = \alpha\mu(I).$$

Proof of the inequality $\mathbb{B}_{c,p} \geq B_{c,p}$. Suppose first that $c \geq p/(p-1)$. If $L \geq \gamma f$, then the estimate is obvious: in the definition of $\mathbb{B}_{c,p}$ consider the constant function $\phi \equiv f$. When $L < \gamma f$, the construction of the extremal function is more involved. The idea is to find ϕ such that for each n, both sides of (2.7) are asymptotically equal. To give the precise definition, pick arbitrary $\gamma' \in (L/f, \gamma)$ and $\delta > 0$. By an inductive use of Lemma 2.3, there is a sequence $X = A_0 \supset A_1 \supset A_2 \supset \ldots$ satisfying the following properties:

(i) For each k, A_k is a union of certain pairwise disjoint subsets from $\mathcal{T}$: we have $A_k = \bigcup F_k$ for some $F_k \subset \mathcal{T}$.

(ii) We have
$$\mu(A_1) = \frac{f\gamma' - L}{L(\gamma' - 1)}.$$

(iii) For any $k \geq 1$ and any $I \in F_k$,
$$\frac{\mu(A_{k+1} \cap I)}{\mu(I)} = \frac{\gamma' - 1}{\gamma' - 1 + \delta\gamma'}.$$

In particular, condition (iii) implies that $\mu(A_{k+1}) = \frac{\gamma'-1}{\gamma'-1+\delta\gamma'}\mu(A_k)$ for $k \geq 1$, so

$$\begin{aligned}\mu(A_k \setminus A_{k+1}) &= \frac{\delta\gamma'}{\gamma' - 1 + \delta\gamma'}\mu(A_k) \\ &= \frac{\delta\gamma'}{\gamma' - 1 + \delta\gamma'}\left(\frac{\gamma' - 1}{\gamma' - 1 + \delta\gamma'}\right)^{k-1}\mu(A_1) \\ &= \frac{\delta\gamma'}{\gamma' - 1 + \delta\gamma'}\left(\frac{\gamma' - 1}{\gamma' - 1 + \delta\gamma'}\right)^{k-1}\frac{f\gamma' - L}{L(\gamma' - 1)}.\end{aligned} \tag{2.8}$$

Next, consider the function
$$\phi = \frac{L}{\gamma'}\chi_{A_0 \setminus A_1} + \frac{L}{\gamma'}\sum_{k=1}^{\infty}(1+\delta)^{k-1}\chi_{A_k \setminus A_{k+1}}.$$

Then we compute that
$$\begin{aligned}\int_X \phi^p \mathrm{d}\mu &= \frac{L - f}{L(1 - 1/\gamma')}\left(\frac{L}{\gamma'}\right)^p \\ &\quad + \frac{f\gamma' - L}{L(\gamma' - 1)}\left(\frac{L}{\gamma'}\right)^p\sum_{k=1}^{\infty}(1+\delta)^{(k-1)p}\left(\frac{\gamma' - 1}{\gamma' - 1 + \delta\gamma'}\right)^{k-1}\frac{\delta\gamma'}{\gamma' - 1 + \delta\gamma'} \\ &\xrightarrow{\delta \to 0} \frac{L\gamma' - f\gamma'}{L(\gamma' - 1)}\left(\frac{L}{\gamma'}\right)^p + \frac{f\gamma' - L}{L(\gamma' - 1)}\left(\frac{L}{\gamma'}\right)^p\frac{\gamma'}{\gamma' - p(\gamma' - 1)}.\end{aligned}$$

The same chain of calculations shows that $\int_X \phi \mathrm{d}\mu = f$ (this time for all δ, without passing to the limit). Furthermore, for any $n \geq 1$,
$$\begin{aligned}\int_{A_n}\phi \mathrm{d}\mu &= \frac{L}{\gamma'}\sum_{k=n}^{\infty}(1+\delta)^{k-1}\frac{\delta\gamma'}{\gamma' - 1 + \delta\gamma'}\left(\frac{\gamma' - 1}{\gamma' - 1 + \delta\gamma'}\right)^{k-1}\frac{f\gamma' - L}{L(\gamma' - 1)} \\ &= \frac{f\gamma' - L}{\gamma' - 1}(1+\delta)^{n-1}\left(\frac{\gamma' - 1}{\gamma' - 1 + \delta\gamma'}\right)^{n-1}\end{aligned}$$

and, by (2.8),
$$\mu(A_n) = \left(\frac{\gamma' - 1}{\gamma' - 1 + \delta\gamma'}\right)^{n-1}\frac{f\gamma' - L}{L(\gamma' - 1)},$$

so we get that
$$\frac{1}{\mu(A_n)}\int_{A_n}\phi \mathrm{d}\mu = L(1+\delta)^{n-1}.$$

This implies a slightly stronger statement. Namely, for any $n \geq 1$ and any $I \in F_n$,
$$\frac{1}{\mu(I)}\int_I \phi \, \mathrm{d}\mu = L(1+\delta)^{n-1}.$$

Indeed, by (ii), (iii) and the definition of ϕ, the conditional distribution of ϕ is the same on each $I \in F_n$ (i.e., $\mu(\{x \in I : \phi(x) \geq \lambda\})/\mu(I)$ does not depend on I, but only on the "level" n to which I belongs). Therefore the above equality holds, and it implies that $\mathcal{M}_\mathcal{T}\phi \geq L(1+\delta)^{n-1}$ on A_n, $n \geq 1$. Hence, comparing this to the definition of ϕ, we obtain the pointwise bound $\mathcal{M}_\mathcal{T}\phi \geq \gamma'\phi$ on A_1, and thus

$$\max\{\mathcal{M}_\mathcal{T}\phi, L\} \geq \gamma'\phi \qquad \text{on } X \tag{2.9}$$

(note that the use of the function $\max\{\cdot, L\}$ is necessary to guarantee this inequality on the set $X \setminus A_1$). Consequently,

$$\int_X \Big[\max\{\mathcal{M}_\mathcal{T}\phi, L\}^p - c^p\phi^p\Big]\mathrm{d}\mu \geq ((\gamma')^p - c^p)\int_X \phi^p \mathrm{d}\mu.$$

Plugging the above formula for $\int_X \phi^p\mathrm{d}\mu$ and letting $\delta \to 0$, we conclude that

$$\mathbb{B}_{c,p}(f, L) \geq \frac{\left(1 - \left(\frac{c}{\gamma'}\right)^p\right) L^{p-1}\gamma'}{\gamma' - 1}\left[L - f + \left(f - \frac{L}{\gamma'}\right)\frac{\gamma'}{p - (p-1)\gamma'}\right]. \tag{2.10}$$

Now let $\gamma' \to \gamma$, and combine it with the equality

$$\left(\frac{c}{\gamma}\right)^p = \frac{1}{(p-1)(\gamma-1)},$$

which follows from (2.3). Then, after some tedious, but straightforward calculations, we obtain $\mathbb{B}_{c,p}(f, L) \geq B_{c,p}(f, L)$.

It remains to consider the case $c < p/(p-1)$. Let us first assume that $L < pf/(p-1)$. Consider the same examples as previously; then, by (2.10), we get $\mathbb{B}_{c,p}(f, L) = \infty$, simply by letting $\gamma' \to p/(p-1)$. To see that $\mathbb{B}_{c,p}$ is infinite for $L \geq pf/(p-1)$, use the fact that for any fixed f, the function $L \mapsto \mathbb{B}_{c,p}(f, L)$ is nondecreasing (which follows from the very definition of $\mathbb{B}_{c,p}$) and the fact that $\mathbb{B}_{c,p}(f, L) = \infty$ for $L \in [f, pf/(p-1))$, as we have just shown. The claim follows. □

Remark 2.4. Let us briefly describe the *informal* reasoning which leads to the discovery of the formula for $\mathbb{B}_{c,p}$. We start from the observation that this function is homogeneous of order p (which follows from the very definition). How to proceed further? The key is to search for a function which satisfies the properties listed in Lemma 2.2. Then (2.4) implies that

$$\text{for any } z > 0,\ \mathbb{B}_{c,p}(\cdot, z) \text{ is convex on } [0, z] \tag{2.11}$$

and (taking $x = z$ and letting $y \downarrow z$)

$$\frac{\partial \mathbb{B}_{c,p}}{\partial z}(z, z) \leq 0 \qquad \text{for any } z > 0. \tag{2.12}$$

The first idea is just to take $\mathbb{B}_{c,p}(x, z) = z^p - c^p x^p$: this function satisfies the majorization and (2.11). Unfortunately, the condition (2.12) does not hold and hence the function $\mathbb{B}_{c,p}$ must be raised a little bit in the neighborhood of the diagonal $\{(z, z) : z > 0\}$. By homogeneity, we have only one reasonable candidate: there must be some $\gamma > 1$ such that for any $z > 0$,

$$\mathbb{B}_{c,p}(x, z) = \begin{cases} z^p - c^p x^p & \text{if } z \geq \gamma x, \\ \text{linear in } x & \text{if } z \leq \gamma x. \end{cases}$$

Since $\mathbb{B}_{c,p}$ is expected to be of class C^1, we compute that

$$\mathbb{B}_{c,p}(x,z) = z^p\left(1-\left(\frac{c}{\gamma}\right)^p\right) - pc^p\left(\frac{z}{\gamma}\right)^{p-1}\left(x-\frac{z}{\gamma}\right) \qquad \text{for } z\leq \gamma x.$$

Now it can be easily verified that (2.12) is equivalent to

$$\left(\frac{\gamma}{c}\right)^p - (p-1)(\gamma-1)\leq 0.$$

If $c<p/(p-1)$, then no $\gamma>1$ satisfies this estimate: this suggests that $\mathbb{B}_{c,p}$ is infinite. On the other hand, if $c\geq p/(p-1)$, we assume that we actually have equality above (i.e., (2.3) holds), and this brings us to the right-hand side of (2.2).

3. The formula for $\mathfrak{B}_p$

We are ready to deduce the formula for the three-dimensional Bellman function $\mathfrak{B}_p$. Recall that $H_p:[1,p/(p-1)]\to[0,1]$ is given by $H_p(z)=-(p-1)z^p+pz^{p-1}$ and w_p stands for its inverse. Denote the right-hand side of (1.2) by $\mathcal{B}_p$. For any ϕ satisfying $\int_X\phi=f$ and $\int_X\phi^p\leq F$, and any $c\geq p/(p-1)$, we may write

$$\begin{aligned}\int_X \max\{\mathcal{M}_{\mathcal{T}}\phi,L\}^p\mathrm{d}\mu &= \int_X\Big[\max\{\mathcal{M}_{\mathcal{T}}\phi,L\}^p-c^p\phi^p\Big]\mathrm{d}\mu + c^p\int_X\phi^p\mathrm{d}\mu\\ &\leq \mathbb{B}_{c,p}(f,L)+c^pF,\end{aligned}$$

which implies

$$\mathfrak{B}_p(f,F,L)\leq \inf_{c\geq p/(p-1)}\Big\{\mathbb{B}_{c,p}(f,L)+c^pF\Big\}.$$

We will check that the right-hand side is precisely $\mathcal{B}_p(f,F,L)$. First we will deal with the case $L\geq pf/(p-1)$. Then $\mathbb{B}_{c,p}(f,L)=L^p-c^pf^p$ and hence

$$\begin{aligned}\inf_{c\geq p/(p-1)}\Big\{\mathbb{B}_{c,p}(f,L)+c^pF\Big\} &= \inf_{c\geq p/(p-1)}\Big\{L^p+c^p(F-f^p)\Big\}\\ &= L^p+\left(\frac{p}{p-1}\right)^p(F-f^p)=\mathcal{B}_p(f,F,L),\end{aligned}$$

where in the second equality we have used the bound $F\geq f^p$. Next, let $L<pf/(p-1)$. By (2.3), we have $c^p=\gamma^p(p-1)^{-1}(\gamma-1)^{-1}$ for $\gamma\in(1,p/(p-1)]$, so

$$\mathbb{B}_{c,p}(f,L)+c^pF=\begin{cases} L^p+\frac{\gamma^p(F-f^p)}{(p-1)(\gamma-1)} & \text{if } \gamma<L/f,\\ \frac{\gamma}{\gamma-1}\Big[L^p-\frac{p}{p-1}L^{p-1}f\Big]+\frac{\gamma^pF}{(p-1)(\gamma-1)} & \text{if } \gamma\geq L/f.\end{cases}$$

We must find the minimal value of the expression on the right, when γ is assumed to run over the interval $(1,p/(p-1)]$. The right-hand side is a continuous function of γ, and the function $\gamma\mapsto\gamma^p/(\gamma-1)$ is nonincreasing (for $\gamma\leq p/(p-1)$); therefore, the minimum is attained on the interval $[L/f,p/(p-1)]$. A direct differentiation shows that

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}\gamma}\left[\frac{\gamma}{\gamma-1}\left[L^p-\frac{p}{p-1}L^{p-1}f\right]+\frac{\gamma^pF}{(p-1)(\gamma-1)}\right]\\ &\qquad=-\frac{1}{(\gamma-1)^2}\left[L^p-\frac{p}{p-1}L^{p-1}f\right]+\frac{(p-1)\gamma^p-p\gamma^{p-1}}{(\gamma-1)^2}\frac{F}{p-1}.\end{aligned}$$

The latter expression is negative for $\gamma\to L/f$, and positive for $\gamma\to p/(p-1)$ (in both cases, this follows from the estimate $L<pf/(p-1)$). Furthermore, it has a unique root γ equal to $w_p\left((pL^{p-1}f-(p-1)L^p)/F\right)$. Consequently, the minimum

is attained for this choice of γ, and after some easy computations we get that this minimal value equals $F\gamma^p$. This completes the proof of the inequality $\mathfrak{B}_p \leq \mathcal{B}_p$.

We turn our attention to the reverse inequality $\mathfrak{B}_p \geq \mathcal{B}_p$. We will make use of the examples constructed in the previous section. Suppose first that $L < pf/(p-1)$ and let ϕ be the function of Section 2, corresponding to a certain $\gamma' \in [L/f, p/(p-1))$ and $\delta > 0$. If δ is sufficiently small, then $\int_X \phi^p \mathrm{d}\mu$ can be made arbitrarily close to

$$\begin{aligned}&\frac{L\gamma' - f\gamma'}{L(\gamma'-1)}\left(\frac{L}{\gamma'}\right)^p + \frac{f\gamma' - L}{L(\gamma'-1)}\left(\frac{L}{\gamma'}\right)^p \frac{\gamma'}{\gamma' - p(\gamma'-1)}\\ &\qquad = \frac{p-1}{p-(p-1)\gamma'}\left(\frac{L}{\gamma'}\right)^{p-1}\left[\frac{p}{p-1}f - L\right] = \frac{pL^{p-1}f - (p-1)L^p}{H_p(\gamma')}.\end{aligned}$$

If $\gamma' = L/f$, then the expression equals f^p; if $\gamma' \to p/(p-1)$, then it tends to infinity. Thus, there is γ' for which the expression equals F: this γ' is precisely $w_p((pL^{p-1}f - (p-1)L^p)/F)$. Hence, for this choice of γ',

$$\mathfrak{B}_p(f, F, L) \geq \int_X \max\{\mathcal{M}_{\mathcal{T}}\phi, L\}^p \mathrm{d}\mu.$$

But $\max\{\mathcal{M}_{\mathcal{T}}\phi, L\} \geq \gamma'\phi$ (see (2.9)), so

$$\mathfrak{B}_p(f, F, L) \geq (\gamma')^p \int_X \phi^p \mathrm{d}\mu = F(\gamma')^p = F w_p\left(\frac{pL^{p-1}f - (p-1)L^p}{F}\right)^p.$$

It remains to handle the case $L \geq pf/(p-1)$. If $f^p = F$, we get the estimate $\mathfrak{B}_p(f, F, L) \geq L^p$, which is obvious. Suppose then that $f^p < F$, and let us use a certain modification of the example of the previous section. Namely, take $\varepsilon > 0$, $\gamma' \in [1, p/(p-1))$ and consider the sequence $X = A_0 \supset A_1 \supset A_2 \supset \ldots$ satisfying (i), (iii) and

(ii') We have $\mu(A_1) = \varepsilon/(L - f + \varepsilon)$.

Introduce the function

$$\phi = (f - \delta)\chi_{A_0 \setminus A_1} + \frac{L}{\gamma'}\sum_{k=1}^{\infty}(1+\delta)^{k-1}\chi_{A_k \setminus A_{k+1}}.$$

Repeating the preceding calculations, we derive that $\int_X \phi \mathrm{d}\mu = f$ and

$$\int_X \phi^p \mathrm{d}\mu = \frac{L-f}{L-f+\varepsilon}\cdot(f-\varepsilon)^p + \frac{\varepsilon}{L-f+\varepsilon}\frac{L^p}{H_p(\gamma')}.$$

If $\varepsilon \to 0$, then the right-hand side tends to $f^p < F$; thus, it is less than F for sufficiently small ε. Moreover, if we let $\gamma' \to p/(p-1)$, then $H_p(\gamma') \to 0$ and the right-hand side explodes. Hence, if ε is sufficiently small, then there is $\gamma' = \gamma'_\varepsilon$ for which $\int_X \phi^p \mathrm{d}\mu = F$ and $\lim_{\varepsilon \to 0} \gamma'_\varepsilon = p/(p-1)$. However, $\max\{\mathcal{M}_{\mathcal{T}}\phi, L\} \geq L$ on $A_0 \setminus A_1$ and, as one easily verifies, $\max\{\mathcal{M}_{\mathcal{T}}\phi, L\} \geq \gamma'_\varepsilon\phi$ on A_1 (repeat the proof of (2.9)), so

$$\begin{aligned}\int_X \max\{\mathcal{M}\phi, L\}^p \mathrm{d}\mu &\geq \frac{L-f}{L-f+\varepsilon}\cdot L^p + \frac{\varepsilon}{L-f+\varepsilon}\frac{(\gamma'_\varepsilon)^p L^p}{H_p(\gamma'_\varepsilon)}\\ &= \frac{L-f}{L-f+\varepsilon}L^p + (\gamma'_\varepsilon)^p\left(F - \frac{L-f}{L-f+\varepsilon}(f-\varepsilon)^p\right).\end{aligned}$$

It remains to let $\varepsilon \to 0$ to obtain that

$$\mathfrak{B}_p(f,F,L) \geq L^p + \left(\frac{p}{p-1}\right)^p (F - f^p) = \mathcal{B}_p(f,F,L).$$

The proof is complete.

4. Further examples and questions

The considerations presented above concern the analysis of the L^p estimate for the maximal operator. We would like to conclude the paper by presenting several related problems for which the above method might work. For the sake of consistence, we will focus on maximal operators; for the probabilistic aspects of these statements, we refer the interested reader to the survey [13].

Let us take a look at the following four estimates for maximal operators. Firstly, note that the L^p bounds fail to hold for $p = 1$, so, as a substitute, one can ask about sharp versions of the $L \log L$ inequalities

$$\int_X \mathcal{M}_{\mathcal{T}}\phi \, \mathrm{d}\mu \leq K \int_X (|\phi| + 1)\log(|\phi|+1)\mathrm{d}\mu + L.$$

To be more precise, two questions can be formulated:

(i) For which K there is a universal $L < \infty$ such that the above bound holds?
(ii) For K as in (i), what is the best value $L = L(K)$?

These questions were answered by Melas [5], who identified the explicit expression for the associated Bellman function given by

$$\mathfrak{B}(f,F,L,k) = \sup\left\{ \int_E \max\{\mathcal{M}_{\mathcal{T}}\phi, L\}\,\mathrm{d}\mu \; : \; \int_X \phi\,\mathrm{d}\mu = f, \right. \\ \int_X (\phi+1)\log(\phi+1)\mathrm{d}\mu \leq F, \\ \left. E \subset X \text{ is measurable, } \mu(E) = k \right\}$$

(as in the L^p case, one easily sees that it is enough to handle nonnegative functions ϕ only. We will also exploit this observation in the two remaining bounds as well).

A second problem concerns the related weak-type (p,p) inequalities for $\mathcal{M}_{\mathcal{T}}$:

$$||\mathcal{M}_{\mathcal{T}}\phi||_{p,\infty} \leq c_p ||\phi||_p, \qquad 1 \leq p < \infty.$$

Here $||\phi||_{p,\infty} = \sup_{\lambda>0} \lambda(\mu(\{x \in X : |\phi(x)| \geq \lambda\}))^{1/p}$ denotes the weak p-th norm. The above inequality was studied by Melas and Nikolidakis in [7] by means of the associated Bellman function

$$\mathfrak{B}_p(f,F,L) = \sup\left\{ \mu\big(\max\{\mathcal{M}_{\mathcal{T}}\phi(x), L\} \geq 1\big) \; : \; \int_X \phi\,\mathrm{d}\mu = f, \; \int_X \phi^p \mathrm{d}\mu \leq F \right\}.$$

The third result is the sharp localized $L^p \to L^q$ estimates

$$\left(\int_E (\mathcal{M}_{\mathcal{T}}\phi)^q \mathrm{d}\mu\right)^{1/q} \leq C_{p,q} \left(\int_X |\phi|^p \mathrm{d}\mu\right)^{1/p} \mu(E)^{1/q-1/p},$$

where $1 \leq q < p$ and E runs over all measurable subsets of X. This estimate was studied by Melas [6]: he found the explicit expression for

$$\mathfrak{B}(F,f,L,k) = \sup\left\{ \int_E (\max\{\mathcal{M}_{\mathcal{T}}\phi, L\})^q \mathrm{d}\mu \,:\, \int_X \phi \mathrm{d}\mu = f,\ \int_X \phi^p \mathrm{d}\mu \leq F,\right.$$
$$\left. E \subset X \text{ is measurable, } \mu(E) = k \right\}$$

(for a related estimate corresponding to $q < 1$, see [8]). Finally, let us look at the following sharp comparison between the weak norms:

$$||\mathcal{M}_{\mathcal{T}}\phi||_{p,\infty} \leq \frac{p}{p-1}||\phi||_{p,\infty}, \qquad 1 < p < \infty.$$

This bound was obtained by Nikolidakis [12] with the use of the corresponding Bellman function

$$\mathfrak{B}_p(f,F,L) = \sup\left\{ \mu\big(\max\{\mathcal{M}_{\mathcal{T}}\phi(x), L\} \geq 1\big) \,:\, \int_X \phi \, \mathrm{d}\mu = f,\ ||\phi||_{p,\infty}^p \leq F \right\}.$$

All the Bellman functions written above were found by the exploitation of certain combinatorial properties of $\mathcal{M}_{\mathcal{T}}$ and optimization arguments. Can these objects be identified with the use of the approach we have developed in the preceding sections? Some initial calculations made by the author suggest that the answer may be affirmative.

We can ask similar question concerning the sharp comparison of Lorentz norms

$$||\mathcal{M}_{\mathcal{T}}\phi||_{p,q} \leq C_{p,q}||\phi||_{p,q}, \qquad 1 \leq p < \infty,\ 1 \leq q < \infty.$$

To the best of our knowledge, almost nothing is known about the optimal values of $C_{p,q}$; see [14] for some partial statements. While one can easily write down the Bellman function corresponding to this estimate, it is absolutely not clear how to identify the explicit expression for this object. The main problem which makes our approach fail is that the Lorentz norms are not integral. So, if we consider the simplified function

$$\mathbb{B}_{c,p}(f,L) = \sup\left\{ \big|\big| \max\{\mathcal{M}_{\mathcal{T}}\phi(x), L\}\big|\big|_{p,q}^p - c||\phi||_{p,q}^p \,:\, \int_X \phi \, \mathrm{d}\mu = f \right\},$$

we cannot express the optimized difference as an integral over X of some object, and hence we are unable to proceed any further.

However, we would like to mention a paper [15] which does contain some positive results in the above direction. It was shown here how to adapt the above approach to obtain the sharp inequality

$$||\mathcal{M}_{\mathcal{T}}\phi||_q \leq \left(\frac{p}{p-q}\right)^{1/q} \frac{p}{p-1}||\phi||_{p,\infty}, \qquad 1 \leq q < p < \infty.$$

As we hope, the arguments used there may be helpful in the study of the above problems involving the Lorentz norms, but we have been unable to push the calculations through.

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