SHARP LORENTZ-NORM ESTIMATES FOR DYADIC-LIKE MAXIMAL OPERATORS

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Abstract. For any $1 < p \leq q_1 < q_2 < \infty$, we identify the norm of the dyadic maximal operator on $\mathbb{R}^n$ as an operator from L^{p,q_1} to L^{p,q_2}. A related statement for general measure spaces equipped with tree-like structure is also established. The proof rests on the identification of an explicit formula for the associated Bellman function, which requires novel ideas due to the non-integral form of Lorentz norms.

1. Introduction

The motivation for the results of this paper comes from a natural question about sharp versions of certain Lorentz-norm inequalities for the dyadic maximal operator on $\mathbb{R}^n$. Recall that this operator is given by the formula

$$\mathcal{M}_d\phi(x) = \sup\left\{\frac{1}{|Q|}\int_Q |\phi(u)|\mathrm{d}u : x \in Q,\, Q \subset \mathbb{R}^n \text{ is a dyadic cube}\right\},$$

where ϕ is a locally integrable function on $\mathbb{R}^n$ and the dyadic cubes are those formed by the grids $2^{-N}\mathbb{Z}^n$, $N = 0, 1, 2, \ldots$. The maximal operator is a fundamental object in analysis and the theory of PDEs (cf. [4, 19]), and the question about its action on various function spaces (and the size of the associated norms) has gained a lot of interest in the literature. For example, $\mathcal{M}_d$ satisfies the weak-type $(1,1)$ inequality

$$\lambda\Big|\{x \in \mathbb{R}^n : \mathcal{M}_d\phi(x) \geq \lambda\}\Big| \leq \int_{\{\mathcal{M}_d\phi \geq \lambda\}} |\phi(u)|\mathrm{d}u \tag{1.1}$$

for any $\phi \in L^1(\mathbb{R}^n)$ and any $\lambda > 0$. This bound is sharp: it is easy to construct an exemplary non-zero ϕ for which both sides are equal. Integrating the above estimate, we obtain the L^p bound

$$||\mathcal{M}_d\phi||_{L^p(\mathbb{R}^n)} \leq \frac{p}{p-1}||\phi||_{L^p(\mathbb{R}^n)}, \qquad 1 < p \leq \infty, \tag{1.2}$$

in which the constant $p/(p-1)$ is also the best possible. These two results form a natural starting point for various extensions and numerous applications. Due to the immense number of results in this direction, we will only mention below some statements which are closely related to the subject of this paper. First, both (1.1) and (1.2) hold true in the more general setting of maximal operators $\mathcal{M}_\mathcal{T}$ on measure spaces equipped with tree-like structure $\mathcal{T}$. Let us briefly introduce

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the necessary notions. We assume that (Ω, μ) is a nonatomic measure space with $\mu(\Omega) > 0$. Two measurable subsets A, B of Ω are said to be almost disjoint if $\mu(A \cap B) = 0$.

Definition 1.1. A set $\mathcal{T}$ of measurable subsets of Ω will be called a tree if the following conditions are satisfied:

(i) $\Omega \in \mathcal{T}$ and for every $I \in \mathcal{T}$ we have $\mu(I) > 0$.

(ii) For every $I \in \mathcal{T}$ there is a finite subset $C(I) \subset \mathcal{T}$ such that
  (a) the elements of $C(I)$ are pairwise almost disjoint subsets of I,
  (b) $I = \bigcup C(I)$.

(iii) $\mathcal{T} = \bigcup_{m \geq 0} \mathcal{T}^m$, where $\mathcal{T}^0 = \{\Omega\}$ and $T^{m+1} = \bigcup_{I \in \mathcal{T}^m} C(I)$.

(iv) We have $\lim_{m \to \infty} \sup_{I \in \mathcal{T}^m} \mu(I) = 0$.

If $\mu(\Omega) = 1$, i.e., if (Ω, μ) is a probability space, there is one-to-one correspondence between tree structures and atomic filtrations $(\sigma(\mathcal{T}^n))_{n \geq 0}$, and all the results discussed below can be interpreted in terms of martingales and their maximal functions. However, we have decided to present the results from an analytic point of view only. Although we will formulate the stochastic analogue of our main result (see Theorem 1.3 below), we restrain ourselves from any further discussion or applications in this context.

Any measure space equipped with a tree gives rise to the corresponding maximal operator $\mathcal{M}_{\mathcal{T}}$, given by

$$\mathcal{M}_{\mathcal{T}}\phi(x) = \sup \left\{ \frac{1}{\mu(I)} \int_I |\phi(u)| \mathrm{d}\mu(u) : x \in I, I \in \mathcal{T} \right\}.$$

It is easy to describe the interplay between the tree setting and the dyadic counterpart discussed above. Observe that in the dyadic case, it is enough to study (1.1) and (1.2) for functions supported on the cube $[0,1)^n$; the passage to general locally integrable functions follows immediately from straightforward dilation and translation arguments. It remains to note that the class of dyadic cubes contained in $[0,1)^n$ forms a tree, and the associated maximal operator coincides with the dyadic maximal operator (restricted to the functions supported on $[0,1)^n$). Thus the setting of trees is indeed more general, and from now on we will work in this wider context.

Let us present a number of other estimates for the operators $\mathcal{M}_{\mathcal{T}}$ which have appeared in the last twenty years. It is well-known that if $p = 1$, then the inequality

$$||\mathcal{M}_{\mathcal{T}}\phi||_{L^p(\Omega)} \leq C_p ||\phi||_{L^p(\Omega)}$$

does not hold with any finite constant C_p, even in the dyadic case. This gives rise to the question about an appropriate substitute of this bound. Motivated by the classical results of Zygmund, Melas [6] proposed an answer in terms of sharp LlogL-type estimates. The subsequent work [7] concerns another extension of (1.2): the action of $\mathcal{M}_{\mathcal{T}}$, considered as an operator from $L^p(\Omega)$ to $L^q(\Omega)$ (for $1 \leq q < p$), is studied there. Specifically, among other things, Melas determined the best constant $C_{p,q}$ in the following local inequality: for any $E \in \mathcal{T}$,

$$\left(\int_E (\mathcal{M}_{\mathcal{T}}\phi)^q \mathrm{d}\mu \right)^{1/q} \leq C_{p,q} \left(\int_\Omega |\phi|^p \mathrm{d}\mu \right)^{1/p} \mu(E)^{1/q - 1/p}.$$

The paper [8] by Melas and Nikolidakis extends the above estimate to the wider range of parameters. It is devoted to the following sharp version of Kolmogorov's inequality: for any $0<q<1$ and any $E\in\mathcal{T}$,

$$\left(\int_E |\mathcal{M}_\mathcal{T}\phi|^q \mathrm{d}\mu\right)^{1/q} \leq \left(\frac{1}{1-q}\right)^{1/q} \left(\int_\Omega |\phi|\mathrm{d}\mu\right)\mu(E)^{1/q-1}.$$

Lorentz-norm estimates for $\mathcal{M}_\mathcal{T}$ have also gathered a lot of interest. Let us first provide some necessary background. Recall that if ϕ is a measurable function on (Ω,μ), then its nonincreasing rearrangement $\phi^*:[0,\infty)\to[0,\infty)$ is given by

$$\phi^*(t)=\inf\left\{s>0\,:\,\mu(\{x\in\Omega:|\phi(x)|>s\})\leq t\right\}.$$

Given $0<p,q<\infty$, we define the Lorentz space $L^{p,q}=L^{p,q}(\Omega,\mu)$ as the family of all (equivalence classes of) measurable functions f on Ω such that

$$\|f\|_{L^{p,q}}:=\left(\int_0^\infty \left(t^{1/p}f^*(t)\right)^q \frac{\mathrm{d}t}{t}\right)^{1/q}.$$

The space $L^{p,\infty}=L^{p,\infty}(\Omega,\mu)$ is defined similarly, with the use of the quasinorm

$$\|f\|_{L^{p,\infty}}:=\sup_{t>0} t^{1/p}f^*(t).$$

Melas and Nikolidakis [8] proved that for any $1<p,q<\infty$ we have

$$\|\mathcal{M}_\mathcal{T}\phi\|_{L^{p,q}(\Omega)}\leq \frac{p}{p-1}\|\phi\|_{L^{p,q}}.$$

There is also a related estimate concerning the action of $\mathcal{M}_\mathcal{T}$ between the spaces $L^{p,\infty}\to L^{q,r}$, see [8, 13, 14] for details.

We should point out here that the works cited above contain much more: they actually identify the explicit formulae for the associated Bellman functions. This provides a lot of additional information about the action of maximal operators on the corresponding spaces: for the necessary definitions and the explanation of this fact, consult [5, 9, 10, 12, 17, 18, 20, 21]. See also Section 2 below.

In this paper, we continue this line of research. We will be interested in the explicit formula for the norm of $\mathcal{M}_\mathcal{T}$ as an operator from $L^{p,q_1}(\Omega)$ to $L^{p,q_2}(\Omega)$, where $0<p<\infty$ and $0<q_1,q_2<\infty$. First, observe that if $p<1$, then

$$\|\mathcal{M}_\mathcal{T}\|_{L^{p,q_1}\to L^{p,q_2}}=\infty, \tag{1.3}$$

no matter what q_1 and q_2 are. For $p=1$ the identity (1.3) holds as well, unless $q_1=1$ and $q_2=\infty$ (but this special case has been already discussed in (1.1)). Therefore, from now on we only consider the case $p>1$. Of course, if $q_1>q_2$, then there are functions satisfying $||\phi||_{L^{p,q_1}}<\infty$ and $||\mathcal{M}_\mathcal{T}\phi||_{L^{p,q_2}}\geq||\phi||_{L^{p,q_2}}=\infty$, so in this case (1.3) holds as well. Thus, the only notrivial cases left correspond to $1<p<\infty$ and $0<q_1\leq q_2<\infty$.

Our approach will allow us to study the case $1<p\leq q_1<q_2$, from now on we assume that this condition is satisfied. Set $\alpha=q_1/p-1$, $\beta=q_2/p-1$, $\gamma=q_1(p-1)/(p(q_1-1))$ and define

$$C_{p,q_1,q_2}=q_1^{\frac{1}{q_2}}(q_2(q_1-1))^{-\frac{1}{q_1}}\gamma^{\frac{q_2-q_1}{q_1q_2}-1}\left(\frac{(q_2-q_1)\Gamma\left(\frac{q_1q_2}{q_2-q_1}\right)}{\Gamma\left(\frac{q_2(q_1-1)}{q_2-q_1}\right)\Gamma\left(\frac{q_2}{q_2-q_1}\right)}\right)^{\frac{q_2-q_1}{q_1q_2}}.$$

Our main result can be formulated as follows.

Theorem 1.2. *Suppose that $1 < p \leq q_1 < q_2$ are fixed parameters. Then for any integrable function ϕ on Ω,*

$$\|\mathcal{M}_\mathcal{T}\phi\|_{L^{p,q_2}(\Omega)} \leq C_{p,q_1,q_2}\|\phi\|_{L^{p,q_1}(\Omega)} \tag{1.4}$$

and the constant on the right-hand side is the best possible for each individual tree.

There is a probabilistic analogue of this result, which can be expressed in the language of martingales, and which follows from Theorem 1.2 by straightforward approximation (the approximation is needed to handle the case of arbitrary filtrations). For the necessary definitions and related results, we refer the reader to the classical monograph of Doob [2].

Theorem 1.3. *Suppose that $f = (f_n)_{n\geq 0}$ is a martingale on a certain probability space. Then for any $1 < p \leq q_1 < q_2$ we have*

$$\left\| \sup_{n\geq 0} |f_n| \right\|_{L^{p,q_2}} \leq C_{p,q_1,q_2}\|f\|_{L^{p,q_1}}$$

and the constant on the right-hand side is the best possible.

As in the papers cited above, our approach will allow us to prove much more: we will identify the explicit formula for the Bellman function corresponding to (1.4). It should be emphasized that our proof will not be just a mere repetition of the reasoning from [5]-[11]. More specifically, the argument will exploit a certain novel modification of Bellman function method combined with combinatorial and optimization techniques.

We have organized the paper as follows. In the next section we introduce the abstract special function $\mathbb{B}$ corresponding to (1.4). In Section 3 we present an informal reasoning which leads to the explicit candidate B for the Bellman function. In Section 4 we prove that this candidate satisfies $B \geq \mathbb{B}$ and, in particular, we establish the inequality (1.4) there. Section 5 contains the proof of the reverse estimate $B \leq \mathbb{B}$, which, in particular, allows us to show that the constant C_{p,q_1,q_2} in (1.4) is indeed the best possible. In the final part of the paper, we sketch the alternative proof of the estimate (1.4), relating it to a certain Hardy's inequality.

Throughout the article, we will not indicate the dependence of the maximal operator on the underlying tree (which will be clear from the context) and simply write $\mathcal{M}$ instead of $\mathcal{M}_\mathcal{T}$. However, as we shall work with different measure spaces, we will sometimes use the notation $\mathcal{M}_\Omega$ to emphasize that we study the action of the maximal operator on functions on Ω. This should not lead to any confusion.

2. An abstract Bellman function

In the literature, estimates for maximal operators have been studied with the use of various techniques: these include, for example, covering theorems, Calderón-Zygmund-type decompositions, interpolation, and many more. In this paper we will exploit the so-called Bellman function method which, roughly speaking, reduces the problem of proving a given inequality to that of finding an appropriate special function which enjoys certain concavity and size conditions. This approach often allows to identify best constants involved in the estimate under investigation and it also provides some additional information about the structure of extremizers (i.e., functions on which equality is attained, or almost attained). See e.g. [9, 12] for the detailed discussion on these facts. In this paper, we will use a certain novel modification of the method; let us describe it now.

Suppose that x, y are nonnegative numbers and $T>0$. Assume further that $q_1\geq p$, so that $\alpha\geq 0$. Consider the class $\mathcal{C}(x,y,T)$, which consists of all nonnegative measurable functions f given on some measure space (Ω,μ) with $\mu(\Omega)=T$, such that

$$\frac{1}{T}\int_0^T f^*(t)\mathrm{d}t=x,\qquad \frac{1}{T}\int_0^T t^\alpha\big[f^*(t)\big]^{q_1}\mathrm{d}t\leq y.$$

Note that we have inequality in the second requirement. We emphasize that the measure space (Ω,μ) and the tree structure are allowed to vary. By Hölder's inequality, we see that if the class $\mathcal{C}(x,y,T)$ is nonempty, then

$$T^\alpha x^{q_1}\leq \gamma^{1-q_1}y \tag{2.1}$$

(recall that $\gamma=q_1(p-1)/(p(q_1-1)))$. Actually, the reverse implication is also true, which can be seen by taking any measure space (Ω,μ) and any function $f:\Omega\to[0,\infty)$ satisfying $f^*(t)=\gamma x(T/t)^{\alpha/(q_1-1)}$ (for the existence of a function with a prescribed nonincreasing rearrangement, see [4, p. 65] or Lemma 2.3 in [13]). Note that if equality holds in (2.1), then this is the only choice for f^*.

The abstract Bellman function related to the estimate (1.4) is given by

$$\mathbb{B}(x,y,T)=\sup\left\{\int_0^T t^\beta\big[(\mathcal{M}f)^*(t)\big]^{q_2}\mathrm{d}t\,:\,f\in\mathcal{C}(x,y,T)\right\}$$

for $(x,y,T)\in[0,\infty)^2\times(0,\infty)$ satisfying (2.1). In the next three sections, we will identify the explicit formula for $\mathbb{B}$. We would like to emphasize here that our proof will yield a stronger fact. One might consider the above definition of $\mathcal{C}(x,y,T)$ and $\mathbb{B}$ for a *fixed* measure space (Ω,μ) and a tree structure $\mathcal{T}$. We will actually show that for any such individual choice, the resulting Bellman function is the same. However, as it will be useful for us to switch the measure spaces and trees at some points of the proof, we have decided to work under the above definitions.

3. A candidate for the Bellman function

Throughout, we assume that $q_1\geq p$. We start our search by proving the following estimate, which can be regarded as a version of main inequality.

Lemma 3.1. *For any $S,T>0$ and any x, y, $c\geq 0$ we have*

$$\begin{split}&\mathbb{B}\left(\frac{Tx+Sc}{T+S},(T+S)^{-1}\left(Ty+c^{q_1}\cdot\frac{(T+S)^{\alpha+1}-T^{\alpha+1}}{\alpha+1}\right),T+S\right)\\&\geq \mathbb{B}(x,y,T)+\left(\frac{Tx+Sc}{T+S}\right)^{q_2}\frac{(T+S)^{\beta+1}-T^{\beta+1}}{\beta+1}.\end{split}\tag{3.1}$$

Proof. Take arbitrary measure spaces (Ω,μ_Ω), (Λ,μ_Λ) satisfying $\Omega\cap\Lambda=\emptyset$, $\mu_\Omega(\Omega)=T$, $\mu_\Lambda(\Lambda)=S$, equipped with some tree structures $\mathcal{T}^\Omega$, $\mathcal{T}^\Lambda$, respectively. Let $\mu_{\Omega\cup\Lambda}$ be the measure on the space $\Omega\cup\Lambda$, given by $\mu_{\Omega\cup\Lambda}(A\cup B)=\mu_\Omega(A)+\mu_\Lambda(B)$ for all measurable $A\subseteq\Omega$, $B\subseteq\Lambda$. Let $c\geq 0$ be a positive number. Suppose that $f:\Omega\to[0,\infty)$ satisfies

$$\frac{1}{T}\int_0^T f^*(t)\mathrm{d}t=x,\qquad \frac{1}{T}\int_0^T t^\alpha\big[f^*(t)\big]^{q_1}\mathrm{d}t\leq y \tag{3.2}$$

and consider its extension $\tilde{f} = f\chi_\Omega + c\chi_\Lambda$, a nonnegative function on the measure space $(\Omega \cup \Lambda, \mu_{\Omega\cup\Lambda})$. We compute directly that

$$\frac{1}{\mu_{\Omega\cup\Lambda}(\Omega\cup\Lambda)} \int_{\Omega\cup\Lambda} \tilde{f} \mathrm{d}\mu_{\Omega\cup\Lambda} = \frac{Tx+Sc}{T+S} \tag{3.3}$$

and, since $\alpha \geq 0$ (here is the place where we use the assumption $q_1 \geq p$),

$$\begin{aligned} &\frac{1}{\mu_{\Omega\cup\Lambda}(\Omega\cup\Lambda)} \int_0^{T+S} t^\alpha [\tilde{f}^*(t)]^{q_1} \mathrm{d}t \\ &\leq \frac{1}{\mu_{\Omega\cup\Lambda}(\Omega\cup\Lambda)} \left[\int_0^T t^\alpha [f^*(t)]^{q_1} \mathrm{d}t + \int_T^{T+S} t^\alpha c^{q_1} \mathrm{d}t \right] \\ &= (T+S)^{-1} \left(Ty + c^{q_1} \cdot \frac{(T+S)^{\alpha+1} - T^{\alpha+1}}{\alpha+1} \right). \end{aligned} \tag{3.4}$$

In other words, we have the inclusion

$$\tilde{f} \in \mathcal{C}\left(\frac{Tx+Sc}{T+S}, (T+S)^{-1} \left(Ty + c^{q_1} \cdot \frac{(T+S)^{\alpha+1} - T^{\alpha+1}}{\alpha+1} \right), T+S \right).$$

Now let us study the appropriate Lorentz norm of the maximal function of $\tilde{f}$. To this end, we equip the space $(\Omega\cup\Lambda, \mu_{\Omega\cup\Lambda})$ with the tree $\mathcal{T}^{\Omega\cup\Lambda}$ given by $\mathcal{T}_0^{\Omega\cup\Lambda} = \{\Omega\cup\Lambda\}$ and $\mathcal{T}_n^{\Omega\cup\Lambda} = \mathcal{T}_{n-1}^\Omega \cup \mathcal{T}_{n-1}^\Lambda$ for $n \geq 1$. To avoid confusion, we will denote by $\mathcal{M}_\Omega$ and $\mathcal{M}_{\Omega\cup\Lambda}$ the maximal operators on (Ω, μ_Ω) and $(\Omega\cup\Lambda, \mu_{\Omega\cup\Lambda})$. Of course, we may write

$$\begin{aligned} &\int_0^{T+S} t^\beta \left[(\mathcal{M}_{\Omega\cup\Lambda}\tilde{f})^*(t) \right]^{q_2} \mathrm{d}t \\ &\qquad = \int_0^T t^\beta \left[(\mathcal{M}_{\Omega\cup\Lambda}\tilde{f})^*(t) \right]^{q_2} \mathrm{d}t + \int_T^{T+S} t^\beta \left[(\mathcal{M}_{\Omega\cup\Lambda}\tilde{f})^*(t) \right]^{q_2} \mathrm{d}t. \end{aligned}$$

Next, observe that on Ω,

$$\mathcal{M}_{\Omega\cup\Lambda}\tilde{f} = \max\left\{ \mathcal{M}_\Omega f, \frac{1}{T+S} \int_{\Omega\cup\Lambda} \tilde{f} \mathrm{d}\mu_{\Omega\cup\Lambda} \right\} \geq \mathcal{M}_\Omega f.$$

Hence $(\mathcal{M}_{\Omega\cup\Lambda}\tilde{f})^* \geq (\mathcal{M}_\Omega f)^*$ on $(0,T]$ and the first integral on the right is not smaller than $\int_0^T t^\beta \left[(\mathcal{M}f)^*(t) \right]^{q_2} \mathrm{d}t$. To deal with the second integral, note that

$$\mathcal{M}_{\Omega\cup\Lambda}\tilde{f} \geq \frac{1}{\mu(\Omega\cup\Lambda)} \int_{\Omega\cup\Lambda} \tilde{f} \mathrm{d}\mu_{\Omega\cup\Lambda} = \frac{Tx+Sc}{T+S} \qquad \text{on } \Omega\cup\Lambda,$$

and hence

$$\int_T^{T+S} t^\beta \left[(\mathcal{M}_{\Omega\cup\Lambda}\tilde{f})^*(t) \right]^{q_2} \geq \left(\frac{Tx+Sc}{T+S} \right)^{q_2} \frac{(T+S)^{\beta+1} - T^{\beta+1}}{\beta+1}.$$

Thus, taking into account the above estimates for $\mathcal{M}_{\Omega\cup\Lambda}\tilde{f}$ and the conditions (3.3), (3.4), we obtain, by the very definition of $\mathbb{B}$,

$$\begin{aligned} &\mathbb{B}\left(\frac{Tx+Sc}{T+S}, (T+S)^{-1} \left(Ty + c^{q_1} \cdot \frac{(T+S)^{\alpha+1} - T^{\alpha+1}}{\alpha+1} \right), T+S \right) \\ &\qquad \geq \int_0^T t^\beta \left[(\mathcal{M}f)^*(t) \right]^{q_2} \mathrm{d}t + \left(\frac{Tx+Sc}{T+S} \right)^{q_2} \frac{(T+S)^{\beta+1} - T^{\beta+1}}{\beta+1}. \end{aligned}$$

Since (Ω, μ_Ω) was an arbitrary measure space and f was an arbitrary nonnegative function on Ω satisfying (3.2), we get the claim. □

In what follows, we will also need a certain homogeneity-type property of $\mathbb{B}$.

Lemma 3.2. *We have*

$$\mathbb{B}(x,y,T) = x^{q_2}T^{\beta+1}\varphi\left(\frac{y}{x^{q_1}T^\alpha}\right), \tag{3.5}$$

where $\varphi(s) = \mathbb{B}(1,s,1)$.

Proof. Fix an arbitrary measure space (Ω,μ) satisfying $\mu(\Omega)=T$ and an arbitrary function $f:\Omega\to[0,\infty)$ satisfying

$$\frac{1}{T}\int_0^T f^*(t)\mathrm{d}t = x, \qquad \frac{1}{T}\int_0^T t^\alpha\big[f^*(t)\big]^{q_1}\mathrm{d}t \le y.$$

Then for any $\lambda>0$, the function $\tilde{f}=\lambda f$ satisfies

$$\frac{1}{T}\int_0^T \tilde{f}^*(t)\mathrm{d}t = \lambda x, \qquad \frac{1}{T}\int_0^T t^\alpha\big[\tilde{f}^*(t)\big]^{q_1}\mathrm{d}t \le \lambda^{q_1}y$$

and

$$\int_0^T t^\beta\big[(\mathcal{M}\tilde{f})^*(t)\big]^{q_2}\mathrm{d}t = \lambda^{q_2}\int_0^T t^\beta\big[(\mathcal{M}f)^*(t)\big]^{q_2}\mathrm{d}t,$$

so by the very definition of $\mathbb{B}$ we obtain

$$\mathbb{B}(\lambda x,\lambda^{q_1}y,T) \ge \lambda^{q_2}\int_0^T t^\beta\big[(\mathcal{M}f)^*(t)\big]^{q_2}\mathrm{d}t.$$

Since Ω and f were arbitrary, this gives $\mathbb{B}(\lambda x,\lambda^{q_1}y,T)\ge\lambda^{q_2}\mathbb{B}(x,y,T)$. Replacing x, y, λ with λx, $\lambda^{q_1}y$ and λ^{-1}, respectively, we get the reverse bound. Consequently, we may write

$$\mathbb{B}(x,y,T) = x^{q_2}\mathbb{B}(1,y/x^{q_1},T). \tag{3.6}$$

Next, consider the space $(\Omega,\tilde{\mu}) := (\Omega,\mu/\lambda)$ with the same tree structure and let f be as above. We compute that

$$\frac{1}{\tilde{\mu}(\Omega)}\int_\Omega f\mathrm{d}\tilde{\mu} = x$$

and

$$\int_0^{T/\lambda} t^\alpha(f^*_{\tilde{\mu}}(t))^{q_1}\mathrm{d}t = \frac{\lambda^{-\alpha}}{T}\int_0^{T/\lambda} t^\alpha(f^*_{\tilde{\mu}}(t/\lambda))^{q_1}\mathrm{d}t = \frac{\lambda^{-\alpha}}{T}\int_0^T t^\alpha(f^*(t))^{q_1}\mathrm{d}t \le \lambda^{-\alpha}y.$$

Since $\mathcal{M}$ acts identically on the spaces (Ω,μ) and $(\Omega,\tilde{\mu})$, we have

$$\int_0^{T/\lambda} t^\beta((\mathcal{M}f)^*_{\tilde{\mu}}(t))^{q_2}\mathrm{d}t = \lambda^{-\beta-1}\int_0^T t^\beta((\mathcal{M}f)^*_\mu(t))^{q_2}\mathrm{d}t$$

and therefore, by the definition of $\mathbb{B}$,

$$\mathbb{B}(x,y/\lambda^\alpha,T/\lambda) \ge \lambda^{-\beta-1}\int_0^T t^\beta((\mathcal{M}f)^*_\mu(t))^{q_2}\mathrm{d}t.$$

Since f was arbitrary, we get $\mathbb{B}(x,y/\lambda^\alpha,T/\lambda)\ge\lambda^{-\beta-1}\mathbb{B}(x,y,T)$. Replacing y, T, λ with $y\lambda^{-\alpha}$, T/λ and λ^{-1}, we obtain the reverse estimate. Combining this with (3.6), we finally arrive at

$$\mathbb{B}(x,y,T) = x^{q_2}\mathbb{B}(1,y/x^{q_1},T) = x^{q_2}T^{\beta+1}\mathbb{B}(1,x^{-q_1}yT^{-\alpha},1),$$

which is the desired identity. □

To find the candidate for $\mathbb{B}$, we will exploit the "infinitesimal" version of the main inequality (3.1), which combined with the identity (3.5) will yield a certain ordinary differential inequality for φ. From now on we *assume* that $\mathbb{B}$ is of class C^1. We would like to stress that at this point we may impose any regularity assumption, since our main purpose is to *guess* the explicit formula; the rigorous verification will be postponed to the next section.

Lemma 3.3. *The function* $\varphi = \mathbb{B}(1,\cdot,1)$ *satisfies*

$$\varphi\left(\gamma^{q_1-1}\right) = \frac{q_1}{q_2\gamma} \tag{3.7}$$

and the differential inequality

$$(q_1-1)\left[\gamma - \left(\frac{s\varphi'(s) - \frac{q_2}{q_1}\varphi(s)}{\varphi'(s)}\right)^{1/(q_1-1)}\right]\left(s\varphi'(s) - \frac{q_2}{q_1}\varphi(s)\right) \geq 1. \tag{3.8}$$

Proof. To show (3.7), note that the class $\mathcal{C}(1,\gamma^{q_1-1},1)$ contains only one element: see the discussion below (2.1) (formally: all the elements from the class have the same nonincreasing rearrangements) and hence the Bellman function can be directly evaluated. We turn our attention to the differential inequality. Put $T = x = 1$ and rewrite (3.1) in the form

$$\frac{1}{S}\left[\mathbb{B}\left(\frac{1+Sc}{1+S},(1+S)^{-1}\left(y + c^{q_1}\cdot\frac{(1+S)^{\alpha+1}-1}{\alpha+1}\right),1+S\right) - \mathbb{B}(1,y,1)\right]$$
$$\geq \left(\frac{1+Sc}{1+S}\right)^{q_2}\frac{(1+S)^{\beta+1}-1}{(\beta+1)S}.$$

Letting $S\to 0$ (and using the assumption that $\mathbb{B}$ is of class C^1), we get the partial differential inequality

$$(-1+c)\mathbb{B}_x(1,y,1) + (-y+c^{q_1})\mathbb{B}_y(1,y,1) + \mathbb{B}_T(1,y,1) \geq 1, \tag{3.9}$$

or equivalently

$$(q_2\varphi(y) - q_1y\varphi'(y))(c-1) + \varphi'(y)(c^{q_1}-y) + (\beta+1)\varphi(y) - \alpha y\varphi'(y) \geq 1.$$

Since $q_2/q_1 = (1+\beta)/(1+\alpha)$, this can be rewritten in the form

$$(q_2\varphi(y) - q_1y\varphi'(y))\left(c + \frac{\alpha+1}{q_1} - 1\right) + \varphi'(y)c^{q_1} \geq 1.$$

This estimate holds for all c, we may optimize over this parameter. Putting

$$c = \left(y - \frac{q_2\varphi(y)}{q_1\varphi'(y)}\right)^{1/(q_1-1)},$$

we obtain the desired differential inequality. □

Now, let us *assume* that the differential inequality (3.8) is actually an equality. This leads us to the following candidate for the Bellman function. Namely, let φ be the solution of the differential *equation* (3.8) with the initial condition (3.7) (of course, we need to show that such a solution exists; this will be done below). Then the candidate B is obtained via the identity (3.5), i.e.,

$$B(x,y,T) = x^{q_2}T^{\beta+1}\varphi\left(\frac{y}{x^{q_1}T^{\alpha}}\right). \tag{3.10}$$

4. Proof of $\mathbb{B} \leq B$

We start the formal analysis by showing that B is well-defined. To this end, we need the rigorous definition of φ. This will be proved with the help of the following statement.

Lemma 4.1. *For any $s > \gamma^{q_1-1}$ there is a unique $u = u(s) \in (0, \gamma)$ which satisfies the identity*

$$\frac{q_2(q_1-1)}{q_2-q_1}\int_u^\gamma (\gamma - w)^{q_1/(q_2-q_1)}\, w^{q_1(q_1-1)/(q_2-q_1)+q_1-2}dw \\ = \left(s - u^{q_1-1}\right)(\gamma-u)^{q_1/(q_2-q_1)}\, u^{q_1(q_1-1)/(q_2-q_1)}. \tag{4.1}$$

Furthermore, $\lim_{s\to\gamma^{q_1-1}} u(s) = \gamma$ *and* $\lim_{s\to\infty} u(s) = 0$.

Proof. For a fixed s, consider the difference of the left- and the right-hand side as a function of $u \in (0,\gamma)$ and denote it by $F(u)$. A bit lengthy computation shows that

$$F'(u) = \frac{q_1}{q_2-q_1}(\gamma-u)^{q_2/(q_2-q_1)-1}\, u^{q_1(q_1-1)/(q_2-q_1)-1}G(u),$$

where $G(u) = s(q_1u - q_1 + 1 + \alpha) - u^{q_1}$. Since $G'(u) = q_1(s - u^{q_1-1})$, the function G is increasing on the interval $(0,\gamma)$. Note that $G(0) = s(-q_1+1+\alpha) < 0$ and

$$G(\gamma) = \gamma\left(s - \gamma^{q_1-1}\right) > 0,$$

so there is a unique u_0 such that the function G is negative on $(0, u_0)$ and positive on (u_0, γ). This implies that F decreases on $(0,u_0)$ and increases on (u_0,γ); since $F(0) > 0$ and $F(\gamma) = 0$, the existence of $u(s)$ is proved. The limiting behavior of this function as $s \to \gamma^{q_1-1}$ or $s \to \infty$ follows quickly from the definition (4.1). □

Letting $s\to\infty$ in (4.1) and using the fact that $u(s) \to 0$, we see that

$$\frac{q_2(q_1-1)}{q_2-q_1}\int_0^\gamma (\gamma-w)^{q_1/(q_2-q_1)}\, w^{q_1(q_1-1)/(q_2-q_1)+q_1-2}\mathrm{d}w \\ = \gamma^{q_1/(q_2-q_1)} \lim_{s\to\infty} su(s)^{q_1(q_1-1)/(q_2-q_1)},$$

or equivalently,

$$\lim_{s\to\infty} s^{q_2/q_1-1}u(s)^{q_1-1} \\ = \left[\frac{q_2(q_1-1)}{q_2-q_1}\cdot\frac{\Gamma\left(\frac{q_2}{q_2-q_1}\right)\Gamma\left(\frac{q_2(q_1-1)}{q_2-q_1}\right)}{\Gamma\left(\frac{q_1q_2}{q_2-q_1}\right)}\right]^{q_2/q_1-1}\gamma^{q_2(q_1-1)/q_1}, \tag{4.2}$$

by the properties of beta function. We are ready for the proof of the existence of the function φ.

Lemma 4.2. *There is an increasing function $\varphi : \left[\gamma^{q_1-1},\infty\right) \to \mathbb{R}$, satisfying the differential equation*

$$(q_1-1)\left[\gamma - \left(\frac{s\varphi'(s) - \frac{q_2}{q_1}\varphi(s)}{\varphi'(s)}\right)^{1/(q_1-1)}\right]\left(s\varphi'(s) - \frac{q_2}{q_1}\varphi(s)\right) = 1 \tag{4.3}$$

for $s > \gamma^{q_1-1}$ and the initial condition $\varphi\left(\gamma^{q_1-1}\right) = \frac{q_1}{q_2\gamma}$. Furthermore, we have $\varphi(s) \leq C^{q_2}_{p,q_1,q_2}s^{q_2/q_1}$ for all s.

Proof. Define φ by the formula

$$\varphi(s)=\frac{q_1(s-u^{q_1-1}(s))}{q_2(q_1-1)u^{q_1-1}(s)(\gamma-u(s))},\qquad s>\gamma^{q_1-1},$$

where u comes from the previous lemma. Some lengthy calculations show that

$$\varphi'(s)=\frac{1}{(q_1-1)u^{q_1-1}(s)(\gamma-u(s))}=\frac{\frac{q_2}{q_1}\varphi(s)}{s-u^{q_1-1}(s)}.$$

Consequently, we have $u(s)=(s-\frac{q_2}{q_1}\varphi(s)/\varphi'(s))^{1/(q_1-1)}$ and (4.3) follows. To prove the initial condition, recall that by the previous lemma,

$$\lim_{s\to\gamma^{q_1-1}}u(s)=\gamma$$

and hence, by the definitions of φ and u,

$$\lim_{s\to\gamma^{q_1-1}}\varphi(s)=\lim_{s\to\gamma^{q_1-1}}\frac{q_1\int_u^\gamma(\gamma-w)^{q_1/(q_2-q_1)}\,w^{q_1(q_1-1)/(q_2-q_1)+q_1-2}\mathrm{d}w}{(q_2-q_1)\,(\gamma-u(s))^{q_2/(q_2-q_1)}\,(u(s))^{q_2(q_1-1)/(q_2-q_1)}}=\frac{q_1}{q_2\gamma},$$

where in the last line we have used de l'Hospital rule. Finally, to establish the majorization $\varphi(s)\leq C_{p,q_1,q_2}^{q_2}s^{q_2/q_1}$, one easily shows that the function $s\mapsto\varphi(s)/s^{q_2/q_1}$ is increasing and converges to $C_{p,q_1,q_2}^{q_2}$ as $s\to\infty$. Indeed, by differentiation, the monotonicity follows from the estimate $\varphi'(s)s\geq\frac{q_2}{q_1}\varphi(s)$ (which obviously holds), and the formula for the limit is a consequence of the definition of φ and the identity (4.2). □

Thus we have shown that the candidate B given by (3.10) is well-defined. We turn our attention to its properties.

Lemma 4.3. *We have*

$$B_x(x,y,T)\cdot\frac{c-x}{T}+B_y(x,y,T)\cdot\frac{c^{q_1}T^\alpha-y}{T}+B_T(x,y,T)\geq x^{q_2}T^\beta. \tag{4.4}$$

Proof. We will use certain formulas obtained in the previous section. First, note that we have the following analogue of (3.9):

$$(-1+c)B_x(1,s,1)+(-s+c^{q_1})B_y(1,s,1)+B_T(1,s,1)\geq 1. \tag{4.5}$$

To show this, observe that $B_y(1,s,1)>0$ (since φ is an increasing function) and

$$B_x(1,s,1)=q_2\varphi(s)-q_1s\varphi'(s)\leq 0.$$

Hence the expression on the left of (4.5), considered as a function of $c\geq 0$, attains its minimum at $c=((-B_x(1,s,1)/(q_1B_y(1,s,1))))^{1/(q_1-1)}$. But this minimal value is equal to 1: this is equivalent to the differential equation (4.3), as we have already checked in the previous section. Hence (4.5) holds; replacing c with cx, we get

$$(-1+cx)B_x(1,s,1)+(-y+c^{q_1}x^{q_1})B_y(1,s,1)+B_T(1,s,1)\geq 1. \tag{4.6}$$

Put $s=x^{-q_1}yT^{-\alpha}$. It follows directly from the definition of B that

$$B_x(x,y,T)=x^{q_2-1}T^{\beta+1}B_x(1,x^{-q_1}yT^{-\alpha},1),$$

$$B_y(x,y,T)=x^{q_2-q_1}T^{\beta+1-\alpha}B_y(1,x^{-q_1}yT^{-\alpha},1)$$

and

$$B_T(x,y,T)=x^{q_2}T^\beta B_T(1,x^{-q_1}yT^{-\alpha},1).$$

Combining these identities with (4.6) yields the claim. Let us also record that if

$$c = \left(-\frac{B_x(x,y,T)}{q_1 T^\alpha B_y(x,y,T)}\right)^{1/(q_1-1)}, \tag{4.7}$$

then both sides if (4.4) are equal. This follows from the proof above. □

Now we will show that B satisfies the following main inequality.

Lemma 4.4. *For any $S, T > 0$, any x, y and $c \in [0, x]$ we have*

$$\begin{aligned} &B\left(\frac{Tx+Sc}{T+S}, (T+S)^{-1}\left(Ty + c^{q_1}\cdot\frac{(T+S)^{\alpha+1}-T^{\alpha+1}}{\alpha+1}\right), T+S\right)\\ &\geq B(x,y,T) + \left(\frac{Tx+Sc}{T+S}\right)^{q_2}\frac{(T+S)^{\beta+1}-T^{\beta+1}}{\beta+1}. \end{aligned} \tag{4.8}$$

Proof. Define auxiliary functions X, $Y : [T, S+T] \to [0,\infty)$ by the formulas

$$X(t) = \frac{Tx+(t-T)c}{t}, \qquad Y(t) = \frac{1}{t}\left(Ty + c^{q_1}\cdot\frac{t^{\alpha+1}-T^{\alpha+1}}{\alpha+1}\right).$$

We compute that

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}B(X(t),Y(t),t)\\ &= B_x(X(t),Y(t),t)\cdot\frac{T(c-x)}{t^2}\\ &\quad + B_y(X(t),Y(t),t)\cdot\left(-\frac{Ty+(\alpha+1)^{-1}c^{q_1}(t^{\alpha+1}-T^{\alpha+1})}{t^2} + c^{q_1}t^{\alpha-1}\right)\\ &\quad + B_T(X(t),Y(t),t). \end{aligned} \tag{4.9}$$

However, by (4.4), the expression

$$B_x(X(t),Y(t),t)\cdot\frac{c-X(t)}{t} + B_y(X(t),Y(t),t)\cdot\frac{c^{q_1}t^\alpha - Y(t)}{t} + B_T(X(t),Y(t),t)$$

is not smaller than $X(t)^{q_2}t^\beta$. In addition, we have

$$\frac{c-X(t)}{t} = \frac{T(c-x)}{t^2}$$

and

$$\frac{c^{q_1}t^\alpha - Y(t)}{t} = -\frac{Ty+(\alpha+1)^{-1}c^{q_1}(t^{\alpha+1}-T^{\alpha+1})}{t^2} + c^{q_1}t^{\alpha-1},$$

so by (4.9), we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}B(X(t),Y(t),t) \geq X(t)^{q_2}t^\beta \geq \left(\frac{Tx+Sc}{T+S}\right)^{q_2}t^\beta.$$

Here in the last line we have used the inequality $X(t) \geq (Tx+Sc)/(T+S)$, which is a direct consequence of the assumption $c \leq x$. This proves that

$$B(X(T+S),Y(T+S),T+S) \geq B(X(T),Y(T),T) + \int_T^{T+S}\left(\frac{Tx+Sc}{T+S}\right)^{q_2}t^\beta \mathrm{d}t,$$

and it remains to use the identities $(X(T), Y(T), T) = (x, y, T)$ and

$$\begin{aligned}&(X(T+S), Y(T+S), T+S)\\&= \left(\frac{Tx+Sc}{T+S}, (T+S)^{-1}\left(Ty + c^{q_1}\cdot\frac{(T+S)^{\alpha+1}-T^{\alpha+1}}{\alpha+1}\right), T+S\right).\end{aligned}$$

The claim is established. □

Remark 4.5. Later on, we will need to know when both sides of (4.8) are almost equal. Of course, this is true if we let $S \to 0$, but actually the reasoning from the previous section allows to extract an „infinitesimal" version of this statement: namely, if $S \to 0$ and we let

$$c = \left(\frac{-B_x(X(T), Y(T), T)}{q_1 T^\alpha B_y(X(T), Y(T), T)}\right)^{1/(q_1-1)},$$

then the difference of the left- and the right-hand side is of order $o(S)$. This follows from the proof of Lemma 4.3.

We are ready for the proof of the key estimate.

Proof of $\mathbb{B} \leq B$. Let (Ω, μ) be an arbitrary measure space with $\mu(\Omega) = T$ and let $f : \Omega \to [0, \infty)$ be a measurable function belonging to the class $\mathcal{C}(x, y, T)$.

Step 1. Reductions. If equality holds in (2.1), then there is nothing to prove: we already know that $\mathbb{B} = B$ at such point. So, suppose that we have strict inequality in (2.1); then by a simple approximation, we may assume that $\frac{1}{T}\int_0^T t^\alpha [f^*(t)]^{q_1}\mathrm{d}t < y$. Next, we replace f by an appropriate simple function. To this end, let N be a huge integer and let $g = g^N$ be the conditional expectation of f with respect to $\mathcal{T}^N$: that is, g is constant on each element Q of $\mathcal{T}^N$ and equal to $\frac{1}{\mu(Q)}\int_Q f\mathrm{d}\mu$ there. Clearly, g has the same average as f; furthermore, by Doob's martingale convergence theorem (and the assumption (iv) on the tree), we have $g^N \to f$ μ-almost everywhere and hence also $\|g^N\|_{L^{p,q_1}} \to \|f\|_{L^{p,q_1}}$; thus in particular $g^N \in \mathcal{C}(x, y, T)$ provided N is large enough. Furthermore, $\mathcal{M}g^N \leq \mathcal{M}f$ and $\mathcal{M}g^N \uparrow \mathcal{M}f$. Thus, an upper estimate for $\|\mathcal{M}g^N\|_{L^{p,q_2}}$ will also imply the same bound for $\|\mathcal{M}f\|_{L^{p,q_2}}$. So, let N be fixed. Our final reduction is that we may assume that $g^N = \sum_{k=1}^M a_k\chi_{A_k}$ for some pairwise disjoint sets A_k of the same measure $\mu(\Omega)/M = T/M$: this can be seen by modifying the generation $\mathcal{T}^N$ so that its elements have (almost) equal measures and discarding the generations $\mathcal{T}^{N+1}$, $\mathcal{T}^{N+2}$, ... From now on, we will write g instead of g^N. We need to prove that

$$\int_0^T t^\beta [(\mathcal{M}g)^*(t)]^{q_2}\mathrm{d}t \leq B(x, y, T). \tag{4.10}$$

This will be done by induction.

Step 2. Proof of (4.10) *for* $M = 1$. Then both g and $\mathcal{M}g$ are constant and equal to x on Ω and, in addition,

$$y \geq \frac{1}{T}\int_0^T t^\alpha (g^*(t))^{q_1}\mathrm{d}t = c^{q_1}T^\alpha/(\alpha+1). \tag{4.11}$$

Note that $B \geq 0$, so (4.8) implies

$$B\left(\frac{Tx+Sc}{T+S}, (T+S)^{-1}\left(Ty + c^{q_1}\cdot\frac{(T+S)^{\alpha+1}-T^{\alpha+1}}{\alpha+1}\right), T+S\right) \geq \left(\frac{Tx+Sc}{T+S}\right)^{q_2}\frac{(T+S)^{\beta+1}-T^{\beta+1}}{\beta+1}.$$

So, letting $T \to 0$ we get, by the continuity of B,

$$B(c, c^{q_1}S^{\alpha}/(\alpha+1), S) \geq c^{q_2}S^{\beta+1}/(\beta+1).$$

Now replace c with x, S with T and use the inequality (4.11) together with the monotonicity of B with respect to the variable y to get

$$x^{q_2}T^{\beta+1}/(\beta+1) \leq B(x,y,T).$$

This is precisely (4.10) (for $M=1$).

Step 3. Induction step. It follows from the weak-type inequality for $\mathcal{M}$ that there exists $k \in \{1,2,\ldots,M\}$ such that $\mathcal{M}g = \frac{1}{\mu(\Omega)}\int_\Omega f\mathrm{d}\mu = x$ on A_k. Consider the space $\tilde{\Omega} = \Omega \setminus A_k$ equipped with the restricted measure μ and the tree $\tilde{\mathcal{T}}$ which consists of all sets of the form $A \setminus A_k$, $A \in \mathcal{T}$, provided that the difference is nonempty. Denote the associated maximal operator by $\tilde{\mathcal{M}}$.

Obviously, there is an index m such that $g = \min g$ on A_m. If $k \neq m$, then we replace g with

$$\tilde{g} = a_k\chi_{A_m} + a_m\chi_{A_k} + \sum_{r\notin\{k,m\}} a_r\chi_{A_r},$$

i.e., we switch the values of g at the sets A_k and A_m. Since $\mu(A_k) = \mu(A_m)$, this modification does not change the nonincreasing rearrangement of g. On the other hand, note that on A_k we have

$$\mathcal{M}\tilde{g} \geq \frac{1}{\mu(\Omega)}\int_\Omega \tilde{g}\mathrm{d}\mu = \frac{1}{\mu(\Omega)}\int_\Omega g\mathrm{d}\mu = \mathcal{M}g.$$

Furthermore, we have

$$\tilde{M}\tilde{g} \geq \mathcal{M}g \quad \text{on } \Omega\setminus A_k. \tag{4.12}$$

Indeed, suppose that $u \in \Omega \setminus A_k$ and let A be the element of $\mathcal{T}$ containing u such that

$$\mathcal{M}g = \frac{1}{\mu(A)}\int_A g\mathrm{d}\mu. \tag{4.13}$$

There may be many sets A with this property; if this is the case, we choose A which belongs to $\mathcal{T}^j$ with j as small as possible. If $A \cap A_k = \emptyset$, then $\tilde{g} \geq g$ on A and hence $\tilde{\mathcal{M}}\tilde{g}(u) \geq \frac{1}{\mu(A)}\int_A \tilde{g}\mathrm{d}\mu \geq \mathcal{M}g(u)$. On the other hand, if $A_k \subset A$, then $\frac{1}{\mu(A)}\int_A g\mathrm{d}\mu \leq x$, by the very definition of A_k, and hence we must actually have equality: see (4.13). Hence

$$\tilde{\mathcal{M}}\tilde{g}(u) \geq \frac{1}{\mu(\Omega\setminus A_k)}\int_{\Omega\setminus A_k}\tilde{g}\mathrm{d}\mu \geq \frac{1}{\mu(\Omega)}\int_\Omega g\mathrm{d}\mu = x = \mathcal{M}g(u)$$

and the desired majorization is established. Note that we may apply induction hypothesis to $\tilde{g}$, obtaining

$$\int_0^{T(M-1)/M} t^\beta\left[(\tilde{\mathcal{M}}\tilde{g})^*(t)\right]^{q_2}\mathrm{d}t \leq B\left(\tilde{x}, \tilde{y}, T(M-1)/M\right),$$

where

$$\tilde{x} = \frac{1}{\mu(\tilde{\Omega})}\int_{\tilde{\Omega}} \tilde{g}\mathrm{d}\mu, \qquad \tilde{y} = \frac{1}{\mu(\tilde{\Omega})}\int_0^{\mu(\tilde{\Omega})} t^\alpha(\tilde{g}^*(t))^{q_1}\mathrm{d}t.$$

Hence

(4.14)

$$\begin{aligned}
&\int_0^T t^\beta \left[(\mathcal{M}g)^*(t)\right]^{q_2}\mathrm{d}t \\
&= \int_0^{T(M-1)/M} t^\beta \left[(\mathcal{M}g)^*(t)\right]^{q_2}\mathrm{d}t + \int_{T(M-1)/M}^T t^\beta \left[(\mathcal{M}g)^*(t)\right]^{q_2}\mathrm{d}t \\
&\leq \int_0^{T(M-1)/M} t^\beta \left[(\tilde{\mathcal{M}}\tilde{g})^*(t)\right]^{q_2}\mathrm{d}t + x^{q_2}(\beta+1)^{-1}T^{\beta+1}\left(1-\left(\frac{M-1}{M}\right)^{\beta+1}\right) \\
&\leq B(\tilde{x},\tilde{y},T(M-1)/M) + x^{q_2}(\beta+1)^{-1}T^{\beta+1}\left(1-\left(\frac{M-1}{M}\right)^{\beta+1}\right).
\end{aligned}$$

In the light of (4.8) (applied with $x := \tilde{x}$, $y := \tilde{y}$, $S := T/M$, $T := T(M-1)/M$ and $c := a_m = \min g$), the latter expression is not bigger than $B(x, y, T)$. This completes the proof of (4.10) and the inequality $\mathbb{B} \leq B$ follows. □

Proof of (1.4). Take any measure space (Ω, μ), any measurable function $f : \Omega \to \mathbb{R}$ and set

$$T = \mu(\Omega), \qquad x = \frac{1}{T}\int_0^T f^*(t)\mathrm{d}t, \qquad y = \frac{1}{T}\int_0^T t^\alpha(f^*(t))^{q_1}\mathrm{d}t.$$

Then by Lemma 4.2,

$$\begin{aligned}
\|\mathcal{M}f\|^{q_2}_{L^{p,q_2}(\Omega,\mu)} &= \int_0^T t^\beta[(\mathcal{M}f(t))^*]^{q_2}\mathrm{d}t \\
&\leq B(x,y,T) \leq x^{q_2}T^{\beta+1}\cdot C^{q_2}_{p,q_1,q_2}\left(\frac{y}{x^{q_1}T^\alpha}\right)^{q_2/q_1} = C^{q_2}_{p,q_1,q_2}\|f\|^{q_2}_{L^{p,q_1}}.
\end{aligned}$$

This completes the proof. □

5. The inequality $\mathbb{B} \geq B$

It is convenient to split the reasoning into two parts.

5.1. **On the search of the extremizer.** First we will sketch some steps which lead to the discovery of extremal function. Let us emphasize here that the argumentation will be informal and brief, its purpose is to discover the formula for the nonincreasing rearrangement of the appropriate function. From the formal point of view, the reader might skip this subsection and proceed to the next one; however, we believe that the contents of this subsection is helpful as it explains the origins of the complicated formulas which will appear later. The idea is very simple: we will inspect carefully the above proof of the inequality $\mathbb{B} \leq B$ and try to find a function g for which all the inequalities become (almost) equalities. Fix a huge integer N (it will be sent to infinity in a moment). First, we will consider a special measure space (Ω, μ): the interval $(0, 1]$ with the Lebesgue measure, and equip it with the tree $\mathcal{T}$, where for any $0 \leq n \leq N$, the family $\mathcal{T}^n$ contains the intervals $(0,(N-n)/N]$, $((N-n)/N,(N-n+1)/N]$, ..., $(1-1/N, 1]$. In what follows, we will assume that g is a nonincreasing function. Then $\mathcal{M}g$ also has this property, and hence the function $\tilde{g}$, appearing in the proof of $\mathbb{B} \leq B$, coincides with g on its domain (therefore in (4.12) we will have equality). Thus the only inequalities which must be turned into (almost) equalities is the last passage in (4.14) and the fact that the final expression in (4.14) is not bigger than $B(x, y, T)$. Let us look at the second estimate: to see when both sides become almost equal, we go back to Remark 4.5. This statement suggests that on the interval $(m/N, (m+1)/N]$, g should equal

$$\left(\frac{-B_x(X(m/N), Y(m/N), m/N)}{q_1(m/N)^\alpha B_y(X(m/N), Y(m/N), m/N)}\right)^{1/(q_1-1)} = X(m/N)\left(s - \frac{q_2\varphi(s)}{q_1\varphi'(s)}\right)^{1/(q_1-1)} = X(m/N)u(s),$$

where

$$X(m/N) = \frac{1}{m/N}\int_0^{m/N} g(t)\mathrm{d}t, \qquad Y(m/N) = \frac{1}{m/N}\int_0^{m/N} t^\alpha g(t)^{q_1}\mathrm{d}t$$

and $s = X^{-q_1}(m/N)Y(m/N)(m/N)^{-\alpha}$. Now let $N \to \infty$: we obtain that for any $t \in (0, 1]$, we should have

$$\xi(t) := \frac{g(t)}{\frac{1}{t}\int_0^t g(r)\mathrm{d}r} = u\left(\left(\frac{1}{t}\int_0^t g(r)\mathrm{d}r\right)^{-q_1}\left(\frac{1}{t}\int_0^t r^\alpha g(r)^{q_1}\mathrm{d}r\right)t^{-\alpha}\right). \tag{5.1}$$

Plug this into the definition of u: we get

$$\frac{q_2(q_1-1)}{q_2-q_1}\int_{\xi(t)}^{\gamma}(\gamma - w)^{q_1/(q_2-q_1)}\, w^{q_1(q_1-1)/(q_2-q_1)+q_1-2}\mathrm{d}w = \left(\frac{Y(t)}{t^\alpha X(t)} - \xi(t)^{q_1-1}\right)(\gamma - \xi(t))^{q_1/(q_2-q_1)}\,\xi(t)^{q_1(q_1-1)/(q_2-q_1)}.$$

Now we differentiate both sides with respect to t. After some lengthy and tedious computations, we get the equivalent equality $I \cdot II = 0$, where

$$I = \xi'(t) + \frac{q_2-q_1}{q_1}\cdot\frac{\xi(t)\,(\gamma - \xi(t))}{t}$$

and II is a certain complicated expression. Assuming that the term I vanishes, we obtain a simple differential equation for ξ, whose general solution is

$$\xi(t) = \gamma\left(1 + dt^{\gamma\frac{q_2-q_1}{q_1}}\right)^{-1}.$$

Here d is an arbitrary real number. Having identified ξ, we easily find X and g: since $X'(t) = g(t)/t - X(t)/t$, (5.1) implies

$$\frac{\mathrm{d}}{\mathrm{d}t}X(t) = -\frac{X(t)}{t} + \frac{X(t)\xi(t)}{t}.$$

This is easily solved:

$$X(t) = ct^{-\alpha/(q_1-1)}\left(1 + dt^{\gamma\frac{q_2-q_1}{q_1}}\right)^{-q_1/(q_2-q_1)},$$

(where c is an arbitrary number) and hence we obtain the following candidate for the extremizer:

$$g(t) = ct^{-\alpha/(q_1-1)}\left(1 + dt^{\gamma\frac{q_2-q_1}{q_1}}\right)^{-q_2/(q_2-q_1)}. \tag{5.2}$$

Now, we can choose c and d so that

$$\int_0^1 g(t)\mathrm{d}t = x \quad \text{and} \quad \int_0^1 t^\alpha (g(t))^{q_1}\mathrm{d}t = y. \tag{5.3}$$

Indeed: we compute that

$$R(d) := \frac{\int_0^1 t^\alpha (g(t))^{q_1}\mathrm{d}t}{\left(\int_0^1 g(t)\mathrm{d}t\right)^{q_1}} = \frac{\int_0^1 t^{-\alpha/(q_1-1)}\left(1+dt^{\gamma\frac{q_2-q_1}{q_1}}\right)^{-q_2q_1/(q_2-q_1)}\mathrm{d}t}{\left(\int_0^1 t^{-\alpha/(q_1-1)}\left(1+dt^{\gamma\frac{q_2-q_1}{q_1}}\right)^{-q_2/(q_2-q_1)}\mathrm{d}t\right)^{q_1}}$$

is a continuous function of $d \in [0,\infty)$ and

$$R(0) = \gamma, \qquad \lim_{d\to\infty} R(d) = \infty.$$

Therefore, there is d for which $R(d) = y/x^{q_1}$, and then we choose c so that $\int_0^1 g = x$.

5.2. **A formal verification.** Now we can prove rigorously the bound $\mathbb{B}(x,y,T) \geq B(x,y,T)$. By homogeneity, we may assume that $T = 1$: that is, we assume that (Ω,μ) is a probability space. We repeat the above arguments in the reverse direction. Let g be given by (5.2), where c, d are chosen so that (5.3) holds. Then a careful inspection of the above arguments (or a direct calculation) shows that the function

$$G(t) := B\left(\frac{1}{t}\int_0^t g(r)\mathrm{d}r, \frac{1}{t}\int_0^t r^\alpha (g(r))^{q_1}\mathrm{d}r, t\right) + \int_t^1 r^\beta \left(\frac{1}{r}\int_0^r g(w)\mathrm{d}w\right)^{q_2}\mathrm{d}r$$

is constant. We have $G(1) = B(x,y,1)$; let us check how G behaves in the neighborhood of 0. Note that

$$B\left(\frac{1}{t}\int_0^t g(r)\mathrm{d}r, \frac{1}{t}\int_0^t r^\alpha (g(r))^{q_1}\mathrm{d}r, t\right) = \left(\frac{1}{t}\int_0^t g(r)\mathrm{d}r\right)^{q_2} t^{\beta+1}\varphi(s),$$

where

$$s = \left(\frac{1}{t}\int_0^t r^\alpha g(r)^{q_1}\mathrm{d}r\right)\left(\frac{1}{t}\int_0^t g(r)\mathrm{d}r\right)^{-q_1} t^{-\alpha}.$$

Now if we let $t \to 0$, then $s \to \gamma^{q_1-1}$ as $t \to 0$, and the factor

$$\left(\frac{1}{t}\int_0^t g(r)\mathrm{d}r\right)^{q_2} t^{\beta+1}$$

converges to zero. Therefore

$$\lim_{t\to 0} G(t) = \int_0^1 r^\beta \left(\frac{1}{r}\int_0^r g(w)\mathrm{d}w\right)^{q_2}\mathrm{d}r$$

and hence we have proved that

$$\int_0^1 r^\beta \left(\frac{1}{r}\int_0^r g(w)\mathrm{d}w\right)^{q_2}\mathrm{d}r = B(x,y,1).$$

Now we return to the general context. Let (Ω,μ) be a nonatomic probability space equipped with an arbitrary tree structure $\mathcal{T}$. The idea is very simple: we will construct a random variable f such that the distributions of f and g coincide, while the distributions of $\mathcal{M}f$ and the function $t \mapsto \frac{1}{t}\int_0^t g$ are arbitrarily close. Let us recall a notion which is frequently used in probability theory.

Definition 5.1. Suppose that f_1, f_2 are two measurable functions on some measure spaces (Ω_i, μ_i) with $\mu_i(\Omega_i) > 0$, $i = 1, 2$.

(i) Suppose that $\mu_1(\Omega_1) = \mu_2(\Omega_2)$. The measurable functions $f_1 : \Omega_1 \to \mathbb{R}$ and $f_2 : \Omega_2 \to \mathbb{R}$ are said to have the same distribution, if their nonincreasing rearrangements coincide: $f^* = g^*$.

(ii) Without the assumption $\mu_1(\Omega_1) = \mu_2(\Omega_2)$, the functions f_1 and f_2 are said to have the same *conditional* distribution, if their nonincreasing rearrangements, with respect to the normalized measures $\mu_1/\mu_1(\Omega_1)$, $\mu_2/\mu_2(\Omega_2)$, coincide.

We will freely use the fact that if (Ω_1, μ_1), (Ω_2, μ_2) are nonatomic measure spaces with $\mu_i(\Omega_i) > 0$, $i = 1, 2$, then for any measurable function f_1 on Ω_1, there exists a measurable function f_2 on Ω_2 with the same conditional distribution. See [4, p. 65] or Lemma 2.3 in [13].

We will also need the following simple structural fact proved in [5].

Lemma 5.2. *For every $Q \in \mathcal{T}$ and every $\beta \in (0, 1)$ there is a subfamily $F(Q) \subset \mathcal{T}$ consisting of pairwise almost disjoint subsets of Q such that*

$$\mu\left(\bigcup_{R \in F(Q)} R\right) = \sum_{R \in F(Q)} \mu(R) = \beta\mu(Q).$$

Now we proceed to the construction. Let $N \geq 2$ be a fixed integer.

Step 1. First we use Lemma 5.2 inductively, in order to construct an appropriate nonincreasing sequence $\Omega = E_0 = E_1 \supset E_2 \supset \ldots \supset E_{N-1} \supset E_N = \emptyset$ with $\mu(E_j) = 1 - j/N$ for all j, possessing an additional fractal structure. This sequence corresponds to the sequence $[0, 1) \supset [0, 1-N^{-1}) \supset [0, 1-2N^{-1}) \supset \ldots \supset [0, N^{-1}) \supset \emptyset$ which appears in the above analysis of g. We use the following recursive argument. We set $E_0 = \Omega$. Suppose that we have constructed the set E_j, for some $j \in \{0, 1, 2, \ldots, N-2\}$, which is a union of a pairwise almost disjoint family $\mathcal{Q}^j \subset \mathcal{T}$ (clearly, E_0 has this property, as it itself belongs to $\mathcal{T}$). Then, for each element $Q \in \mathcal{Q}^j$, we apply Lemma 5.2 with the parameter $\beta = 1 - (N-j)^{-1}$, obtaining an appropriate family $F(Q)$. Then we set $\mathcal{Q}^{j+1} = \bigcup_{Q \in \mathcal{Q}^j} F(Q)$ and define E_{j+1} as the union of all elements from $\mathcal{Q}^{j+1}$: then

$$\mu(E_{j+1}) = \sum_{Q \in \mathcal{Q}^j} \mu\left(\bigcup F(Q)\right) = (1 - (N-j)^{-1})\mu(E_j) = 1 - \frac{j+1}{N}$$

and the sets from $\mathcal{Q}^{j+1}$ are pairwise almost disjoint. This procedure gives us the nonincreasing sequence $(E_j)_{j=0}^N$ as above. Note that we can actually prove a stronger fractal property. Namely, by the construction, for any $Q \in \mathcal{Q}^j$ we have $\mu(Q \cap E_{j+1}) = \mu(Q) \cdot (1 - (N-j)^{-1})$ and this implies, for each $k \geq j$,

$$\frac{\mu(Q \cap E_k)}{\mu(Q)} = \frac{N-k}{N-j} = \frac{|[0, 1-k/N)|}{|[0, 1-j/N)|}. \tag{5.4}$$

The straightforward inductive proof with respect to k is left to the reader.

Step 2. By the construction, for each j the set $E_j \setminus E_{j+1}$ is the union of pairwise almost disjoint sets $Q \setminus E_{j+1}$, $Q \in \mathcal{Q}^j$. Let $f : \Omega \to \mathbb{R}$ be a function whose distribution is uniquely determined by the following requirement: for any j and any $Q \in \mathcal{Q}^j$, the function f restricted to $Q \setminus E_{j+1}$ and the function g restricted to $[1 - (j+1)/N, 1 - j/N)$ have the same conditional distributions. Hence, if we fix j and sum over all $Q \in Q^j$, we see that the distribution of f restricted to

$E_j \setminus E_{j+1}$ and the distribution of g restricted to $[1-(j+1)/N, 1-j/N)$ coincide. Consequently, f and g have the same distribution and hence $f \in \mathcal{C}(x,y,1)$.

It remains to handle the maximal function $\mathcal{M}f$, and this is the place where the fractal properties will be used. An important observation is that for any j and any $Q \in \mathcal{Q}^j$ the distribution of f restricted to Q and the distribution of g restricted to $[0, 1-j/N)$ conditionally coincide; this follows from (5.4). So, in particular,

$$\frac{1}{\mu(Q)} \int_Q f \mathrm{d}\mu = \frac{N}{N-j} \int_{[0,1-j/N)} g(r)\mathrm{d}r.$$

Consequently, by the definition of the maximal function, we obtain

$$\mathcal{M}f \geq \frac{N}{N-j} \int_{[0,1-j/N)} g(r)\mathrm{d}r \qquad \text{on } Q,$$

and since $Q \in \mathcal{Q}^j$ was arbitrary, the above estimate holds on the whole E_j. By the very definition of the nonincreasing rearrangement, this yields

$$(\mathcal{M}f)^*(t) \geq \frac{1}{t+N^{-1}} \int_0^{t+N^{-1}} g(r)\mathrm{d}r,$$

since $\mu(E_j) = 1 - j/N$. Therefore,

$$\int_0^1 t^\beta \left[(\mathcal{M}f)^*(t)\right]^{q_2} \mathrm{d}t \geq \int_0^1 t^\beta \left(\frac{1}{t+N^{-1}} \int_0^{t+N^{-1}} g(r)\mathrm{d}r \right)^{q_2} \mathrm{d}r.$$

By Lebesgue's monotone convergence theorem, the expression on the right converges, as $N \to \infty$, to

$$\int_0^1 t^\beta \left(\frac{1}{t} \int_0^t g(r)\mathrm{d}r \right)^{q_2} \mathrm{d}t = B(x,y,1).$$

This, by the very definition of $\mathbb{B}$, shows that $\mathbb{B}(x,y,1) \geq B(x,y,1)$ and completes the proof.

6. On an alternative proof of (1.4)

It was pointed out by the Referee that the Lorentz-norm estimate (1.4) can be established directly, without referring to the Bellman function method. The purpose of this section is to sketch briefly the main steps of the argumentation.

We start from an observation concerning the weak-type bound (1.1). Namely, it is well-known that the probabilistic version of this estimate is equivalent to

$$(\mathcal{M}_\mathcal{T}\phi)^*(t) \leq \frac{1}{t} \int_0^t \phi^*(s)\mathrm{d}s \qquad \text{for all } t \in (0,1].$$

This inequality is extremely sharp: as we have seen above, for any nonincreasing and integrable function g and any probability space (Ω, μ) equipped with a tree $\mathcal{T}$, there exists a random variable f such that the distributions of f and g coincide, while the distributions of $\mathcal{M}_\mathcal{T} f$ and $t \mapsto \frac{1}{t}\int_0^t g$ are as close as we wish. (For the probabilistic version of this sharpness, see Dubins and Gilat [3]). This observation allows to reduce the problem of finding the sharp constant in (1.4) to the question about the best constant in a modified Hardy's inequality

$$\left(\int_0^1 t^{1/p} \left(\frac{1}{t} \int_0^t g(s)\mathrm{d}s \right)^{q_1} \frac{\mathrm{d}t}{t} \right)^{1/q_1} \leq C_{p,q_1,q_2} \left(\int_0^1 t^{1/p} g^{q_2}(t) \frac{\mathrm{d}t}{t} \right)^{1/q_2} \tag{6.1}$$

tested against non-increasing functions g; the two constants coincide. The complete analysis of the latter estimate, for the full range of parameters p, q_1, q_2 can be found, for example, in the paper by Persson and Samko [16]. Interestingly, they studied the inequality for general (i.e., not necessarily monotone) functions and it turns out that the extremizers are nonincreasing if and only if $q_1 \leq q_2$. In other words, both approaches - exploiting the Bellman function and that above - allow to obtain the sharp version of the estimate (1.4) only in this limited range of q_1 and q_2.

Several comments are in order. The proof of the estimate (6.1) presented in [16] rests on a number of clever observations and substitutions which reduce the claim to the classical Bliss' inequality

$$\left(\int_0^\infty \left(\int_0^x g(t)\mathrm{d}t\right)^q x^{-q/p'-1}\mathrm{d}x\right)^{1/q} \leq c_{p,q}\left(\int_0^\infty g^p(x)\mathrm{d}x\right)^{1/p},$$

for $1 < p \leq q < \infty$. This estimate was established in [1] with the use of the calculus of variations (see also [15] for an alternative proof). We would like to emphasize that the reasoning presented in this paper - i.e., the explicit formula for the Bellman function $\mathbb{B}$ - gives more information about the action of maximal operators on Lorentz spaces: it yields the sharp bound for $\|\mathcal{M}f\|_{L^{p,q_2}}$ assuming that $\|f\|_{L^{p,q_1}}$ and $\|f\|_{L^1}$ are known. In addition, we strongly believe that our approach to (1.4) is of independent interest and connections. To the best of our knowledge, this is the first time when the Bellman function method has been successfully applied in the study of Lorentz-norm estimates. The approach seems very flexible, which might enable the unified treatment of related inequalities in other important contexts of harmonic analysis.

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