INEQUALITIES FOR MARTINGALES TAKING VALUES IN 2-CONVEX BANACH SPACES

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Abstract. We study sharp square function inequalities for martingales taking values in a 2-convex Banach space $\mathbb{B}$. We show that an appropriate weak-type bound holds true if and only if $\mathbb{B}$ is isometric to a Hilbert space.

1. Introduction

As evidenced in numerous papers, martingale theory is a convenient tool in the investigation of the structure and the geometry of Banach spaces. See e.g. [2], [3], [6], [7], [9], [10] and references therein. The purpose of this paper is to establish a novel characterization of Hilbert spaces in terms of a certain square function inequality.

We start with introducing the necessary background and notation. Assume that $(\mathbb{B}, ||\cdot||)$ is a real or complex Banach space and let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, equipped with $(\mathcal{F}_n)_{n\geq 0}$, a nondecreasing sequence of sub-σ-algebras of $\mathcal{F}$. Let $f = (f_n)_{n\geq 0}$ be an adapted martingale taking values in $\mathbb{B}$, with the corresponding difference sequence $d = (d_n)_{n\geq 0}$ given by $d_0 = f_0$ and $d_n = f_n - f_{n-1}$ for $n \geq 1$. We say that f is conditionally symmetric if for each $n \geq 1$, the conditional distributions of d_n and $-d_n$ given $\mathcal{F}_{n-1}$ coincide. Such martingales arise naturally in several contexts. We will only mention here one important example, related to the Haar system $h = (h_n)_{n\geq 0}$ on $[0, 1]$. Suppose that the probability space is the interval $[0, 1]$ equipped with its Borel subsets and Lebesgue measure, and let $(\mathcal{F}_n)_{n\geq 0}$ be the natural filtration of h. Then for any coefficients a_0, a_1, a_2, $\ldots$ from $\mathbb{B}$, the sequence $(\sum_{k=0}^n a_k h_k)_{n\geq 0}$ forms a martingale. Such a process is called *a dyadic martingale* or *Paley-Walsh martingale*, and it is easy to see that it is conditionally symmetric.

We define the square function $S(f)$ associated to f by $S(f) = \left(\sum_{n=0}^\infty ||d_n||^2\right)^{1/2}$ and, throughout, use the notation $S_N(f) = \left(\sum_{n=0}^N ||d_n||^2\right)^{1/2}$ for $N = 0, 1, 2, \ldots$.

We will be mainly interested in the weak-type $(1, 1)$ inequality

$$\mathbb{P}(S(f) \geq 1) \leq \beta ||f||_1 \tag{1.1}$$

under the assumption that f is conditionally symmetric. Here $||f||_1 = \sup_{n\geq 0} ||f_n||_1$ is the first norm of f. There is a natural question about the class of those Banach spaces $\mathbb{B}$, for which the inequality (1.1) holds true with some absolute β. It follows from the results of [4] and [10] ([5] is also a convenient reference) that this class

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coincides with the class of 2-convex spaces. Recall that a Banach space $\mathbb{B}$ is 2-convex, if it can be renormed so that for all x, $y \in \mathbb{B}$,

$$||x+y||^2 + ||x-y||^2 \geq 2||x||^2 + 2\kappa||y||^2$$

for some universal positive κ (cf. [10]). For instance, one can easily show that $L_p([0,1])$ is 2-convex if and only if $1 \leq p \leq 2$. What can be said about the optimal (i.e., the least) value $\beta(\mathbb{B})$ of the constant β in (1.1)? [1] proved that

$$\beta(\mathbb{R}) \leq e^{-1/2} + \int_0^1 e^{-t^2/2}\mathrm{d}t = 1.4622\ldots$$

and [8] showed that actually we have equality here. We will strengthen this result as follows.

Theorem 1.1. *If $\mathbb{B}$ is a Hilbert space, then $\beta(\mathbb{B}) = \beta(\mathbb{R})$.*

It should be pointed out here that this passage from real to Hilbert-space-valued setting is absolutely not automatic. There are several basic inequalities for square functions of conditionally symmetric martingales, for which the constants in the one- and higher-dimensional settings differ. See [11] for a careful analysis of this phenomenon.

Our second result is the converse to Theorem 1.1.

Theorem 1.2. *If $\mathbb{B}$ is a Banach space which is not isometric to a Hilbert space, then there is a dyadic martingale f such that $\mathbb{P}(S(f) \geq 1) > \beta(\mathbb{R})||f||_1$.*

The two above theorems yield the following characterization: a Banach space $\mathbb{B}$ is a Hilbert space if and only if $\beta(\mathbb{B}) = \beta(\mathbb{R})$.

We have organized this note as follows. Theorem 1.1 is established in the next section. Section 3 is devoted to the proof of Theorem 1.2.

2. Sharp inequality for Hilbert-space-valued martingales

We start by defining $\varphi : \mathbb{R} \to \mathbb{R}$ by the formula

$$\varphi(s) = e^{-s^2/2} + s\int_0^s e^{-t^2/2}\mathrm{d}t.$$

Note that $\varphi(1) = \beta(\mathbb{R}) = 1.4622\ldots$. Furthermore, we easily check that

$$\varphi(s) = s\varphi'(s) + \varphi''(s) \qquad \text{for } s \in \mathbb{R}. \tag{2.1}$$

Introduce $U : \mathbb{R} \times [0,\infty) \to \mathbb{R}$ by

$$U(x,y) = \begin{cases} 1 - \sqrt{1-y^2}\,\varphi\left(\frac{|x|}{\sqrt{1-y^2}}\right) & \text{if } x^2+y^2 \leq 1, \\ 1 - \varphi(1)|x| & \text{if } x^2+y^2 > 1. \end{cases}$$

The function U appears, in a slightly different form, in [1]. One can extract from that paper (see pages 381 and 382 there) that U has the following property: if x, d are real numbers and $y \geq 0$, then

$$U(x+d, \sqrt{y^2+d^2}) + U(x-d, \sqrt{y^2+d^2}) \leq 2U(x,y). \tag{2.2}$$

Furthermore, we easily check that

$$1 - \varphi(1)|x| \geq U(x,y) \geq 1_{\{x^2+y^2 \geq 1\}} - \varphi(1)|x|. \tag{2.3}$$

Indeed, if $x^2+y^2 \geq 1$, then we have equality throughout; if $x^2+y^2<1$, the inequality can be rewritten in the form

$$\frac{1}{\sqrt{1-y^2}} \geq \varphi\left(\frac{|x|}{\sqrt{1-y^2}}\right) - \frac{\varphi(1)|x|}{\sqrt{1-y^2}} \geq 0.$$

However, the function $s \mapsto \varphi(s)-\varphi(1)s$ is decreasing on $[0,1]$: its derivative equals $-\int_s^1 e^{-t^2/2}\mathrm{d}t - e^{-1/2}$. Consequently,

$$\frac{1}{\sqrt{1-y^2}} \geq 1 = \varphi(0) \geq \varphi\left(\frac{|x|}{\sqrt{1-y^2}}\right) - \frac{\varphi(1)|x|}{\sqrt{1-y^2}} \geq \varphi(1)-\varphi(1) = 0.$$

The key ingredient of the proof is the following vector-valued version of (2.2).

Lemma 2.1. *Suppose that $\mathbb{B}$ is a Hilbert space. Pick x, $d \in \mathbb{B}$ and $y \geq 0$. Then*

$$U(||x+d||, \sqrt{y^2+||d||^2}) + U(||x-d||, \sqrt{y^2+||d||^2}) \leq 2U(||x||, y). \tag{2.4}$$

Proof. It is convenient to split the reasoning into a few parts.

Step 1. Assume that $||x||^2+y^2 \geq 1$. Using the left bound in (2.3), we may write

$$\begin{aligned} &U(||x+d||, \sqrt{y^2+||d||^2}) + U(||x-d||, \sqrt{y^2+||d||^2}) \\ &\quad \leq 2 - \varphi(1)\big[||x+d|| + ||x-d||\big] \leq 2 - 2\varphi(1)||x|| = 2U(||x||, y). \end{aligned}$$

Step 2. Now, assume that $||x||^2+y^2<1$ and $||x \pm d||^2 + y^2 + ||d||^2 < 1$. The left-hand side of (2.4) can be rewritten in the form $F(2\Re\langle x, d\rangle)$, where

$$F(s) = U(\sqrt{A+s}, \sqrt{y^2+||d||^2}) + U(\sqrt{A-s}, \sqrt{y^2+||d||^2})$$

and $A = ||x||^2 + ||d||^2$. The function F is nondecreasing on $[0, 2||x|| \cdot ||d||]$; to see this, take $s>0$ and compute that

$$F'(s) = \frac{U_x(\sqrt{A+s}, \sqrt{y^2+||d||^2})}{2\sqrt{A+s}} - \frac{U_x(\sqrt{A-s}, \sqrt{y^2+||d||^2})}{2\sqrt{A-s}}.$$

Now, we have $F'(0)=0$; in addition, if u, v are positive numbers satisfying $u^2+v^2<1$, and we denote the ratio $u/\sqrt{1-v^2}$ by z, then

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}u}(U_x(u,v)/u) &= (uU_{xx}(u,v) - U_x(u,v))u^{-2} \\ &= \left(-z\varphi''(z) + \varphi'(z)\right)u^{-2} = \left(\int_0^z e^{-t^2/2}\mathrm{d}t - ze^{-z^2/2}\right)u^{-2} \geq 0, \end{aligned}$$

since $t \mapsto e^{-t^2/2}$ is nonincreasing. This shows that $F'(s)>0$ for positive s. Furthermore, F is even; thus, it suffices to prove (2.4) for $\Re\langle x, d\rangle = \pm||x|| \cdot ||d||$, i.e. in the case when x and d are linearly dependent. This follows immediately from the real-valued version (2.2).

Step 3. Next, suppose that $||x||^2+y^2<1$ and that exactly one of the inequalities

$$||x-d||^2+y^2+||d||^2 \geq 1, \qquad ||x+d||^2+y^2+||d||^2 \geq 1$$

holds true. We may assume that the first of them is true, replacing x by $-x$ if necessary. Furthermore, we may assume that $y=0$: otherwise we divide throughout by $\sqrt{1-y^2}$ and substitute $X = x/\sqrt{1-y^2}$, $D = d/\sqrt{1-y^2}$. The inequality (2.4) takes the form

$$\sqrt{1-||d||^2}\varphi\left(\frac{||x+d||}{\sqrt{1-||d||^2}}\right) + \varphi(1)||x-d|| \geq 2\varphi(||x||).$$

Carry out the following optimization procedure. Fix $||x||$, $||x-d||$, put $s = ||d||$ and minimize the left-hand side over s. By parallelogram identity, we have $||x+d|| = \sqrt{2||x||^2 + 2s^2 - ||x-d||^2}$, so the derivative with respect to s of the left-hand side is

$$\begin{aligned} &-\frac{s}{\sqrt{1-s^2}}\varphi(X) + \frac{\sqrt{1-s^2}}{2X}\varphi'(X)\cdot\frac{s(2||x||^2+2-||x-d||^2)}{(1-s^2)^2} \\ &= \frac{s}{X\sqrt{1-s^2}}\big[-\varphi(X)X + \varphi'(X)(X^2+2)\big], \end{aligned}$$

where $X = \sqrt{(2||x||^2 + 2s^2 - ||x-d||^2)/(1-s^2)} \in [0,1]$. But the expression in the square brackets is nonnegative: indeed, it equals 0 for $X = 0$, and its derivative is

$$\varphi''(X)(X^2+2) + \varphi'(X)X - \varphi(X) = \varphi''(X)(X^2+1) \geq 0,$$

in view of (2.1). Consequently, to prove the desired estimate (2.4), it suffices to consider the smallest possible $||d||$ (for which the assumptions of Step 3 are valid). That is, we must prove the claim for d such that $||d|| = ||x-d|| - ||x||$ or such that $||x-d||^2 + ||d||^2 = 1$. In the first case, we get that x and d are linearly dependent, so (2.4) follows from its real version (2.2); the second case follows immediately from continuity and Step 2 above.

Step 4. Finally, assume that $||x||^2 + y^2 < 1$ and $||x \pm d||^2 + y^2 + ||d||^2 > 1$. As previously, it suffices to establish the desired bound for $y = 0$ only; it reads

$$\varphi(1)||x+d|| + \varphi(1)||x-d|| \geq 2\varphi(||x||).$$

Observe that the function $G(s) = ||x+ds|| + ||x-ds||$ is nondecreasing on $[0,1]$: this follows immediately from the triangle inequality. Consequently, we may write

$$\varphi(1)||x+d|| + \varphi(1)||x-d|| \geq \varphi(1)||x+ds_0|| + \varphi(1)||x-ds_0||,$$

where s_0 is the largest number such that at least one of the inequalities

$$||x-ds_0||^2 + ||ds_0||^2 \leq 1, \qquad ||x+ds_0||^2 + ||ds_0||^2 \leq 1$$

holds true. It remains to note that the inequality $\varphi(1)||x+ds_0|| + \varphi(1)||x-ds_0|| \geq 2\varphi(||x||)$ has been already established in Step 2 or Step 3 above. □

Proof. It suffices to show that $\beta(\mathbb{B}) \leq \varphi(1)$; the sharpness of this bound has been proved in [8]. By easy approximation, we will be done if we show the bound for simple martingales (a martingale f is simple, if for any n the random variable f_n takes only a finite number of values and there is a deterministic N such that $f_N = f_{N+1} = f_{N+2} = \ldots = f_\infty$). We will prove a slightly stronger statement

$$\mathbb{P}(S_n(f)^2 + ||f_n||^2 \geq 1) \leq \varphi(1)||f_n||_1, \qquad n = 0,\, 1,\, 2,\, \ldots.$$

The key observation is that the process $(U(||f_n||, S_n(f))_{n\geq 0}$ is a supermartingale: indeed, by the conditional symmetry of f,

$$\begin{aligned}\mathbb{E}\big[U(||f_{n+1}||, S_{n+1}(f))|\mathcal{F}_n\big] &= \mathbb{E}\big[U(||f_n + d_{n+1}||, \sqrt{S_n^2(f) + ||d_{n+1}||^2}|\mathcal{F}_n\big]\\ &= \frac{1}{2}\Big\{\mathbb{E}\big[U(||f_n + d_{n+1}||, \sqrt{S_n^2(f) + ||d_{n+1}||^2}|\mathcal{F}_n\big]\\ &\qquad + \mathbb{E}\big[U(||f_n - d_{n+1}||, \sqrt{S_n^2(f) + ||d_{n+1}||^2}|\mathcal{F}_n\big]\Big\}\\ &\leq U(||f_n||, S_n(f)),\end{aligned}$$

where in the last passage we have exploited (2.4). So, by the right bound in (2.3),

$$\begin{aligned}\mathbb{P}(S_n^2(f) + ||f_n||^2 \geq 1) - \varphi(1)||f_n||_1 &= \mathbb{E}\Big\{1_{\{S_n^2(f)+||f_n||^2\geq 1\}} - \varphi(1)||f_n||\Big\}\\ &\leq \mathbb{E}U(||f_n||, S_n(f))\\ &\leq \mathbb{E}U(||f_0||, S_0(f))\\ &= \frac{1}{2}\mathbb{E}\Big\{U(||f_0||, S_0(f)) + U(||-f_0||, S_0(f))\Big\}\\ &\leq U(0,0) = 0,\end{aligned}$$

where in the last inequality we used (2.4) again. This completes the proof. □

3. Characterization of Hilbert spaces

Throughout this section, we assume that the underlying probability space is the interval $[0,1]$ equipped with its Borel subsets and Lebesgue measure. Let $(\mathbb{B}, ||\cdot||)$ be a Banach space such that for any dyadic $\mathbb{B}$-valued martingale f,

$$\mathbb{P}(S(f) \geq 1) \leq \varphi(1)||f||_1. \tag{3.1}$$

For $x \in \mathbb{B}$ and $y \geq 0$, let $M(x,y)$ denote the class of all simple dyadic $\mathbb{B}$-valued martingales f satisfying $f_0 \equiv x$ and

$$y^2 - ||x||^2 + S^2(f) \geq 1 \qquad \text{almost surely.} \tag{3.2}$$

Here the filtration may vary. Consider the function $U^0 : \mathbb{B} \times [0,\infty) \to \mathbb{R}$, given by

$$U^0(x,y) = \inf\{\mathbb{E}||f_\infty||\},$$

where the infimum is taken over all $f \in M(x,y)$ and f_∞ is the pointwise limit of f.

Lemma 3.1. *The function U^0 enjoys the following properties.*

1° *For any $x \in \mathbb{B}$ and $y \geq 0$ we have $U^0(x,y) \geq ||x||$.*

2° *For any x, $d \in \mathbb{B}$ and $y \geq 0$ we have*

$$\frac{1}{2}\Big[U^0(x+d, \sqrt{y^2 + ||d||^2}) + U^0(x-d, \sqrt{y^2 + ||d||^2}))\Big] \geq U^0(x,y). \tag{3.3}$$

3° *For any $x \in \mathbb{B}$ we have $U^0(x, ||x||) \geq \varphi(1)^{-1}$.*

Proof. The first property is obvious: if $f \in M(x,y)$, then $||f_n||_1 \geq ||f_0||_1 = ||x||$ for all n. To establish 2°, we exploit the so-called "splicing argument": see e.g. [3]. Pick a martingale $f^\pm$ from the class $M(x \pm d, \sqrt{y^2 + ||d||^2})$ and define a simple sequence f by $f_0 \equiv x$ and

$$f_n(\omega) = \begin{cases} f_{n-1}^-(2\omega) & \text{if } \omega \in [0,1/2],\\ f_{n-1}^+(2\omega - 1) & \text{if } \omega \in (1/2, 1].\end{cases}$$

Then f is a dyadic martingale. Furthermore, if $\omega \in [0, 1/2]$, then

$$y^2 - ||x||^2 + S^2(f)(\omega) = y^2 + ||d||^2 - ||x - d||^2 + S^2(f^-)(2\omega) \geq 1$$

unless ω belongs to a set of probability 0: this is due to $f^- \in M(x-d, \sqrt{y^2 + ||d||^2})$. Similarly, $y^2 - ||x||^2 + S^2(f)(\omega) \geq 1$ for almost all $w \in (1/2, 1]$. Therefore (3.2) holds, so by the definition of U^0, we have $||f_\infty||_1 \geq U^0(x, y)$. However, the left hand side equals $\frac{1}{2}||f_\infty^-||_1 + \frac{1}{2}||f_\infty^+||_1$, which, by the proper choice of $f^\pm$, can be made arbitrarily close to the left hand side of (3.3). This gives 2°. Finally, the condition 3° follows at once from (3.1) and the definition of U^0. □

The further properties of U^0 are studied in the next lemma.

Lemma 3.2. *(i) The function U^0 has the homogeneity-type property*

$$U^0(x, y) = \sqrt{1 - y^2}\, U^0\left(\frac{x}{\sqrt{1 - y^2}}, 0\right) \qquad \text{for all } x \in \mathbb{B} \text{ and } y \in [0, 1).$$

(ii) The function U^0 is continuous on $\mathbb{B} \times [0, 1)$.

Proof. (i) This follows immediately from the definition of U^0 and the fact that $f \in M(x, y)$ if and only if $f/\sqrt{1 - y^2} \in M(x/\sqrt{1 - y^2}, 0)$.

(ii) If $f \in M(x, y)$ and $\bar{x} \in \mathbb{B}$, then $\bar{x} - x + f \in M(\bar{x}, y)$ and

$$||\bar{x} - x|| \geq \mathbb{E}\big[||\bar{x} - x + f_n|| - ||f_n||\big] \geq U^0(\bar{x}, y) - \mathbb{E}||f_n||,$$

so taking infimum over f gives $U^0(\bar{x}, y) - U^0(x, y) \leq ||\bar{x} - x||$. By the symmetry of x and $\bar{x}$, we see that for a fixed y, the function $U^0(\cdot, y)$ is Lipschitz. An application of (i) yields the desired continuity. □

Now, put $\psi(x) = U^0(x, 0)$ for $x \in \mathbb{B}$. The next step is to prove the following.

Lemma 3.3. *We have $\varphi(1)\psi(x) = \varphi(||x||)$ for all $x \in \mathbb{B}$ with $||x|| \leq 1$.*

Proof. It is convenient to split the reasoning into a few parts.

Step 1. First we will show that if $||x|| = 1$, then $U^0(x, 0) = 1$. Indeed, the inequality "≥" follows directly from part 1° of Lemma 3.1; to get the reverse, consider the dyadic martingale f given by $f_0 \equiv x$ and $f_1 = f_2 = \ldots = 2x \cdot 1_{[0,1/2]}$. It satisfies $-||x||^2 + S^2(f) = ||x||^2 \geq 1$ almost surely; thus $f \in M(x, 0)$ and $U^0(x, 0) \leq \mathbb{E}||f_1|| = 1$. So, $\varphi(1)\psi(x) = \varphi(1)U^0(x, 0) = \varphi(1) = \varphi(||x||)$.

Step 2. Next we will show that $\varphi(1)\psi(x) \leq \varphi(||x||)$ for $||x|| < 1$. If f is a conditionally symmetric martingale with $f_0 \equiv x$, then

$$\mathbb{E}U^0(f_n, \sqrt{-||x||^2 + S^2(f)}) \geq U^0(x, 0).$$

To see this, use (3.3) and repeat the argumentation from the end of Section 2. By a straightforward approximation and Lemma 3.2 (ii), the bound above leads to the following inequality for Brownian motion B. Namely, pick a stopping time τ satisfying $\tau < 1$ almost surely and let $x \in \mathbb{B}$. Apply the above bound to the conditionally symmetric martingale $(x + x' B_{\tau \wedge k2^{-N}})_{k=0}^{2^N}$ and let $N \to \infty$ to get $\mathbb{E}U^0(x + x' B_\tau, \sqrt{\tau}) \geq U^0(x, 0)$, or, by Lemma 3.2 (i),

$$\psi(x) \leq \mathbb{E}\sqrt{1 - \tau}\,\psi\left(\frac{x + x' B_\tau}{\sqrt{1 - \tau}}\right). \tag{3.4}$$

If we put $\tau = \inf\{t : ||x + x' B_t|| = \sqrt{1 - t}\}$, then by Step 1 we obtain $\psi(x) \leq \mathbb{E}\sqrt{1 - \tau}$. On the other hand, by (2.1), the function $V : \mathbb{R} \times (-1, 1) \to \mathbb{R}$ given by

$V(s,t) = \sqrt{1-t}\varphi(s/\sqrt{1-t})$ satisfies the heat equation $V_t + \frac{1}{2}V_{ss} = 0$. Therefore, by Itô's formula, if τ is as above, we get

$$\varphi(||x||) = V(||x||,0) = \mathbb{E}V(||x||+B_\tau,\tau) = \mathbb{E}\sqrt{1-\tau}\varphi\left(\frac{||x||+B_\tau}{\sqrt{1-\tau}}\right) = \varphi(1)\mathbb{E}\sqrt{1-\tau}.$$

This yields the claimed bound $\varphi(1)\psi(x) \leq \varphi(||x||)$.

Step 3. Suppose now that there is a vector z of norm smaller than 1 for which we have the strict estimate $\varphi(1)\psi(z) < \varphi(||z||)$. By the property 3° of Lemma 3.1 (used with $x = 0$), we must have $z \neq 0$. Consider the stopping time $\tau = \inf\{t : |B_t| = ||z||\sqrt{1-t}\}$ and apply (3.4) with $x = 0$ (as x', take the vector z') to get

$$\varphi(1)\psi(0) \leq \varphi(1)\mathbb{E}\sqrt{1-\tau}\psi(z) < \varphi(||z||)\mathbb{E}\sqrt{1-\tau}.$$

However, as previously, Itô's formula gives

$$\varphi(0) = V(0,0) = \mathbb{E}V(B_\tau,\tau) = \mathbb{E}\sqrt{1-\tau}\varphi\left(\frac{B_\tau}{\sqrt{1-\tau}}\right) = \varphi(||z||)\mathbb{E}\sqrt{1-\tau}.$$

Therefore $\varphi(1)\psi(0) < \varphi(0) = 1$, which contradicts part 3° of Lemma 3.1. □

We are ready to establish the main result.

Proof. Assume $\mathbb{B}$ is a Banach space for which the weak-type constant for the dyadic square function equals $\varphi(1)$. By Lemmas 3.2 and 3.3, we have

$$U^0(x,y) = \varphi(1)^{-1}\sqrt{1-y^2}\varphi\left(\frac{||x||}{\sqrt{1-y^2}}\right)$$

provided $||x||^2 + y^2 < 1$. Pick vectors x, $d \in \mathbb{B}$ and a small positive number t (so that $||x \pm d||^2 + ||d||^2 < t^{-2}$). By (3.3), applied to tx, td and $y = 0$, we get

$$2\varphi(t||x||) \leq \sqrt{1-t^2||d||^2}\left[\varphi\left(\frac{t||x-d||}{\sqrt{1-t^2||d||^2}}\right) + \varphi\left(\frac{t||x+d||}{\sqrt{1-t^2||d||^2}}\right)\right].$$

This can be rewritten in the form

$$\begin{aligned} &2\varphi(t||x||) - 2\varphi(0) \\ &\leq \sqrt{1-t^2||d||^2}\left[\varphi\left(\frac{t||x-d||}{\sqrt{1-t^2||d||^2}}\right) + \varphi\left(\frac{t||x+d||}{\sqrt{1-t^2||d||^2}}\right) - 2\varphi(0)\right] \\ &\quad + 2\varphi(0)\big(\sqrt{1-t^2||d||^2} - 1\big). \end{aligned} \tag{3.5}$$

Divide throughout by t^2 and let $t \to 0$. Then

$$\frac{2\varphi(t||x||) - 2\varphi(0)}{t^2} = \frac{\varphi(t||x||) + \varphi(-t||x||) - 2\varphi(0)}{t^2} \to \varphi''(0)||x||^2 = ||x||^2,$$

and similarly, the right-hand side of (3.5) tends to $\frac{1}{2}\left[||x-d||^2 + ||x+d||^2\right] - ||d||^2$. So,

$$||x-d||^2 + ||x+d||^2 \geq 2||x||^2 + 2||d||^2,$$

and replacing x, d by $x+d$ and $x-d$ yields the reverse bound. Consequently, parallelogram identity holds and hence $\mathbb{B}$ is isomeric to a Hilbert space. □

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