LOGARITHMIC INEQUALITIES FOR SECOND-ORDER RIESZ TRANSFORMS AND RELATED FOURIER MULTIPLIERS

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Abstract. We study logarithmic estimates for a class of Fourier multipliers which arise from a nonsymmetric modulation of jumps of Lévy processes. In particular, this leads to corresponding tight bounds for second-order Riesz transforms on $\mathbb{R}^d$.

1. Introduction

As evidenced in [1], [2], [3], [4], [6], [11], [15] and many other papers, the martingale theory plays a fundamental role in obtaining various bounds for many important singular integrals and Fourier multipliers. So far, the martingale methods have constituted a particularly efficient tool in the proofs of L^p bounds. In [15] the author proposed a novel approach which enabled the study of logarithmic estimates and used it to obtain some tight bounds for Beurling-Ahlfors operator. This paper is a continuation of that work and contains, among other things, a "fine-tuning" of the martingale methods which leads to the improvement of several results from [15], and indicates various interesting connections between certain classes of Fourier multipliers and special pairs of differentially subordinated martingales.

We start with recalling the necessary background and notation. Let $d \geq 1$ be a fixed integer. For any bounded function $m : \mathbb{R}^d \to \mathbb{C}$, there is a unique bounded linear operator T_m on $L^2(\mathbb{R}^d)$, called *the Fourier multiplier with the symbol* m, which is given by the identity $\widehat{T_m f} = m\hat{f}$. By Plancherel's theorem, the norm of T_m on $L^2(\mathbb{R}^d)$ is equal to $||m||_{L^\infty(\mathbb{R}^d)}$ and there is a classical problem to characterize those m, for which the corresponding Fourier multiplier extends to a bounded linear operator on $L^p(\mathbb{R}^d)$, $1 < p < \infty$. This question is motivated by the analysis of the classical example, a the collection of Riesz transforms $\{R_j\}_{j=1}^d$ on $\mathbb{R}^d$ [18]. Here, for any j, the transform R_j is a Fourier multiplier corresponding to the symbol $m(\xi) = -i\xi_j/|\xi|$, $\xi \neq 0$. The remarkable feature is that R_j can be alternatively defined via the singular integrals

$$R_j f(x) = \frac{\Gamma\left(\frac{d+1}{2}\right)}{\pi^{(d+1)/2}} \text{ p.v.} \int_{\mathbb{R}^d} \frac{x_j - y_j}{|x-y|^{d+1}} f(y)\mathrm{d}y, \qquad j = 1, 2, \ldots, d.$$

It is well-known that singular integral operators are important in the theory of partial differential equations and have been used, in particular, in the study of the higher integrability theory of the gradient of weak solutions. In addition, the

2010 *Mathematics Subject Classification.* Primary: 42B15, 60G44. Secondary: 42B20.

Key words and phrases. Fourier multiplier, singular integral, martingale, differential subordination.

Partially supported by Polish Ministry of Science and Higher Education (MNiSW) grant IP2011 039571 'Iuventus Plus'.

exact information on the size of such operators (e.g. on the p-norms) provides the insight into the degrees of improved regularity and other geometric properties of solutions and their gradients. This gives rise to another classical problem for Fourier multipliers: for a given m, provide tight bounds for the size of the multiplier T_m in terms of some characteristics of the symbol.

We will study this question for the following class of symbols, introduced by Bañuelos and Bogdan in [1]. Let ν be a Lévy measure on $\mathbb{R}^d$, i.e., a nonnegative Borel measure on $\mathbb{R}^d$ such that $\nu(\{0\}) = 0$ and

$$\int_{\mathbb{R}^d} \min\{|x|^2, 1\}\nu(\mathrm{d}x) < \infty.$$

Assume further that μ is a finite nonnegative Borel measure on the unit sphere $\mathbb{S}$ of $\mathbb{R}^d$ and fix two Borel functions ϕ on $\mathbb{R}^d$ and ψ on $\mathbb{S}$ which take values in the unit ball of $\mathbb{C}$. We define the associated multiplier $m = m_{\phi,\psi,\mu,\nu}$ on $\mathbb{R}^d$ by

$$m(\xi) = \frac{\frac{1}{2}\int_{\mathbb{S}}\langle \xi, \theta\rangle^2 \psi(\theta)\mu(\mathrm{d}\theta) + \int_{\mathbb{R}^d}[1 - \cos\langle \xi, x\rangle]\phi(x)\nu(\mathrm{d}x)}{\frac{1}{2}\int_{\mathbb{S}}\langle \xi, \theta\rangle^2 \mu(\mathrm{d}\theta) + \int_{\mathbb{R}^d}[1 - \cos\langle \xi, x\rangle]\nu(\mathrm{d}x)} \tag{1.1}$$

if the denominator is not 0, and $m(\xi) = 0$ otherwise. Here $\langle \cdot, \cdot \rangle$ stands for the scalar product in $\mathbb{R}^d$. The Fourier multipliers corresponding to these symbols can be given a martingale representation by the use of transformations of jumps of Lévy processes (see [1] and [2]). Combining this representation with Burkholder's martingale inequalities, Bañuelos and Bogdan [1] and Bañuelos, Bielaszewski and Bogdan [2] obtained the following L^p bound.

Theorem 1.1. *Let $1 < p < \infty$ and let $m = m_{\phi,\psi,\mu,\nu}$ be given by* (1.1). *Then for any $f \in L^p(\mathbb{R}^d)$ we have*

$$||T_m f||_{L^p(\mathbb{R}^d)} \leq (p^* - 1)||f||_{L^p(\mathbb{R}^d)}, \tag{1.2}$$

where $p^* = \max\{p, p/(p-1)\}$.

It turns out that the above constant $p^* - 1$ cannot be replaced by a smaller number, which has been shown recently by Geiss, Montgomery-Smith and Saksman [11] (see also [5]). In the limit case $p = 1$, the L^p estimate does not hold with any finite constant, but we have the following substitute. Throughout the paper, the functions Φ, $\Psi : [0, \infty) \to [0, \infty)$ are given by the formulas

$$\Phi(x) = e^x - 1 - x \quad \text{and} \quad \Psi(x) = (x + 1)\log(x + 1) - x$$

and the LlogL class is defined by

$$\mathrm{LlogL}(\mathbb{R}^d) = \left\{ f : \mathbb{R}^d \to \mathbb{C} : \int_{\mathbb{R}^d} \Psi(|f(x)|)\mathrm{d}x < \infty \right\}.$$

Using a standard density argument and Corollary 1.3 from [15], one can define the action of a multiplier T_m (with m coming from (1.1)) on the class $\mathrm{LlogL}(\mathbb{R}^d)$. In addition, we have the following bound, the main result of [15].

Theorem 1.2. *Let $m = m_{\phi,\psi,\mu,\nu}$ be given by* (1.1), *with μ, ν, ϕ and ψ satisfying the above assumptions. Then for any $K > 1$, any $f \in LlogL(\mathbb{R}^d)$ and any Borel subset A of $\mathbb{R}^d$ we have*

$$\int_A |T_m f(x)|\,dx \leq K \int_{\mathbb{R}^d} \Psi(|f(x)|)\,dx + \frac{|A|}{2(K - 1)}. \tag{1.3}$$

Furthermore, for any $K > 2/\pi$ there is a multiplier $m : \mathbb{C} \to \mathbb{R}$ from the class (1.1)*, a Borel subset A of $\mathbb{C}$ and a function $f \in LlogL(\mathbb{C})$ for which*

$$\int_A |T_m f(z)|dz = K \int_{\mathbb{C}} \Psi(|f(z)|)dz + \frac{|A|}{\pi(K\pi - 2)}.$$

In particular, the above theorems give quite precise information for second-order Riesz transforms R_iR_j, as well as for $\sum_{i,j=1}^d a_{ij}R_iR_j$, the linear combinations of such operators (cf. [4], [5], see also Section 4 below), which have further important connections to the Beurling-Ahlfors operator and Iwaniec' conjecture [12].

It turns out that for a certain natural and wide subclass of (1.1) the estimate (1.3) can be considerably improved. Specifically, we will restrict ourselves to the symbols of the form (1.1) in which the functions ϕ and ψ take values in the interval $[0, 1]$ (some examples which motivate this restriction are presented in Section 3). For such multipliers, we will prove the following. For $K > 1/2$, define

$$C_K = \frac{1}{K}\left[\int_0^1 \frac{e^{2\lambda} - 1}{2\lambda} \mathrm{d}\lambda + \int_1^{3/2} \frac{e^{2\lambda} - 1}{2(2\lambda - 1)^2} \mathrm{d}\lambda\right] + \frac{e^3}{16(K - 1/2)}. \tag{1.4}$$

Computer simulations show that $C_K \le 3.1325/K + 1.2554/(K - 1/2)$.

Theorem 1.3. *Fix $K > 1/2$ and let $m = m_{\phi,\psi,\mu,\nu}$ be given by* (1.1)*, with μ, ν as above and ϕ, ψ taking values in the interval* $[0, 1]$*. Then for any $f \in LlogL(\mathbb{R}^d)$ and any Borel subset A of $\mathbb{R}^d$ we have*

$$\int_A |T_m f(x)|dx \le K \int_{\mathbb{R}^d} \Psi(|f(x)|)dx + C_K|A|. \tag{1.5}$$

Furthermore, for any $d \ge 2$ and any $K > 1/2$ there is a multiplier $m : \mathbb{R}^d \to \mathbb{R}$ from the class (1.1)*, a Borel subset A of $\mathbb{R}^d$ and a function $f \in LlogL(\mathbb{R}^d)$ for which*

$$\int_A |T_m f(z)|dz \ge K \int_{\mathbb{R}^d} \Psi(|f(z)|)dz + \frac{2K - 1}{2(4K - 1)^2} \cdot |A|. \tag{1.6}$$

Comparing the assertions of the two above theorems, we see that the restriction to $[0, 1]$-valued functions ϕ and ψ results in the improvement of the integrability of the multiplier: the threshold $K > 1$ is reduced to $K > 1/2$, while the multiplicative constant appearing in front of $|A|$ remains of order $O(K^{-1})$ as $K \to \infty$. We believe, but have been unable to prove, that for $K \le 1/2$ the inequality (1.5) does not hold in general with any finite C_K (however, we have managed to prove the probabilistic version of this statement: see Section 2 below).

A few words about the proof and the organization of the paper are in order. It should be stressed here that the proof of (1.5) is not just a mere repetition of the arguments from [15]. The passage to the above special subclass of the symbols m requires an appropriate adjustment in the martingale setting, which leads to an exponential estimate which is much more challenging than its counterpart in [15]. This exponential bound will be established by means of the corresponding weak-type inequality, which is of independent interest. All these probabilistic facts will be presented in the next section. Section 3 is devoted to the proof of (1.5) and contains some examples and applications. We also discuss there the possibility of extending (1.5) to the vector valued setting. In the final part of the paper we study the lower bound for the constant C_K in (1.5); first we show (1.6) in the two-dimensional case, exploiting the properties of the Beurling-Ahlfors operator, and then pass to the general setting, using an appropriate transference method.

2. A martingale inequality

2.1. **Background and statement of the results.** Assume that $(\Omega, \mathcal{F}, \mathbb{P})$ is a complete probability space, equipped with $(\mathcal{F}_t)_{t\geq 0}$, a nondecreasing family of sub-σ-fields of $\mathcal{F}$, such that $\mathcal{F}_0$ contains all the events of probability 0. Suppose that X, Y are two adapted martingales taking values in a certain separable Hilbert space $\mathcal{H}$ with norm $|\cdot|$ and scalar product $\langle\cdot,\cdot\rangle$; in fact, we may take $\mathcal{H}$ to be equal to the subspace of ℓ_2. We impose the usual regularity conditions on the trajectories of the processes: we assume that the paths are right-continuous and have limits from the left. Then X^*, the maximal function of X, is defined by $X^* = \sup_{t\geq 0}|X_t|$. The symbol $[X,Y]$ will stand for the quadratic covariance process of X and Y. See e.g. Dellacherie and Meyer [10] for details in the case when the processes are real-valued, and extend the definition to the vector setting by $[X,Y] = \sum_{k=0}^\infty [X^k, Y^k]$, where X^k, Y^k are the k-th coordinates of X, Y. Following Bañuelos and Wang [6] and Wang [19], we say that Y is differentially subordinate to X, if the process $([X,X]_t - [Y,Y]_t)_{t\geq 0}$ is nonnegative and nondecreasing as a function of t. For example, if X is a standard one-dimensional Brownian motion, stopped at the set $\{-1,1\}$, H is a predictable process taking values in $[-1,1]$ and $Y = H\cdot X$ is the Itô integral of H with respect to X, then Y is differentially subordinate to X: this follows from the identity

$$[X,X]_t - [Y,Y]_t = X_0^2(1-H_0^2) + \int_{0+}^t (1-H_s^2)\mathrm{d}s, \qquad t\geq 0.$$

As exhibited in [1], [2], martingales X, Y satisfying the differential subordination arise naturally in the martingale study of Fourier multipliers (1.1). In order to investigate the subclass studied in this paper (i.e., corresponding to ϕ, ψ taking values in $[0,1]$), we will work with pairs X, Y satisfying a slightly different condition:

$$([X,Y]_t - [Y,Y]_t)_{t\geq 0} \text{ is nondecreasing and nonnegative as a function of } t, \tag{2.1}$$

which can be regarded as "non-symmetric differential subordination". For instance, this holds in the above setting of stochastic integrals, if we assume that the integrand H takes values in $[0,1]$. Inequalities for such martingales were studied by a number of authors (see e.g. Burkholder [7], Choi [9] and the author [13], [14]). We refer the interested reader to those papers and mention here only result, which will be needed later. It was proved for martingale transforms by Burkholder [7] and in the general case by the author in [14]. Throughout, we use the notation $||X||_p = \sup_{t\geq 0}||X_t||_p$, $1\leq p\leq\infty$.

Theorem 2.1. *Let X, Y be two Hilbert-space-valued martingales satisfying* (2.1). *Then for any $\lambda > 0$ we have*

$$\lambda\mathbb{P}(Y^*\geq\lambda) \leq ||X||_1. \tag{2.2}$$

For each λ the inequality is sharp.

The main result of this section is the following.

Theorem 2.2. *Suppose that X, Y are Hilbert-space-valued martingales satisfying* (2.1) *and $||X||_\infty \leq 1$. Then for any $K > 1/2$ we have*

$$\sup_{t\geq 0}\mathbb{E}\Phi(|Y_t|/K) \leq \frac{C_K}{K}||X||_1, \tag{2.3}$$

where C_K is given by (1.4). *The above inequality does not hold with any finite constant C_K when $K \leq 1/2$. Furthermore, the constant C_K/K is of optimal order $O((K-1/2)^{-1})$ as $K \to 1/2$ and $O(K^{-2})$ as $K \to \infty$.*

The analogous statement for differentially subordinated martingales, with the best constant, was established in [15]. In the above "non-symmetric" setting the inequality is much more difficult and, in particular, we did not manage to obtain the optimal value of C_K. The key ingredient of the proof is the following weak type estimate, which is of independent interest (and for which we have found the best constants).

Theorem 2.3. *Suppose that X, Y are Hilbert-space-valued martingales satisfying* (2.1) *and $||X||_\infty \leq 1$. Then for any $\lambda > 0$ we have*

$$\mathbb{P}(Y^* \geq \lambda) \leq P(\lambda)||X||_1, \tag{2.4}$$

where

$$P(\lambda) = \begin{cases} \lambda^{-1} & \text{if } 0 < \lambda \leq 1, \\ (2\lambda - 1)^{-2} & \text{if } 1 < \lambda \leq 3/2, \\ e^{3-2\lambda}/4 & \text{if } \lambda > 3/2. \end{cases}$$

The bound on the right-hand side of (2.4) *is the best possible for each λ.*

For related results for differentially subordinated martingales, see Sections 8 and 9 in [8].

2.2. **On the method of proof.** Let us describe the method which will be used to establish the weak-type estimate of Theorem 2.3. We will restrict ourselves to the case in which the dimension of the Hilbert space $\mathcal{H}$ is finite (this will be sufficient for our purposes), but the reasoning can be easily extended to the infinite-dimensional setting, by the use of standard approximation arguments (see e.g. Wang [19]). So, assume that $\mathcal{H} = \mathbb{R}^d$ for some positive integer d. Let $\lambda > 0$ be given and fixed, and suppose that there is a real-valued function U_λ, defined on the strip $S = \{(x, y) \in \mathcal{H} \times \mathcal{H} : |x| \leq 1\}$, which satisfies the following four properties (here and below, A^o and $\overline{A}$ stand for the interior and the closure of the set A, respectively):

1° U_λ is of class C^2 on S^o.

2° For all $(x, y) \in S$ we have the majorization

$$U_\lambda(x, y) \geq 1_{\{|y| \geq \lambda\}} - P(\lambda)|x|. \tag{2.5}$$

3° There is a Borel function $c : S^o \to [0, \infty)$ with the following property: for any $(x, y) \in S^o$ and any h, $k \in \mathcal{H}$ such that $|x + h| \leq 1$, we have

$$\langle U_{\lambda xx}(x, y)h, h\rangle + 2\langle U_{\lambda xy}(x, y)h, k\rangle + \langle U_{\lambda yy}(x, y)k, k\rangle \leq c(x, y)(|k|^2 - \langle h, k\rangle). \tag{2.6}$$

4° For any $(x, y) \in S$ with $|y|^2 \leq \langle x, y\rangle$ we have $U_\lambda(x, y) \leq 0$.

Then (2.4) follows. To see this, it is convenient to split the reasoning into a few parts.

Step 1. A stopping time argument. It suffices to prove the following weaker form of (2.4): for all X, Y as in the statement and any $t \geq 0$,

$$\mathbb{P}(|Y_t| > \lambda) \leq P(\lambda)||X||_1. \tag{2.7}$$

Indeed, suppose that we have established (2.7). Fix $\varepsilon \in (0, \lambda/2)$ and consider the stopping time $\tau = \inf\{t : |Y_t| \geq \lambda - \varepsilon\}$. Since

$$\{Y^* \geq \lambda\} \subseteq \{|Y_t| \geq \lambda - \varepsilon \text{ for some } t\} = \{|Y_{\tau\wedge t}| \geq \lambda - \varepsilon \text{ for some } t\}$$

and the events $\{|Y_{\tau\wedge t}| \geq \lambda - \varepsilon\}$ are monotone with respect to t, we conclude that

$$\mathbb{P}(Y^* \geq \lambda) \leq \lim_{t\to\infty} \mathbb{P}(|Y_{\tau\wedge t}| \geq \lambda - \varepsilon).$$

However, if we apply (2.7) to $\lambda - 2\varepsilon$ and to the stopped martingales $(X_{\tau\wedge t})_{t\geq 0}$, $(Y_{\tau\wedge t})_{t\geq 0}$ (for which (2.1) is still satisfied), we obtain

$$\mathbb{P}(|Y_{\tau\wedge t}| \geq \lambda - \varepsilon) \leq \mathbb{P}(|Y_{\tau\wedge t}| > \lambda - 2\varepsilon) \leq P(\lambda - 2\varepsilon)||X||_1.$$

Consequently,

$$\mathbb{P}(Y^* \geq \lambda) \leq P(\lambda - 2\varepsilon)||X||_1,$$

and letting $\varepsilon \to 0$ yields the claim, since the function P is continuous.

Step 2. An application of Itô's formula. Take martingales X, Y as in the statement and let $Z_t = (X_t, Y_t)$ for $t \geq 0$. An application of Itô's formula to the process $(U_\lambda(Z_t))_{t\geq 0}$ yields

$$U_\lambda(Z_t) = U_\lambda(Z_0) + I_1 + I_2/2 + I_3, \tag{2.8}$$

where

$$\begin{aligned}
I_1 &= \int_{0+}^t U_{\lambda x}(Z_{s-})\mathrm{d}X_s + \int_{0+}^t U_{\lambda y}(Z_{s-})\mathrm{d}Y_s,\\
I_2 &= \int_{0+}^t U_{\lambda xx}(Z_{s-})\mathrm{d}[X,X]_s^c + 2\int_{0+}^t U_{\lambda xy}(Z_{s-})\mathrm{d}[X,Y]_s^c + \int_{0+}^t U_{\lambda yy}(Z_{s-})\mathrm{d}[Y,Y]_s^c,\\
I_3 &= \sum_{0<s\leq t} \Big[U_\lambda(Z_s) - U_\lambda(Z_{s-}) - \langle U_{\lambda x}(Z_{s-}), \Delta X_s\rangle - \langle U_{\lambda y}(Z_{s-}), \Delta Y_s\rangle\Big].
\end{aligned}$$

Now let us analyze each of the terms $I_1 - I_3$ separately. We have $\mathbb{E}I_1 = 0$, by the properties of stochastic integrals. To deal with I_2, let $0 \leq s_0 < s_1 \leq t$. For any $j \geq 0$, let $(\eta_i^j)_{1\leq i\leq i_j}$ be a sequence of nondecreasing finite stopping times with $\eta_0^j = s_0$, $\eta_{i_j}^j = s_1$ such that $\lim_{j\to\infty} \max_{1\leq i\leq i_j - 1} |\eta_{i+1}^j - \eta_i^j| = 0$. Keeping j fixed, we apply, for each $i = 0, 1, 2, \ldots, i_j$, the inequality (2.6) to $x = X_{s_0-}$, $y = Y_{s_0-}$ and $h = h_i^j = X_{\eta_{i+1}^j} - X_{\eta_i^j}$, $k = k_i^j = Y_{\eta_{i+1}^j} - Y_{\eta_i^j}$. Summing the obtained $i_j + 1$ inequalities and letting $j \to \infty$ yields

$$\begin{aligned}
\sum_{m=1}^d \sum_{n=1}^d \Big[&U_{x_m x_n}(Z_{s_0-})[X^m, X^n]_{s_0}^{s_1}\\
&+ 2U_{x_m y_n}(Z_{s_0-})[X^m, Y^n]_{s_0}^{s_1} + U_{y_m y_n}(Z_{s_0-})[Y^m, Y^n]_{s_0}^{s_1}\Big]\\
&\leq c(Z_{s_0-})\left([Y,Y]_{s_0}^{s_1} - [X,Y]_{s_0}^{s_1}\right),
\end{aligned}$$

where we have used the notation $[S,T]_{s_0}^{s_1} = [S,T]_{s_1} - [S,T]_{s_0}$ and X^m, Y^n denote the m-th and n-th coordinates of X and Y, respectively. By (2.1) and the condition $c \geq 0$, the double sum above is nonpositive; hence, if we approximate I_2 by discrete sums, we obtain $I_2 \leq 0$. Finally, I_3 is also nonpositive. To see this, apply the mean-value property: for any $\omega \in \Omega$ we write

$$\begin{aligned} &U_\lambda(Z_s(\omega)) - U_\lambda(Z_{s-}(\omega)) - \langle U_{\lambda x}(Z_{s-}(\omega)), \Delta X_s(\omega)\rangle - \langle U_{\lambda y}(Z_{s-}(\omega)), \Delta Y_s(\omega)\rangle \\ &= \frac{1}{2}\Big[\langle U_{\lambda xx}(\xi)\Delta X_s(\omega), \Delta X_s(\omega)\rangle + 2\langle U_{\lambda xy}(\xi)\Delta X_s(\omega), \Delta Y_s(\omega)\rangle \\ &\quad + \langle U_{\lambda yy}(\xi)\Delta Y_s(\omega), \Delta Y_s(\omega)\rangle\Big], \end{aligned}$$

where ξ is a certain point in S. Using (2.6), this can be bounded from above by $c(\xi)\big[|\Delta Y_s(\omega)|^2 - \langle \Delta X_s(\omega), \Delta Y_s(\omega)\rangle\big]$. However, we have

$$|\Delta Y_s(\omega)|^2 \leq \langle \Delta X_s(\omega), \Delta Y_s(\omega)\rangle,$$

since otherwise the condition (2.1) would not be satisfied, and the inequality $I_3 \leq 0$ follows.

Step 3. The final part. Combining all the above facts and taking expectation of both sides of (2.8) gives $\mathbb{E}U_\lambda(Z_t) \leq \mathbb{E}U_\lambda(Z_0)$. Using (2.5), this estimate implies

$$\mathbb{P}(|Y_t| \geq \lambda) \leq P(\lambda)\mathbb{E}|X_t| + \mathbb{E}U_\lambda(Z_0) \leq P(\lambda)||X||_1 + \mathbb{E}U_\lambda(Z_0).$$

It remains to use the condition 4°: by (2.1), we have $U_\lambda(Z_0) \leq 0$.

As we will see, the method can be modified to the case when U_λ satisfies slightly less restrictive conditions; see below.

2.3. **Proof of** (2.4)**.** We consider the cases $0 < \lambda \leq 1$, $1 < \lambda \leq 3/2$ and $\lambda > 3/2$ separately.

The case $0 < \lambda \leq 1$. This follows immediately from (2.2); in fact, for these λ's the inequality (2.4) is valid without the assumption $||X||_\infty \leq 1$.

The case $1 < \lambda \leq 3/2$. Let $U_\lambda : S \to \mathbb{R}$ be given by

$$U_\lambda(x, y) = \frac{4}{(2\lambda-1)^2}(|y|^2 - \langle y, x\rangle).$$

Then 1° is obvious, since U_λ is of class C^∞ in the interior of S. To check 2°, note that for $|y| < \lambda$,

$$U_\lambda(x, y) \geq -\frac{1}{(2\lambda-1)^2}|x|^2 \geq -P(\lambda)|x| = 1_{\{|y|\geq\lambda\}} - P(\lambda)|x|$$

(in the second passage we have used $|x| \leq 1$). For $|y| \geq \lambda$, we have

$$U_\lambda(x, y) \geq \frac{4(\lambda^2 - \lambda|x|)}{(2\lambda-1)^2} = \frac{(4\lambda-1)(1-|x|)}{(2\lambda-1)^2} + 1 - P(\lambda)|x| \geq 1_{\{|y|\geq 1\}} - P(\lambda)|x|.$$

The condition 3° is obvious: we have

$$\langle U_{\lambda xx}(x, y)h, h\rangle + 2\langle U_{\lambda xy}(x, y)h, k\rangle + \langle U_{\lambda yy}(x, y)k, k\rangle = \frac{8}{(2\lambda-1)^2}(|k|^2 - \langle h, k\rangle),$$

so we can take $c(x, y) = 8(2\lambda - 1)^{-2}$. Finally, 4° is trivial. Therefore, using the above machinery, we get that (2.4) holds true.

The case $\lambda > 3/2$. This is the most difficult part, and the special function will be much more complicated. Introduce the following subsets of the strip S:

$$\begin{aligned} D_1 &= \{(x,y) \in S : |x| + |2y-x| \leq 1\}, \\ D_2 &= \{(x,y) \in S : 1 < |x| + |2y-x| \leq 2\lambda - 2\}, \\ D_3 &= \{(x,y) \in S : |x| + |2y-x| > 2\lambda - 2\} \end{aligned}$$

and let $U_\lambda : S \to \mathbb{R}$ be given by

$$U_\lambda(x,y) = \begin{cases} e^{3-2\lambda}(|y|^2 - \langle x,y \rangle) & \text{if } (x,y) \in D_1, \\ (1-|x|)e^{|x|+|2y-x|-2\lambda+2}/2 - e^{3-2\lambda}/4 & \text{if } (x,y) \in D_2, \\ \big[(|2y-x| - 2\lambda + 3)^2 - |x|^2 + 1 - e^{3-2\lambda}\big]/4 & \text{if } (x,y) \in D_3. \end{cases}$$

This function does not have the necessary smoothness, but this will be overcome with the use of a straightforward mollification. However, let us first verify that U_λ satisfies (2.5), the condition 3° on the large part of S, and the condition 4°.

Let us deal with the majorization (2.5). If $(x,y) \in D_1$, then

$$U_\lambda(x,y) \geq -\frac{1}{4}e^{3-2\lambda}|x|^2 \geq -\frac{1}{4}e^{3-2\lambda}|x| = 1_{\{|y| \geq \lambda\}} - P(\lambda)|x|,$$

where in the second passage we have used the bound $|x| \leq 1$. If $(x,y) \in D_2$, then

$$|y| \leq \frac{1}{2}(|x| + |2y-x|) \leq \lambda - 1,$$

so (2.5) holds trivially. If $(x,y) \in D_3$ and $|y| < \lambda$, we derive that

$$\begin{aligned} U_\lambda(x,y) &\geq \frac{1}{4}(-|x|^2 + 1 - e^{3-2\lambda}) \\ &= \frac{1}{4}(1-|x|)\big(|x| + 1 - e^{3-2\lambda}\big) - P(\lambda)|x| \geq 1_{\{|y| \geq \lambda\}} - P(\lambda)|x|. \end{aligned}$$

Finally, if $(x,y) \in D_3$ and $|y| \geq \lambda$, then $|2y-x| \geq 2\lambda - 1$ and

$$\begin{aligned} U_\lambda(x,y) &\geq \frac{1}{4}(4 - |x|^2 + 1 - e^{3-2\lambda}) \\ &= 1 + \frac{1}{4}(1-|x|)\big(|x| + 1 - e^{3-2\lambda}\big) - P(\lambda)|x| \geq 1_{\{|y| \geq \lambda\}} - P(\lambda)|x|. \end{aligned}$$

The next step is to verify the condition (2.6), under the additional assumption that $(x,y) \in D_1^o \cup D_2^o \cup D_3^o$ and that $|x|$, $|2y-x|$ are nonzero. If $(x,y) \in D_1^o$, then the left-hand side of (2.6) equals $2e^{3-2\lambda}(|k|^2 - \langle h,k \rangle)$, so one can take $c(x,y) = 2e^{3-2\lambda}$. Suppose that (x,y) belongs to the interior of D_2. The left-hand side of (2.6) is equal to the second derivative of $t \mapsto U_\lambda(x+th, y+tk)$ at 0. For $x \neq 0$, we have

$$\frac{d}{dt}|x+th|\Big|_{t=0} = \langle x', h \rangle \qquad \text{and} \qquad \frac{d^2}{dt^2}|x+th|\Big|_{t=0} = \frac{|x|^2|h|^2 - \langle x,h \rangle^2}{|x|^3}$$

(where $x' = x/|x|$). Therefore, the left-hand side of (2.6) equals

$$\frac{1}{2}e^{|x|+|2y-x|-2\lambda+2}(A+B+C),$$

where

$$\begin{aligned} A &= -|x|\left[\langle x', h \rangle + \langle (2y-x)', 2k-h \rangle\right]^2, \\ B &= 4(|k|^2 - \langle k,h \rangle), \\ C &= -\left[|2k-h|^2 - \langle (2y-x)', 2k-h \rangle^2\right](|x| + |2y-x| - 1)/|2y-x|. \end{aligned}$$

Since A and C are nonpositive, we may take $c(x,y) = 2e^{|x|+|2y-x|-2\lambda+2}$. Finally, suppose that (x,y) lies in D_3^o and write the identity

$$(|2y-x|-2\lambda+3)^2 = |2y-x|^2 + (2\lambda-3)^2 + 2(-2\lambda+3)|2y-x|.$$

The function $(x,y) \mapsto 2(-2\lambda+3)|2y-x|$ is concave, so if we omit this term while computing the left-hand side of (2.6), we obtain

$$\langle U_{\lambda xx}(x,y)h,h\rangle + 2\langle U_{\lambda xy}(x,y)h,k\rangle + \langle U_{\lambda yy}(x,y)k,k\rangle \leq 2(|k|^2 - \langle h,k\rangle).$$

Consequently, $c(x,y) = 2$ works fine; note that the function c which we have just introduced is bounded. Finally, we check 4°. If $(x,y) \in D_1$, then the condition is obvious. Suppose that $(x,y) \in D_2$. The inequality $|y|^2 \leq \langle x,y\rangle$ is equivalent to $|2y-x| \leq |x|$, so we have

$$U_\lambda(x,y) \leq \frac{1}{2}(1-|x|)e^{2|x|-2\lambda+2} - \frac{1}{4}e^{3-2\lambda} \leq 0,$$

because of the elementary bound $(1-t)e^{2t} \leq e/2$. If $(x,y) \in D_3$, then, by the definition of D_3, $|2y-x| > 2\lambda-2-|x| \geq 2\lambda-3$; on the other hand, $|2y-x| \leq |x|$ (which follows from $|y|^2 \leq \langle x,y\rangle$), so

$$U_\lambda(x,y) + \frac{1}{4}e^{3-2\lambda} \leq \frac{1}{4}\big[(|x|-2\lambda+3)^2 - |x|^2 + 1\big] = \frac{1}{4}\big[(2\lambda-3)^2 + 1 - 2(2\lambda-3)|x|\big].$$

Using again the definition of D_3 and the inequality $|2y-x| \leq |x|$, we see that $|x| > \lambda - 1$, which implies

$$U_\lambda(x,y) + \frac{1}{4}e^{3-2\lambda} < \frac{1}{4}\big[(2\lambda-3)^2 + 1 - 2(2\lambda-3)(\lambda-1)\big] = \frac{1}{4}\big[1 + (3-2\lambda)\big].$$

The latter expression does not exceed $e^{3-2\lambda}/4$. This completes the proof of 4°.

Now we carry out the mollification argument. Consider a C^∞ function $g : \mathcal{H}\times\mathcal{H} \to [0,\infty)$, supported on the unit ball of $\mathcal{H}\times\mathcal{H}$ and satisfying $\int_{\mathcal{H}\times\mathcal{H}} g = 1$. For a given $\delta \in (0,1/4)$, let $U_\lambda^{(\delta)}$ be defined on $(1-\delta)S = \{(x,y) : |x| \leq 1-\delta\}$ by the convolution

$$U_\lambda^{(\delta)}(x,y) = \int_{[-1,1]^d\times[-1,1]^d} U_\lambda(x+\delta u, y+\delta v)g(u,v)\mathrm{d}u\mathrm{d}v.$$

Of course, $U_\lambda^{(\delta)}$ is of class C^∞ in the interior of its domain. This function inherits the crucial properties from U_λ. Namely, we have the following version of (2.5):

$$U_\lambda^{(\delta)}(x,y) \geq 1_{\{|y|\geq\lambda+\delta\}} - P(\lambda)(|x|+\delta) \tag{2.9}$$

for all $(x,y) \in (1-\delta)S$. Next, note that the function U_λ is of class C^1 on the set $S^o \setminus \{x = 0 \text{ or } 2y-x = 0\}$; therefore, integrating by parts implies

$$U_{\lambda xx}^{(\delta)}(x,y) = \int_{[-1,1]^d\times[-1,1]^d} U_{\lambda xx}(x+\delta u, y+\delta v)g(u,v)\mathrm{d}u\mathrm{d}v,$$

on the set $W = \{(x,y) \in (1-\delta)S : |x| \geq \delta \text{ and } |2y-x| \geq 3\delta\}$. Similar identities hold for $U_{\lambda xy}^{(\delta)}$ and $U_{\lambda yy}^{(\delta)}$, so (2.6) holds true, for all $(x,y) \in W$, with

$$c^{(\delta)}(x,y) = \int_{[-1,1]^d\times[-1,1]^d} c(x+\delta u, y+\delta v)g(u,v)\,\mathrm{d}u\mathrm{d}v \geq 0$$

(recall that c constructed above was bounded, so there is no problem with the integration). To apply the methodology described in §2.2, we need to ensure that the martingale pair takes values in W. To this end, we add one dimension and replace $\mathcal{H}$ by $\mathcal{H}\times\mathbb{R}$. Consider a new pair $Z^{(\delta)}$ of $\mathcal{H}\times\mathbb{R}$-valued martingales $X^{(\delta)} = \big((1-4\delta)X, 3\delta\big)$ and $Y^{(\delta)} = \big((1-4\delta)Y, 0\big)$. Then $Z^{(\delta)} \in W$ almost surely, so we are permitted to repeat the arguments of §2.2 to $U_\lambda^{(\delta)}$ and $Z^{(\delta)}$. As the result, we obtain $\mathbb{E}U_\lambda^{(\delta)}(Z_t^{(\delta)}) \leq \mathbb{E}U_\lambda^{(\delta)}(Z_0^{(\delta)})$, and combining this with (2.9), we arrive at

$$\mathbb{P}(|Y_t^{(\delta)}| \geq \lambda+\delta) \leq P(\lambda)(||X^{(\delta)}||_1 + \delta) + \mathbb{E}U_\lambda^{(\delta)}(Z_0^{(\delta)}).$$

Letting $\delta \to 0$ we obtain (2.7), which immediately leads us to the desired bound.

2.4. **Sharpness of** (2.4)**.** Suppose that B is a standard one-dimensional Brownian motion. We will prove that for any λ there is a nonzero stopping time τ and a predictable process H taking values in $[0,1]$ such that if $X = (B_{\tau\wedge t})_{t\geq 0}$ and $Y = H\cdot X$, then $||X||_\infty \leq 1$ and both sides of (2.4) are either equal, or as close as we wish. As previously, we consider the cases $\lambda \leq 1$, $1 < \lambda \leq 3/2$ and $\lambda > 3/2$ separately.

The case $0 < \lambda \leq 1$. Here the example is straightforward: we take $\tau = \inf\{t : |B_t| = \lambda\}$ and $H \equiv 1$. Then $Y = X$,

$$1 = \mathbb{P}(Y^* \geq \lambda) = ||X||_1/\lambda.$$

The case $1 < \lambda \leq 3/2$. The idea is to construct first an appropriate Markov process taking values in $[-1,1]\times[-\lambda,\lambda]$ and then embed it into (X,Y) as above. Distinguish the following eleven points from $\mathbb{R}^2$: $A_0 = \left(\frac{1}{2},\frac{1}{2}\right)$, $A_1 = \left(1,\frac{1}{2}\right)$, $A_2 = \left(\frac{3}{2}-\lambda,\frac{1}{2}\right)$, $A_3 = (1,\lambda)$, $A_4 = \left(\frac{1}{2}-\lambda,-\frac{1}{2}\right)$ and $A_5 = -A_1$, $A_6 = -A_2$, $A_7 = -A_3$, $A_8 = -A_4$, $A_9 = A_1$, $A_{10} = A_2$. Consider a Markov martingale (f,g), uniquely determined by the following conditions:

(i) We have $(f_0,g_0) \equiv A_0$.
(ii) For any $0 \leq k \leq 4$, the state A_{2k} leads to A_{2k+1} or to A_{2k+2}.
(iii) The states A_1, A_3, A_5 and A_7 are absorbing.

Let us gather some relevant information about the behavior of the pair (f,g). For any $0 \leq k \leq 4$, the line segments $A_{2k}A_{2k+1}$ and $A_{2k}A_{2k+2}$ are of slope 0 or 1: therefore, we may embed the pair (f,g) into the martingale (X,Y), where X is a one-dimensional Brownian motion started at $1/2$ and stopped at τ, its exit time from $[-1,1]$, and $Y = H\cdot X$ for a certain predictable process H taking values in $\{0,1\}$. To be more precise, note that there is a nondecreasing sequence $(\tau_n)_{n\geq 0}$, adapted to the filtration generated by X, such that $(X_{\tau_k})_{k\geq 0}$ and f have the same distribution. Next, put $H_0 \equiv 1$ and

$$H_t = \begin{cases} 0 & \text{if } \tau_{2k} < t \leq \tau_{2k+1}, \\ 1 & \text{if } \tau_{2k+1} < t \leq \tau_{2k+2}, \end{cases}$$

whenever $t < \tau$, and $H_t = 0$ for $t \geq \tau$. This implies $Y_{\tau_{2k+1}} - Y_{\tau_{2k}} = 0$ and $Y_{\tau_{2k+2}} - Y_{\tau_{2k+1}} = X_{\tau_{2k+2}} - X_{\tau_{2k+1}} = f_{2k+2} - f_{2k+1}$, so the pair $(X_{\tau_k}, Y_{\tau_k})_{k\geq 0}$ has the same distribution as (f,g). Now, observe that the terminal value (X_τ, Y_τ) takes values in the set $\{A_1, A_3, A_5, A_7\}$, and hence $2Y_\tau - X_\tau \in \{0, \pm(2\lambda-1)\}$. Furthermore, $Y_\tau \in \{\lambda, -\lambda\}$ if and only if $2Y - X \in \{2\lambda-1, -2\lambda+1\}$. Therefore,

$$\mathbb{P}(Y^* \geq \lambda) = \mathbb{P}(|Y_\tau| = \lambda) = \mathbb{P}(|2Y_\tau - X_\tau| = 2\lambda - 1) = \frac{\mathbb{E}|2Y_\tau - X_\tau|^2}{(2\lambda-1)^2}.$$

However, the martingales X, Y are bounded, so

$$\mathbb{E}|2Y_\tau - X_\tau|^2 = \mathbb{E}\big[2Y_\tau - X_\tau, 2Y_\tau - X_\tau\big] = \mathbb{E}\int_0^\tau |2H_s - 1|^2 \mathrm{d}s = \mathbb{E}\tau = \mathbb{E}|X_\tau|^2 = 1,$$

where in the third passage we have used the equality $|2H_s - 1| \equiv 1$ for all s. This implies that the constant $P(\lambda)$ is indeed the best in the range $1 < \lambda \leq 3/2$.

The case $\lambda > 3/2$. Here the reasoning is similar: first we construct an appropriate discrete-time Markov process. Fix a large positive integer N and let $\delta = (\lambda - 3/2)/N$. Consider the Markov martingale (f, g), given by the following:

(i) We have $(f_0, g_0) \equiv (1/2, 1/2)$.
(ii) Any state of the form (x, y), with $x \in (0, 1)$ and $y > 0$, leads to $(0, y)$ or to $(1, y)$.
(iii) Any state of the form $(0, y)$, where $0 < y < \lambda - 1$, leads to $(\delta, y + \delta)$ or to $(-1, y - 1)$.
(iv) The state $(0, \lambda - 1)$ leads to $(-1, \lambda - 2)$ or to $(1, \lambda)$.
(v) All the states lying on the lines $x = \pm 1$ are absorbing.

Arguing as previously, we may embed (f, g) into a pair (X, Y) such that X is a Brownian motion starting from $1/2$ and Y is an Itô integral, with respect to X, of a certain predictable process with values in $\{0, 1\}$. Directly from the transition probabilities above, we have

$$\begin{aligned}\mathbb{P}(Y^* \geq \lambda) &= \mathbb{P}(g^* \geq \lambda)\\ &= \mathbb{P}((f_0, f_1, \ldots, f_{2N+2}) = (1/2, 0, \delta, 0, \delta, 0, \ldots, \delta, 0, 1))\\ &= \frac{1}{4}\left(\frac{1-\delta}{1+\delta}\right)^N.\end{aligned}$$

Recall that $\delta = (\lambda - 3/2)/N$; thus, if N is sufficiently large, the above probability can be made arbitrarily close to $e^{3-2\lambda}/4$. This proves the desired sharpness of the bound (1.5).

2.5. **Proof of Theorem 2.2.** Finally, we show how to deduce the exponential inequality (2.3) from (2.4) and study the behavior of C_K as $K \to 1/2$ and $K \to \infty$. For X, Y as in the statement and any $t \geq 0$, we have

$$\begin{aligned}&\mathbb{E}\Phi\left(\frac{|Y_t|}{K}\right)\\ &\leq \mathbb{E}\Phi\left(\frac{Y^*}{K}\right)\\ &= \frac{1}{K}\int_0^\infty \Phi'\left(\frac{\lambda}{K}\right)\mathbb{P}(Y^* \geq \lambda)\mathrm{d}\lambda\\ &\leq \frac{1}{K}\left[\int_0^1 \frac{e^{\lambda/K}-1}{\lambda}\mathrm{d}\lambda + \int_1^{3/2}\frac{e^{\lambda/K}-1}{(2\lambda-1)^2}\mathrm{d}\lambda + \int_{3/2}^\infty \frac{(e^{\lambda/K}-1)e^{3-2\lambda}}{4}\mathrm{d}\lambda\right]||X||_1\\ &= \frac{1}{K^2}A_K||X||_1 + \frac{1}{8K(K-1/2)}B_K||X||_1,\end{aligned}$$

where the constants A_K, B_K are given by

$$A_K = \int_0^1 \frac{e^{\lambda/K}-1}{\lambda/K}\mathrm{d}\lambda + \int_1^{3/2}\frac{e^{\lambda/K}-1}{(2\lambda-1)^2/K}\mathrm{d}\lambda \quad\text{and}\quad B_K = Ke^{3/(2K)} - K + 1/2.$$

It suffices to note that A_K, B_K are decreasing functions of K; since $K > 1/2$, this implies

$$A_K \leq \int_0^1 \frac{e^{2\lambda}-1}{2\lambda}\mathrm{d}\lambda + \int_1^{3/2} \frac{e^{2\lambda}-1}{2(2\lambda-1)^2}\mathrm{d}\lambda, \quad B_K \leq e^3/2,$$

and (2.3) follows.

We turn to the lower bounds for the best constant in (2.3). Fix $K \in (1/2, \infty)$. We will use a Markov martingale similar to that used in the case $\lambda > 3/2$ above. Fix $\delta \in (0,1)$ and consider the pair (f,g), satisfying the following conditions:

(i) We have $(f_0, g_0) \equiv (1/2, 1/2)$.
(ii) Any state of the form (x,y), with $x \in (0,1)$ and $y > 0$, leads to $(0,y)$ or to $(1,y)$.
(iii) Any state of the form $(0,y)$, with $y > 0$, leads to $(\delta, y+\delta)$ or to $(-1, y-1)$.
(iv) All the states lying on the lines $x = \pm 1$ are absorbing.

Next, embed it in the pair $(X_t, Y_t)_{t \leq \tau}$ as above. For any positive integer n, the event

$$\{f = (1/2, 0, \delta, 0, \ldots, \delta, 1, 1, 1, 1, \ldots)\},$$

where the first 1 occurs on $2n+1$-st coordinate, has probability

$$\frac{1}{2}\left(\frac{1-\delta}{1+\delta}\right)^{n-1} \frac{\delta}{1+\delta}$$

and is contained in $\{|Y_\tau| = n\delta + 1/2\}$. Consequently,

$$\sup_{t \geq 0} \mathbb{E}\Phi(|Y_t|/K) = \mathbb{E}\Phi(|Y_\tau|/K) \geq \frac{\delta}{2(1+\delta)} \sum_{n=1}^{\infty} \Phi\left(\frac{n\delta}{K}\right)\left(\frac{1-\delta}{1+\delta}\right)^{n-1}.$$

However, if δ is sufficiently small, the latter sum can be made arbitrarily close to

$$\frac{1}{2}\int_0^\infty \Phi\left(\frac{s}{K}\right) e^{-2s}\mathrm{d}s = \frac{1}{16K(K-1/2)}.$$

This immediately yields the assertions concerning the order of C_K for $K \to 1/2$ and $K \to \infty$. Because of the explosion of the constant at $K = 1/2$, (2.3) does not hold with any finite C_K when $K \leq 1/2$.

3. Proof of (1.5)

This section is divided into two parts. The first of them describes the martingale representation of Fourier multipliers with symbols as in (1.1); the material is essentially taken from [1] and [2], and we have decided to include it here for the sake of completeness. The second subsection contains the proof of (1.5).

3.1. **The martingale representation of the Fourier multipliers** (1.1)**.** By the results from [2], we may assume that the Lévy measure ν satisfies the symmetry condition $\nu(B) = \nu(-B)$ for all Borel subsets B of $\mathbb{R}^d$ (more precisely, for any ν there is a symmetric $\bar{\nu}$ which leads to the same multiplier). Assume in addition that $|\nu| = \nu(\mathbb{R}^d)$ is finite and nonzero, and define $\tilde{\nu} = \nu/|\nu|$. Consider the independent random variables T_{-1}, T_{-2}, $\ldots$, Z_{-1}, Z_{-2}, $\ldots$ such that for each $n = -1, -2, \ldots$, T_n has exponential distribution with parameter $|\nu|$ and Z_n takes values in $\mathbb{R}^d$ and has $\tilde{\nu}$ as the distribution. Next, put $S_n = -(T_{-1}+T_{-2}+\ldots+T_n)$ for $n = -1, -2, \ldots$ and let

$$X_{s,t} = \sum_{s<S_j\leq t} Z_j, \qquad X_{s,t-} = \sum_{s<S_j<t} Z_j, \qquad \Delta X_{s,t} = X_{s,t} - X_{s,t-},$$

for $-\infty < s \leq t \leq 0$. For a given $f \in L^\infty(\mathbb{R}^d)$, define its parabolic extension $\mathcal{U}_f$ to $(-\infty, 0] \times \mathbb{R}^d$ by

$$\mathcal{U}_f(s,x) = \mathbb{E}f(x + X_{s,0}).$$

Next, fix $x \in \mathbb{R}^d$, $s < 0$ and let $f, \phi \in L^\infty(\mathbb{R}^d)$. We introduce the processes $F = (F_t^{x,s,f})_{s\leq t\leq 0}$ and $G = (G_t^{x,s,f,\phi})_{s\leq t\leq 0}$ by

$$\begin{aligned} F_t &= \mathcal{U}_f(t, x + X_{s,t}),\\ G_t &= \sum_{s<u\leq t} \big[\Delta F_u \cdot \phi(\Delta X_{s,u})\big]\\ &\quad - \int_s^t \int_{\mathbb{R}^d} \big[\mathcal{U}_f(v, x + X_{s,v-} + z) - \mathcal{U}_f(v, x + X_{s,v-})\big]\phi(z)\nu(\mathrm{d}z)\mathrm{d}v. \end{aligned} \tag{3.1}$$

These processes are martingales adapted to the filtration $\mathcal{F}_t = \sigma(X_{s,t} : t \in [s,0])$ (see [1], [2]). The key fact is the following.

Lemma 3.1. *If ϕ takes values in $[0,1]$, then the pair $(F^{x,s,f}, G^{x,s,f,\phi})$ satisfies* (2.1).

Proof. The assertion follows immediately from the identities

$$[F,G]_t = \sum_{s<u\leq t} |\Delta F_u|^2 \phi(\Delta X_{s,u}) \quad \text{ and } \quad [G,G]_t = \sum_{s<u\leq t} |\Delta F_u|^2 (\phi(\Delta X_{s,u}))^2,$$

which can be established by repeating the reasoning from [1]. $\square$

Now we introduce a family of multipliers. Fix $s < 0$, a function ϕ on $\mathbb{R}^d$ taking values in the unit ball of $\mathbb{C}$ and define the operator $\mathcal{T} = \mathcal{T}^s$ by the bilinear form

$$\int_{\mathbb{R}^d} \mathcal{T}f(x)g(x)\mathrm{d}x = \int_{\mathbb{R}^d} \mathbb{E}\big[G_0^{x,s,f,\phi} g(x + X_{s,0})\big]\mathrm{d}x, \tag{3.2}$$

where $f, g \in C_0^\infty(\mathbb{R}^d)$. We have the following fact, proved in [1].

Lemma 3.2. *Let $1 < p < \infty$ and $d \geq 2$. The operator $\mathcal{T}^s$ is well defined and extends to a bounded operator on $L^p(\mathbb{R}^d)$, which can be expressed as a Fourier multiplier with the symbol*

$$\begin{aligned} M(\xi) &= M_s(\xi)\\ &= \left[1 - \exp\left(2s\int_{\mathbb{R}^d}(1-\cos\langle\xi,z\rangle)\nu(dz)\right)\right] \frac{\int_{\mathbb{R}^d}(1-\cos\langle\xi,z\rangle)\phi(z)\nu(dz)}{\int_{\mathbb{R}^d}(1-\cos\langle\xi,z\rangle)\nu(dz)} \end{aligned}$$

if $\int_{\mathbb{R}^d}(1-\cos\langle\xi,z\rangle)\nu(dz) \neq 0$, and $M(\xi) = 0$ otherwise. Furthermore, (3.2) *holds true for all $f \in C_0^\infty(\mathbb{R}^d)$ and all g belonging to $L^q(\mathbb{R}^d)$ for some $1 < q < \infty$.*

3.2. **Proof of** (1.5)**.** We start with proving the dual version of (1.5).

Theorem 3.3. *Assume that $K > 1/2$ and let $m : \mathbb{R}^d \to \mathbb{C}$ be a multiplier as in Theorem 1.3. Then for any function $f \in L^1(\mathbb{R}^d)$ taking values in the unit ball of $\mathbb{C}$ we have*

$$||\Phi(|T_m f|/K)||_{L^1(\mathbb{R}^d)} \leq \frac{C_K}{K}||f||_{L^1(\mathbb{R}^d)}. \tag{3.3}$$

Proof. We divide the proof into two parts.

Step 1. First we show the estimate for the multipliers of the form

$$M_{\phi,\nu}(\xi) = \frac{\int_{\mathbb{R}^d}(1-\cos\langle\xi,z\rangle)\phi(z)\nu(\mathrm{d}z)}{\int_{\mathbb{R}^d}(1-\cos\langle\xi,z\rangle)\nu(\mathrm{d}z)}. \tag{3.4}$$

In addition, we assume that $0 < \nu(\mathbb{R}^d) < \infty$, so that the above approach using Lévy processes is applicable. Fix $s < 0$ and functions f, $g \in C_0^\infty(\mathbb{R}^d)$ such that f is bounded by 1; then the martingale $F^{x,s,f}$ also takes values in the unit ball of $\mathbb{C}$. By Young inequality, Fubini's theorem, Lemma 3.1 and (2.3), we have

$$\begin{aligned}
&\left|\int_{\mathbb{R}^d}\mathbb{E}\left[\frac{G_0^{x,s,f,\phi}}{K}g(x+X_{s,0})\right]\mathrm{d}x\right| \\
&\qquad \leq \int_{\mathbb{R}^d}\mathbb{E}\Phi\left(\frac{|G_0^{x,s,f,\phi}|}{K}\right)\mathrm{d}x + \int_{\mathbb{R}^d}\mathbb{E}\Psi\big(|g(x+X_{s,0})|\big)\mathrm{d}x \\
&\qquad \leq \frac{C_K}{K}\int_{\mathbb{R}^d}\mathbb{E}|F_0^{x,s,f}|\mathrm{d}x + \int_{\mathbb{R}^d}\Psi(|g(x)|)\mathrm{d}x \\
&\qquad = \frac{C_K}{K}\int_{\mathbb{R}^d}|f(x)|\mathrm{d}x + \int_{\mathbb{R}^d}\Psi(|g(x)|)\mathrm{d}x.
\end{aligned} \tag{3.5}$$

Plugging this into the definition of $\mathcal{T}^s$ (see (3.2)), we obtain

$$\int_{\mathbb{R}^d}\left[\frac{\mathcal{T}^s f(x)}{K}g(x) - \Psi(|g(x)|)\right]\mathrm{d}x \leq \frac{C_K}{K}||f||_{L^1(\mathbb{R}^d)}.$$

Now fix $M > 0$ and put

$$g(x) = \frac{\overline{\mathcal{T}^s f(x)}}{|\mathcal{T}^s f(x)|}\left[\exp\left(\min\left\{\frac{|\mathcal{T}^s f(x)|}{K}, M\right\}\right) - 1\right]$$

(if $\mathcal{T}^s f(x) = 0$, set $g(x) = 0$). We have $|g| \leq c|\mathcal{T}^s f|$ for some constant c depending on M and K; furthermore, $\mathcal{T}^s f \in L^2(\mathbb{R}^d)$, directly from the formula for the symbol of the multiplier (and the fact that $f \in L^2(\mathbb{R}^d)$). Consequently, plugging g into the preceding inequality gives

$$\begin{aligned}
&\int_{\mathbb{R}^d}\Phi\left(\left|\frac{\mathcal{T}^s f(x)}{K}\right|\right)1_{\{|\mathcal{T}^s f(x)|\leq MK\}} \\
&\qquad + \left(\frac{|\mathcal{T}^s f(x)|(e^M-1)}{K} - \Psi(e^M-1)\right)1_{\{|\mathcal{T}^s f(x)|>MK\}}\mathrm{d}x \leq \frac{C_K}{K}||f||_{L^1(\mathbb{R}^d)}
\end{aligned}$$

and hence, by Fatou's lemma, if we let $M \to \infty$, we get

$$\int_{\mathbb{R}^d}\Phi\left(\left|\frac{\mathcal{T}^s f(x)}{K}\right|\right)\mathrm{d}x \leq \frac{C_K}{K}||f||_{L^1(\mathbb{R}^d)}.$$

Now if we let $s \to -\infty$, then M_s converges pointwise to the multiplier $M_{\phi,\nu}$ given by (3.4). By Plancherel's theorem, $\mathcal{T}^s f \to T_{M_{\phi,\nu}} f$ in L^2 and hence there is a sequence

$(s_n)_{n=1}^{\infty}$ converging to $-\infty$ such that $\lim_{n\to\infty}\mathcal{T}^{s_n}f \to T_{M_{\phi,\nu}}f$ almost everywhere. Thus Fatou's lemma yields the desired bound for the multiplier $T_{M_{\phi,\nu}}$.

Step 2. Now we deduce the result for the general multipliers as in (1.1) and drop the assumption $0<\nu(\mathbb{R}^d)<\infty$. For a given $\varepsilon>0$, define a Lévy measure ν_ε in polar coordinates $(r,\theta)\in(0,\infty)\times\mathbb{S}$ by

$$\nu_\varepsilon(\mathrm{d}r\mathrm{d}\theta)=\varepsilon^{-2}\delta_\varepsilon(\mathrm{d}r)\mu(d\theta),$$

where δ_ε denotes Dirac measure on $\{\varepsilon\}$. Next, consider a multiplier m_ε as in (3.4), in which the Lévy measure is $1_{\{|x|>\varepsilon\}}\nu+\nu_\varepsilon$ and the jump modulator is given by $1_{\{|x|>\varepsilon\}}\phi(x)+1_{\{|x|=\varepsilon\}}\psi(x/|x|)$. This yields the claim by applying the previous step to ν_ε and letting $\varepsilon\to 0$. Indeed, we have

$$\begin{aligned}\int_{\mathbb{R}^d}[1-\cos\langle\xi,x\rangle]\psi(x/|x|)\nu_\varepsilon(\mathrm{d}x)&=\int_{\mathbb{S}}\psi(\theta)\frac{1-\cos\langle\xi,\varepsilon\theta\rangle}{\varepsilon^2}\mu(d\theta)\\&\xrightarrow{\varepsilon\to 0}\frac{1}{2}\int_{\mathbb{S}}\langle\xi,\theta\rangle^2\psi(\theta)\mu(\mathrm{d}\theta)\end{aligned}$$

so, as above, it suffices to use Plancherel's theorem and pass to the subsequence which converges almost everywhere. □

Proof of (1.5). Let us skip the lower indices and write m instead of $m_{\phi,\psi,\mu,\nu}$. Fix $f\in L^2(\mathbb{R}^d)$ and put $g=T_mf1_A/|T_mf|$ ($g=0$ if the denominator is zero). We have

$$\begin{aligned}\int_A|T_mf(x)|\mathrm{d}x&=\int_{\mathbb{R}^d}T_mf(x)\overline{g(x)}\mathrm{d}x\\&=\int_{\mathbb{R}^d}\widehat{T_mf}(x)\overline{\widehat{g}(x)}\mathrm{d}x\\&=\int_{\mathbb{R}^d}\widehat{f}(x)\overline{\widehat{T_{\bar{m}}g}(x)}\mathrm{d}x\\&=\int_{\mathbb{R}^d}f(x)\overline{T_{\bar{m}}g(x)}\mathrm{d}x\\&\le K\int_{\mathbb{R}^d}\Psi(|f(x)|)\mathrm{d}x+K\int_{\mathbb{R}^d}\Phi(|T_{\bar{m}}g(x)|/K)\mathrm{d}x\\&\le K\int_{\mathbb{R}^d}\Psi(|f(x)|)\mathrm{d}x+C_K||g||_{L^1(\mathbb{R}^d)}.\end{aligned}\tag{3.6}$$

Here in the fifth line we have exploited Young's inequality and in the latter passage we have used (3.3) and the fact that g takes values in the unit ball of $\mathbb{C}$. It suffices to note that $||g||_{L^1(\mathbb{R}^d)}\le|A|$ to complete the proof for square-integrable f. For general functions from the class LlogL we use a straightforward approximation: there is a sequence $(f_n)_{n\ge1}\subset L^2(\mathbb{R}^d)$ such that $\int_{\mathbb{R}^d}\Psi(|f_n|)\xrightarrow{n\to\infty}\int_{\mathbb{R}^d}\Psi(|f|)$ and $T_mf_n\xrightarrow{n\to\infty}T_mf$ almost everywhere. □

Let us present some examples, following the exposition in [2]. Let $\mu\equiv 0$ and let ν be the Lévy measure of a non-zero symmetric α-stable Lévy process in $\mathbb{R}^d$, $\alpha\in(0,2)$. In polar coordinates we have (see e.g. [17]),

$$\nu(\mathrm{d}r\mathrm{d}\theta)=r^{-1-\alpha}\mathrm{d}r\sigma(\mathrm{d}\theta),\qquad r>0,\ \theta\in\mathbb{S},$$

where the so-called spectral measure σ is finite and non-zero on $\mathbb{S}$. Pick a function $\phi:\mathbb{R}^d\to[0,1]$ bounded by 1 and homogeneous of order 0, that is, satisfying

$\phi(x) = \phi(x/|x|)$ for $x \neq 0$. Let $c_\alpha = \int_0^\infty [1-\cos s] s^{-1-\alpha} \mathrm{d}s$. We have

$$\begin{aligned}\int_{\mathbb{R}^d} [1-\cos\langle \xi, x\rangle]\phi(x)\nu(\mathrm{d}x) &= \int_{\mathbb{S}}\int_0^\infty [1-\cos\langle \xi, r\theta\rangle]\phi(r\theta) r^{-1-\alpha} \mathrm{d}r\sigma(\mathrm{d}\theta)\\ &= c_\alpha \int_{\mathbb{S}} |\langle \xi,\theta\rangle|^\alpha \phi(\theta)\sigma(\mathrm{d}\theta)\end{aligned}$$

and thus Theorem 1.3 gives that the multiplier with the symbol

$$M_\alpha(\xi) = \frac{\int_{\mathbb{S}} |\langle \xi,\theta\rangle|^\alpha \phi(\theta)\sigma(\mathrm{d}\theta)}{\int_{\mathbb{S}} |\langle \xi,\theta\rangle|^\alpha \sigma(\mathrm{d}\theta)} \tag{3.7}$$

satisfies (1.5). In particular, if we take σ to be the probability measure satisfying

$$\sigma(\{(1,0,0,\ldots,0)\}) = \sigma(\{(0,1,0,\ldots,0)\}) = \ldots = \sigma(\{(0,0,\ldots,0,1)\}) = 1/d$$

and ϕ is the indicator function of the j-th axis, we obtain Marcinkiewicz-type multipliers (see Stein [18], p. 110):

$$M_{\alpha,j}(\xi) = \frac{|\xi_j|^\alpha}{|\xi_1|^\alpha + |\xi_2|^\alpha + \ldots + |\xi_d|^\alpha}.$$

If we let $\alpha \uparrow 2$, we obtain the second-order Riesz transforms R_j^2. To give another example, suppose that d is even: $d = 2n$, and let σ be the uniform measure on

$$\{x \in \mathbb{S} : x_1^2 + \ldots + x_n^2 = 1 \text{ or } x_{n+1}^2 + x_{n+2}^2 + \ldots + x_{2n}^2 = 1\}.$$

If ϕ is the indicator function of $\{x \in \mathbb{S} : x_1^2 + \ldots + x_n^2 = 1\}$, then (3.7) becomes

$$M(\xi) = \frac{|\xi_1^2 + \xi_2^2 + \ldots + \xi_n^2|^{\alpha/2}}{|\xi_1^2 + \xi_2^2 + \ldots + \xi_n^2|^{\alpha/2} + |\xi_{n+1}^2 + \xi_{n+2}^2 + \ldots + \xi_{2n}^2|^{\alpha/2}}. \tag{3.8}$$

Finally, we mention an example which is induced by the class of the so-called tempered stable Lévy processes [16]. As previously, take $\mu \equiv 0$ and define the Lévy measure ν in polar coordinates by

$$\nu(\mathrm{d}r\mathrm{d}\theta) = r^{-1}e^{-r}\mathrm{d}r\sigma(\mathrm{d}\theta), \qquad r > 0,\ \theta \in \mathbb{S},$$

where σ is as above. This choice leads to the multiplier

$$M(\xi) = \frac{\int_{\mathbb{S}} \log[1 + \langle \xi,\theta\rangle^2]\phi(\theta)\sigma(\mathrm{d}\theta)}{\int_{\mathbb{S}} \log[1 + \langle \xi,\theta\rangle^2]\sigma(\mathrm{d}\theta)},$$

which, in virtue of Theorem 1.3, satisfies (1.5). (We would like to point out the misprint in the formula for $M(\xi)$ in [2]. The authors of that paper will likely appreciate a corrected version of the formula.) In particular, by choosing ϕ, σ as above, we get the logarithmic estimate for the multipliers

$$M_j(\xi) = \frac{\log(1+\xi_j^2)}{\log(1+\xi_1^2) + \log(1+\xi_2^2) + \ldots + \log(1+\xi_d^2)},$$

for $j = 1, 2, \ldots, d$.

In the remainder of this section we discuss the possibility of extending the assertion of Theorems 1.3 and 3.3 to the vector-valued multipliers. Note that for any bounded function $m = (m_1, m_2, \ldots, m_n) : \mathbb{R}^d \to \mathbb{C}^n$, we may define the associated Fourier multiplier acting on complex valued functions on $\mathbb{R}^d$ by the formula $T_m f = (T_{m_1}f, T_{m_2}f, \ldots, T_{m_n}f)$. The reasoning presented above can be easily modified to yield the following statement.

Theorem 3.4. *Let ν, μ be two measures on $\mathbb{R}^d$ and $\mathbb{S}$, respectively, satisfying the assumptions of Theorem 1.3. Assume further that ϕ, ψ are two Borel functions on $\mathbb{R}^d$ taking values in the cube $[0,1]^n$ and let $m:\mathbb{R}^d\to\mathbb{R}^n$ be the associated symbol given by (1.1). Let $K>1/2$ be a fixed number.*

(i) For any integrable function f on $\mathbb{R}^d$, taking values in the unit ball of $\mathbb{C}$,

$$\int_{\mathbb{R}^d}\Phi\left(\left|\frac{T_mf(x)}{K}\right|\right)dx\leq\frac{C_K}{K}||f||_{L^1(\mathbb{R}^d)}.$$

(ii) For any $f\in LlogL(\mathbb{R}^d)$ and any Borel subset A of $\mathbb{R}^d$,

$$\int_A|T_mf(x)|dx\leq K\int_{\mathbb{R}^d}\Psi(|f(x)|)dx+C_K|A|.$$

The proof is word-by-word repetition of the argumentation from [15] and is omitted.

4. On the lower bound for the constant in (1.5)

4.1. **The case $d=2$.** We start with the two-dimensional setting, in which a very convenient tool, the Beurling-Ahlfors transform, is available. Recall that this operator is a Fourier multiplier with the symbol $m(\xi)=\overline{\xi}/\xi$, $\xi\in\mathbb{C}$; alternatively, it can be defined by the singular integral

$$\mathcal{BA}f(z)=-\frac{1}{\pi}\,\text{p.v.}\int_{\mathbb{C}}\frac{f(w)}{(z-w)^2}\mathrm{d}w$$

(here and below, we identify $\mathbb{R}^2$ with the complex plane $\mathbb{C}$). The fundamental property of this object is that it changes the complex derivative $\overline{\partial}$ to ∂. Precisely, for any f from the Sobolev space $W^{1,2}(\mathbb{C},\mathbb{C})$ we have

$$\mathcal{BA}(\overline{\partial}f)=\partial f, \tag{4.1}$$

where, as usual,

$$\partial f=\frac{1}{2}\left(\frac{\partial f}{\partial x}-i\frac{\partial f}{\partial y}\right),\qquad\overline{\partial}f=\frac{1}{2}\left(\frac{\partial f}{\partial x}+i\frac{\partial f}{\partial y}\right).$$

Directly from the form of the symbol, we infer that $\mathcal{BA}=(R_1^2-R_2^2)+2iR_1R_2$, where R_1, R_2 are planar Riesz transforms, and hence $R_1^2f=(-f+(\operatorname{Re}\mathcal{BA})f)/2$, if we restrict ourselves to the real-valued functions f (here we have used the identity $-\mathrm{Id}=R_1^2+R_2^2$, which follows directly from the passage to Fourier transforms).

We are ready to establish the second part of Theorem 1.3. Consider the following example. For a fixed $\alpha\in(0,1/2)$, let $R>0$ be given by the equation $R^{2\alpha}=1-\alpha$ and define $w:\mathbb{C}\to\mathbb{C}$ by

$$w(z)=\begin{cases}\overline{z}|z|^{-2\alpha}-\overline{z} & \text{if } |z|\leq R,\\ R^{2-2\alpha}z^{-1}-R^2z^{-1} & \text{if } |z|>R.\end{cases}$$

We easily check that $w\in W^{1,2}(\mathbb{C},\mathbb{C})$ and derive that

$$\partial w(z)=\begin{cases}\alpha\overline{z}^2|z|^{-2\alpha-2} & \text{if } |z|<R,\\ -R^{2-2\alpha}z^{-2}+R^2z^{-2} & \text{if } |z|>R\end{cases}$$

and

$$\overline{\partial}w(z)=\begin{cases}(1-\alpha)|z|^{-2\alpha}-1 & \text{if } |z|<R,\\ 0 & \text{if } |z|>R.\end{cases}$$

Finally, put $A=\{z\in\mathbb{C}:|z|\leq R\}$ and $f=\bar{\partial}w$. We have

$$\int_{\mathbb{C}}\Psi(|f(z)|)\mathrm{d}z=\pi R^{2-2\alpha}\left[\log(1-\alpha)+\frac{2\alpha-1}{1-\alpha}-2\alpha\log R\right]+\pi R^2=\frac{\alpha^2|A|}{(1-\alpha)^2}$$

and

$$\begin{aligned}\int_A|R_1^2f(z)|\mathrm{d}z&=\frac{1}{2}\int_A\big|-f(z)+(\operatorname{Re}\mathcal{BA})f(z)\big|\,\mathrm{d}z\\&=\frac{1}{2}\int_A\big|-(1-\alpha)|z|^{-2\alpha}+1+\alpha|z|^{-2\alpha-2}\operatorname{Re}\bar{z}^2\big|\,\mathrm{d}z\\&\geq\frac{1}{2}\int_A\big(1-\alpha-\alpha\operatorname{Re}(\bar{z}^2/|z|^2)\big)\,|z|^{-2\alpha}\,\mathrm{d}z-\frac{|A|}{2},\end{aligned}$$

where in the last line we have used the triangle inequality and the bound $\alpha<1/2$. Passing to polar coordinates and applying the identity $R^{2\alpha}=1-\alpha$, we verify that

$$\int_A|R_1^2f(z)|\mathrm{d}z\geq\pi\int_0^R(1-\alpha)r^{1-2\alpha}\,\mathrm{d}r-\frac{|A|}{2}=\frac{|A|}{2}\cdot\frac{\alpha}{1-\alpha}.$$

Now substitute $\alpha=(4K)^{-1}$ and plug the above facts into (1.5). If we divide both sides by $|A|$, we see that the constant C_K must satisfy

$$C_K\geq\frac{1}{|A|}\left[\int_A|R_1^2f|-K\int_{\mathbb{C}}\Psi(|f|)\right]\geq\frac{1}{2(4K-1)}-\frac{K}{(4K-1)^2}=\frac{2K-1}{2(4K-1)^2}.$$

This yields the claim in the two-dimensional setting.

4.2. **The case $d\geq 3$.** Suppose that for a fixed $K>1/2$ and some $D_K>0$ we have

$$\int_A|R_1^2f(x)|\mathrm{d}x\leq K\int_{\mathbb{R}^d}\Psi(|f(x)|)\mathrm{d}x+D_K\cdot|A| \tag{4.2}$$

for all Borel subsets A of $\mathbb{R}^d$ and all Borel functions $f:\mathbb{R}^d\to\mathbb{R}$. For $t>0$, define the dilation operator δ_t as follows: for any function $g:\mathbb{R}^2\times\mathbb{R}^{d-2}\to\mathbb{R}$, we let $\delta_tg(\xi,\zeta)=g(\xi,t\zeta)$; for any $A\subset\mathbb{R}^2\times\mathbb{R}^{d-2}$, let $\delta_tA=\{(\xi,t\zeta):(\xi,\zeta)\in A\}$. By (4.2), the operator $T_t:=\delta_t^{-1}\circ R_1^2\circ\delta_t$ satisfies

$$\begin{aligned}\int_A|T_tf(x)|\mathrm{d}x&=t^{d-2}\int_{\delta_t^{-1}A}|R_1^2\circ\delta_tf(x)|\mathrm{d}x\\&\leq t^{d-2}\left[K\int_{\mathbb{R}^d}\Psi(|\delta_tf(x)|)\mathrm{d}x+D_K\cdot|\delta_t^{-1}A|\right]\\&=K\int_{\mathbb{R}^d}\Psi(|f(x)|)\mathrm{d}x+D_K\cdot|A|.\end{aligned} \tag{4.3}$$

Now fix $f\in L^2(\mathbb{R}^d)$ satisfying $\int_{\mathbb{R}}\Psi(|f|)<\infty$. It is not difficult to check that the Fourier transform $\mathcal{F}$ satisfies the identity $\mathcal{F}=t^{d-2}\delta_t\circ\mathcal{F}\circ\delta_t$ and hence the operator T_t has the property that

$$\widehat{T_tf}(\xi,\zeta)=-\frac{\xi_1^2}{|\xi|^2+t^2|\zeta|^2}\widehat{f}(\xi,\zeta),\qquad(\xi,\zeta)\in\mathbb{R}^2\times\mathbb{R}^{d-2}.$$

By Lebesgue's dominated convergence theorem, we have

$$\lim_{t\to 0}\widehat{T_tf}(\xi,\zeta)=\widehat{T_0f}(\xi,\zeta)$$

in $L^2(\mathbb{R}^d)$, where $\widehat{T_0 f}(\xi,\zeta) = -\xi_1^2 \widehat{f}(\xi,\zeta)/|\xi|^2$. The convergence in $L^2(\mathbb{R}^d)$ implies the convergence in $L^1(A)$ provided $|A|$ is finite; therefore, (4.3) implies

$$\int_A |T_0 f(x)|\mathrm{d}x \leq K \int_{\mathbb{R}^d} \Psi(|f(x)|)\mathrm{d}x + D_K \cdot |A| \tag{4.4}$$

(if $|A| = \infty$ this is of course also true). Now, recall the function w and the set $B = \{z \in \mathbb{R}^2 : |z| \leq R\}$ from the previous subsection, and define $f : \mathbb{R}^2 \times \mathbb{R}^{d-2} \to \mathbb{R}$ by $f(\xi,\zeta) = \bar{\partial} w(\xi) 1_{[0,1]^{d-2}}(\zeta)$. Denoting by $\mathcal{R}_1$ the first planar Riesz transform, we have $T_0 f(\xi,\zeta) = (\mathcal{R}_1^2 \bar{\partial} w)(\xi) 1_{[0,1]^{d-2}}(\zeta)$, because of the identity

$$\widehat{T_0 f}(\xi,\zeta) = -\frac{\xi_1^2}{|\xi|^2} \widehat{\bar{\partial} w}(\xi) \widehat{1_{[0,1]^{d-2}}}(\zeta).$$

Plug this into (4.4) with the choice $A = B \times [0,1]^{d-2}$ to obtain

$$\int_B |\mathcal{R}_1^2 \bar{\partial} w(\xi)|\mathrm{d}\xi \leq K \int_{\mathbb{R}^2} \Psi(|\bar{\partial} w(\xi)|)\mathrm{d}\xi + D_K \cdot |B|.$$

As we have computed in the previous subsection, this implies

$$D_K \geq \frac{2K-1}{2(4K-1)^2}.$$

The proof is complete.

Acknowledgment

The author would like to express his gratitude to the anonymous Referee for the very careful reading and pointing out several gaps in the first version of the paper.

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