LOGARITHMIC ESTIMATES FOR NONSYMMETRIC MARTINGALE TRANSFORMS

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ABSTRACT. Let $d = (d_0, d_1, d_2, \ldots)$ be a martingale difference sequence and $\theta = (\theta_0, \theta_1, \theta_2, \ldots)$ be a predictable sequence taking values in $[0, 1]$. In the paper we study the inequality

$$\sup_n \mathbb{E}\left|\sum_{k=0}^n \theta_k d_k\right| \leq K \sup_n \mathbb{E}\left|\sum_{k=0}^n d_k\right| \log\left|\sum_{k=0}^n d_k\right| + L(K)$$

and show that it holds with some universal $L(K) < \infty$ if and only if $K > 1/2$. Furthermore, we determine the optimal value of $L(K)$ for $K \geq 1$ and the optimal order of $L(K)$ as $K \to 1/2$. Related estimates for stochastic integrals are also established.

1. Introduction

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, which is equipped with a nondecreasing family $(\mathcal{F}_n)_{n\geq 0}$ of sub-σ-fields of $\mathcal{F}$. Let $d = (d_n)_{n\geq 0}$ be an adapted real-valued martingale difference sequence and suppose $\theta = (\theta_n)_{n\geq 0}$ is a real-valued predictable sequence, that is, for any $n \geq 0$ the random variable θ_n is measurable with respect to $\mathcal{F}_{(n-1)\vee 0}$. As shown by Burkholder in [3], if θ is bounded in absolute value by 1, then for $1 < p < \infty$,

$$\left|\left|\sum_{k=0}^n \theta_k d_k\right|\right|_p \leq C_p \left|\left|\sum_{k=0}^n d_k\right|\right|_p, \quad n = 0, 1, 2, \ldots, \tag{1.1}$$

where $C_p = \max\{p-1, (p-1)^{-1}\}$, and the constant cannot be replaced by a smaller one. The inequality fails to hold for $p = 1$, but, as shown by the author in [6], we have, for $K > 1$,

$$\left|\left|\sum_{k=0}^n \theta_k d_k\right|\right|_1 \leq K\mathbb{E}\left|\sum_{k=0}^n d_k\right| \log\left|\sum_{k=0}^n d_k\right| + L_0(K), \quad n = 0, 1, 2, \ldots, \tag{1.2}$$

where

$$L_0(K) = \begin{cases} \frac{K^2}{2(K-1)} \exp(-K^{-1}) & \text{if } 1 < K < 2, \\ K \exp(K^{-1} - 1) & \text{if } K \geq 2. \end{cases}$$

The constant $L_0(K)$ is the best possible in (1.2); furthermore, the inequality fails to hold for $K \leq 1$ (that is, there is no universal $L_0(K) < \infty$ for which (1.2) is valid). The moment inequality (1.1) was studied by Choi under a different, non-symmetric assumption that the variables θ_n, $n = 0, 1, 2, \ldots$, take values in $[0, 1]$: the paper [5]

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contains the equations determining the optimal constants, for $1 < p < \infty$. There is a natural question about the logarithmic estimate (1.2) in the nonsymmetric setting. Our contribution is stated in the theorem below.

Theorem 1.1. *Let d be a martingale difference sequence and θ be a predictable sequence taking values in $[0, 1]$. Then for $K > 1/2$,*

$$\mathbb{E}\left|\sum_{k=0}^{n}\theta_k d_k\right| \leq K\mathbb{E}\left|\sum_{k=0}^{n} d_k\right| \log\left|\sum_{k=0}^{n} d_k\right| + L(K), \quad n = 0, 1, 2, \ldots, \tag{1.3}$$

where

$$L(K) = \begin{cases} K^2(2K-1)^{-1} & \text{if } 1/2 < K < 1, \\ K\exp(K^{-1}-1) & \text{if } K \geq 1. \end{cases} \tag{1.4}$$

The constant $L(K)$ is the best possible if $K \geq 1$. Furthermore, it is of the best order $O((K - 1/2)^{-1})$ when $K \to 1/2+$. For $K \leq 1/2$ there is no $L(K) < \infty$ for which (1.3) holds.

In fact, it will be shown that for $1/2 < K < 1$, the optimal constant L in (1.3) is not smaller than $\frac{K^2}{2K-1}\exp(1 - K^{-1})$, so we are quite close with our choice of $L(K)$.

Observe that, quite surprisingly, the lack of symmetry in the transforming sequence θ enlarges the set of K's, for which the logarithmic estimate is valid. Nothing like that happens for the moment estimate (1.1), which holds only for $1 < p < \infty$, both in the symmetric and nonsymmetric setting.

Standard approximation arguments yield a related estimate for stochastic integrals (see Section 16 in Burkholder [3], where it is presented how the results of Bichteler [1] can be used to transfer the inequalities from the discrete- to the continuous-time setting). Suppose that $(\Omega, \mathcal{F}, \mathbb{P})$ is complete and filtered by a right-continuous nondecreasing family $(\mathcal{F}_t)_{t\in[0,\infty)}$. In addition, assume that $\mathcal{F}_0$ contains all the events of probability 0.

Theorem 1.2. *Let $X = (X_t)$ be a real-valued right-continuous martingale with limits from the left and let $H = (H_t)$ be a predictable process taking values in $[0, 1]$. Then for any $K > 1/2$,*

$$\mathbb{E}\left|H_0X_0 + \int_{(0,t]} H_s dX_s\right| \leq K\mathbb{E}|X_t|\log|X_t| + L(K), \quad t \geq 0,$$

where $L(K)$ is given by (1.4). The constant is the best possible for $K \geq 1$ and of optimal order for $K \to 1/2+$. For $K \leq 1/2$, the inequality does not hold with any finite $L(K)$.

A few words about the organization of the paper. The inequality (1.3) is established in the next section. The final part of the paper is devoted to the optimality of the constant $L(K)$.

2. The proof of (1.3)

In [3], Burkholder proved that the function $u : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, given by

$$u(x, y) = \begin{cases} 1 + xy & \text{if } |x| \vee |y| \leq 1, \\ |x + y| & \text{if } |x| \vee |y| > 1, \end{cases}$$

satisfies the following condition: if $((X_n, Y_n))$ is an $\mathbb{R}^2$-valued martingale satisfying

$$(X_{n+1} - X_n)(Y_{n+1} - Y_n) \geq 0, \qquad n = 0, 1, 2, \ldots, \tag{2.1}$$

then, for any n,

$$\mathbb{E}u(X_0, Y_0) \leq \mathbb{E}u(X_1, Y_1) \leq \ldots \mathbb{E}u(X_n, Y_n). \tag{2.2}$$

Furthermore, as one easily verifies, we have

$$u(x, y) \geq 1 \quad \text{for all } x,\ y \text{ such that } xy \geq 0. \tag{2.3}$$

A key role in the paper is played by the following function $U_s : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, where $s > 0$ is a fixed parameter. Let

$$\begin{aligned} U_s(x, y) &= \int_s^\infty (1 - u(x/t, y/t))\mathrm{d}t \\ &= \begin{cases} -xy/s & \text{if } |x| \vee |y| \leq s, \\ |x| \vee |y| - s - \frac{xy}{|x|\vee|y|} - |x + y| \log \frac{|x|\vee|y|}{s} & \text{if } |x| \vee |y| > s. \end{cases} \end{aligned} \tag{2.4}$$

For $K > 1/2$, let $V_K : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be given by

$$V_K(x, y) = |y| - K|x + y| \log |x + y| - L(K)$$

and let $\alpha_K = \max\{1, K^{-1}\}$, $s(K) = L(K)/K$. We will need the following majorization property.

Lemma 2.1. *For any $K > 1/2$ and x, $y \in \mathbb{R}$, we have*

$$U_{s(K)}(x, y) \geq \alpha_K V_K(x, y). \tag{2.5}$$

Proof. We will show a stronger estimate

$$F(x, y) := U_{s(K)}(x, y) - \alpha_K(|x| \vee |y| - K|x + y| \log |x + y| - L(K)) \geq 0.$$

We present a detailed proof only in the case $1/2 < K < 1$; if $K \geq 1$, then the estimate can be established essentially in the same manner. Since $F(x, y) = F(y, x)$ and $F(x, y) = F(-x, -y)$ for all x, y, it suffices to prove the majorization on the set $C = \{(x, y) : |x| \leq y\}$. Note that F is of class C^1 in the interior of this set and

$$F(\cdot, y) \text{ is convex and } F_x(-y+, y) \leq 0 \text{ for all } y > 0. \tag{2.6}$$

Furthermore, if $y \geq s(K)$, then

$$F_x(x, y) = \log(x + y) - \log \frac{y}{s(K)},\ F_y(x, y) = \log(x + y) - \log \frac{y}{s(K)} - \frac{x}{y} - \frac{1}{K} + 1,$$

while for $y < s(K)$,

$$F_x(x, y) = -\frac{y}{s(K)} + 1 + \log(x + y),\ F_y(x, y) = -\frac{x}{s(K)} - \frac{1}{K} + 1 + \log(x + y).$$

Therefore,

$$F_x(x, y) = F_y(x, y) = 0 \text{ if and only if } x = \frac{K - 1}{K}y \text{ and } y \geq s(K). \tag{2.7}$$

Moreover,

$$F\left(\frac{K - 1}{K}y, y\right) = 0 \text{ for } y \geq s(K). \tag{2.8}$$

The properties (2.6)–(2.8) imply that $F(x,y) \geq 0$ for $y \geq s(K)$. In virtue of (2.6), it remains to prove that $F(y,y) \geq 0$ for $y < s(K)$. Using the estimate $t \log t \geq t-1$, $t \geq 0$, we may write

$$F(y,y) \geq -\frac{y^2}{s(K)} - \frac{y}{K} + 2y - 1 + \frac{L(K)}{K},$$

and the right-hand side is a concave function of y, taking nonnegative values at 0 and $s(K)$. □

Now we proceed to the proof of our main result.

Proof of (1.3). Using standard approximation, we may assume that the sequences d and θ are simple; that is, for any n, the random variables d_n and θ_n take only a finite number of values and there is a deterministic N such that $d_N = d_{N+1} = d_{N+2} = \ldots = \theta_N = \theta_{N+1} = \theta_{N+2} = \ldots \equiv 0$. This will guarantee the integrability of all the variables appearing below. Consider the pair (X,Y) of martingales defined by

$$X_n = \sum_{k=0}^{n} (1-\theta_k) d_k, \quad Y_n = \sum_{k=0}^{n} \theta_k d_k, \;\; n = 0,\, 1,\, 2,\, \ldots.$$

This pair satisfies (2.1) for any n; this follows from the assumption $\theta_n \in [0,1]$, $n = 1,\, 2,\, \ldots$. Therefore, applying (2.5), and then using (2.2) and the definition of $U_{s(K)}$, we obtain, for any n,

$$\begin{aligned} \alpha_K \mathbb{E} V_K(X_n,Y_n) &\leq \mathbb{E} U_{s(K)}(X_n,Y_n) \leq \mathbb{E} U_{s(K)}(X_{n-1},Y_{n-1}) \\ &\leq \ldots \leq \mathbb{E} U_{s(K)}(X_0,Y_0). \end{aligned} \tag{2.9}$$

Now, by (2.3) and the definition of $U_{s(K)}$, we see that $U_{s(K)}(X_0,Y_0) \leq 0$; this implies $\mathbb{E} V_K(X_n,Y_n) \leq 0$, which is precisely (1.3). □

3. On the lower bound for the constant $L(K)$

We will study the cases $K \geq 1$, $1/2 < K < 1$ and $K \leq 1/2$ separately.

The case $K \geq 1$. The constant $K \exp(K^{-1} - 1)$ is the best possible in (1.3). The equality is attained for $d_0 \equiv \exp(K^{-1}-1)$, $d_1 = d_2 = \ldots \equiv 0$ and $\theta = (1,1,\ldots)$.

The case $1/2 < K < 1$. This is more involved. Suppose that the best constant in (1.3) equals $\beta = \beta(K)$. Let $S = \{(x,y) \in \mathbb{R}^2 : x+y \geq 0\}$ and let $U_0 : S \to \mathbb{R}$ be given by

$$U_0(x,y) = \sup\{\mathbb{E} V_K(X_\infty, Y_\infty) + L(K)\}.$$

Here the supremum is taken over the class $\mathcal{Z}(x,y)$, which consists of all pairs (X,Y) of S-valued simple zigzag martingales starting from (x,y). Here X_∞, Y_∞ denote the almost sure limits $\lim_{n\to\infty} X_n$, $\lim_{n\to\infty} Y_n$ (which exist due to the simplicity assumption) and by the zigzag property we mean that

$$X_{n+1} - X_n \equiv 0 \text{ or } Y_{n+1} - Y_n \equiv 0 \quad \text{for all } n \geq 0. \tag{3.1}$$

We have the following fact.

Lemma 3.1. *(i) For any $y > 0$ we have $U_0(0,y) \leq \beta(K)$.*

(ii) We have $U_0(x,y) < \infty$ for all $(x,y) \in S$.

(iii) The function U_0 is the least biconcave majorant of the function $V_K + L(K)$.

(iv) For any $(x,y) \in S$ and $\lambda > 0$,

$$U_0(\lambda x, \lambda y) = \lambda U_0(x,y) - \lambda(x+y)\log\lambda. \tag{3.2}$$

Proof. (i) Take $(X,Y) \in \mathcal{Z}(0,y)$. Then it follows from (3.1) that the martingale Y is a transform of $X+Y$ by a deterministic (so, in particular, predictable) sequence taking values in $\{0,1\}$. Therefore

$$\mathbb{E}V_K(X_\infty,Y_\infty)+L(K)=\mathbb{E}|Y_\infty|-K\mathbb{E}|(X+Y)_\infty|\log|(X+Y)_\infty|\le\beta(K).$$

It suffices to take supremum over the class $\mathcal{Z}(0,y)$.

(ii) Let $(x,y)\in S$ and $(X,Y)\in\mathcal{Z}(x,y)$. As (2.2) holds, the chain of inequalities (2.9) is valid; this gives $\mathbb{E}V_K(X_\infty,Y_\infty)+L(K)\le\alpha_K^{-1}U_{s(K)}(x,y)+L(K)$ and it remains to take supremum over $\mathcal{Z}(x,y)$.

(iii) This is precisely the first part of Theorem 7.1 in [2].

(iv) We have $(X,Y)\in\mathcal{Z}(x,y)$ if and only if $(\lambda X,\lambda Y)\in\mathcal{Z}(\lambda x,\lambda y)$. Furthermore,

$$\begin{aligned}\mathbb{E}V_K(\lambda X_\infty,\lambda Y_\infty)+L(K)&=\lambda(\mathbb{E}V_K(X_\infty,Y_\infty)+L(K))-\lambda\mathbb{E}(X_\infty+Y_\infty)\log\lambda\\&\le\lambda U_0(x,y)-\lambda(x+y)\log\lambda.\end{aligned}$$

Taking supremum over $\mathcal{Z}(x,y)$ we obtain $U(\lambda x,\lambda y)\le\lambda U(x,y)-\lambda(x+y)\log\lambda$. Applying it to $x:=\lambda x$, $y:=\lambda y$ and $\lambda:=\lambda^{-1}$, we obtain the reverse estimate. □

Now we turn to the lower bound for $\beta(K)$. Let a, δ be fixed positive numbers. By part (ii) of the lemma above, we may write

(3.3)
$$\begin{aligned}U_0(a,a)&\ge\frac{K\delta}{a+K\delta}U_0(a-aK^{-1},a)+\frac{a}{a+K\delta}U_0(a+\delta,a)\\&\ge\frac{K\delta}{a+K\delta}\left[a-K(2a-aK^{-1})\log(2a-aK^{-1})\right]+\frac{a}{a+K\delta}U_0(a+\delta,a)\end{aligned}$$

(note that since $K>1/2$, we have that $(a-aK^{-1},a)$ belongs to S, the domain of U_0). Similarly, we have

(3.4)
$$\begin{aligned}U_0(a+\delta,a)&\ge\frac{K\delta}{a+K\delta}U_0(a+\delta,a-aK^{-1})+\frac{a}{a+K\delta}U_0(a+\delta,a+\delta)\\&\ge\frac{K\delta}{a+K\delta}\left[aK^{-1}-a-K(2a-aK^{-1}+\delta)\log(2a-aK^{-1}+\delta)\right]\\&\quad+\frac{a}{a+K\delta}U_0(a+\delta,a+\delta).\end{aligned}$$

Furthermore, by part (iv), we may write

$$U_0(a+\delta,a+\delta)=\frac{a+\delta}{a}U_0(a,a)-2K(a+\delta)\log\frac{a+\delta}{a}.$$

Put this into (3.4) and plug the obtained lower bound for $U_0(a+\delta,a)$ into (3.3). Then subtract $U_0(a,a)$ from both sides, divide throughout by δ and let δ to 0. As a result, after some manipulations, we get

$$-a-2Ka\log(2a-aK^{-1})\le U_0(a,a).$$

Thus, by part (ii),

$$\begin{aligned}U_0(0,a)&\ge KU_0(a-aK^{-1},a)+(1-K)U_0(a,a)\\&\ge K\left[a-K(2a-aK^{-1})\log(2a-aK^{-1})\right]\\&\quad+(1-K)[-a-2Ka\log(2a-aK^{-1})]\\&=(2K-1)a-Ka\log(2a-aK^{-1}).\end{aligned}$$

Using part (i) of the lemma above and maximizing the left hand-side over a, we get

$$\beta(K) \geq \frac{K^2}{2K-1}\exp(1-K^{-1}). \tag{3.5}$$

This completes the proof.

The case $K \leq 1/2$. Suppose that there is $K \leq 1/2$ for which (1.3) holds with some $L(K) < \infty$. Take $K' > 1$. Since $t\log t + e^{-1} \geq 0$, we have, for any n,

$$\begin{aligned}
\mathbb{E}\left|\sum_{k=0}^n \theta_k d_k\right| &\leq K\left(\mathbb{E}\left|\sum_{k=0}^n d_k\right|\log\left|\sum_{k=0}^n d_k\right| + e^{-1}\right) - Ke^{-1} + L(K)\\
&\leq K'\left(\mathbb{E}\left|\sum_{k=0}^n d_k\right|\log\left|\sum_{k=0}^n d_k\right| + e^{-1}\right) - Ke^{-1} + L(K)\\
&= K'\mathbb{E}\left|\sum_{k=0}^n d_k\right|\log\left|\sum_{k=0}^n d_k\right| + (K'-K)e^{-1} + L(K),
\end{aligned}$$

a contradiction with (3.5), since $(K'-K)e^{-1} + L(K)$ does not tend to ∞ as $K' \to 1/2+$.

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