COMPARISON-TYPE THEOREMS FOR ITÔ PROCESSES AND DIFFERENTIALLY SUBORDINATED SEMIMARTINGALES

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ABSTRACT. Let $\alpha \geq 0$ and let X, Y be Itô processes

$$\mathrm{d}X_t = \phi_t \,\mathrm{d}B_t + \psi_t \,\mathrm{d}t, \qquad \mathrm{d}Y_t = \zeta_t \,\mathrm{d}B_t + \xi_t \,\mathrm{d}t$$

such that $X_0 = x$, $Y_0 = y$, $|\phi| \geq |\zeta|$ and $\alpha\psi \geq |\xi|$. We determine the best universal constant $U_\alpha(x, y)$ such that

$$\mathbb{P}\big(\sup_t Y_t \geq 0\big) \leq ||X^+||_1 + U_\alpha(x, y).$$

As an application, we compute, for any $t \in [0, 1]$ and $\beta \in \mathbb{R}$, the number

$$L(x, y, t) = \inf\{||X^+||_1 : \mathbb{P}(Y^* \geq \beta) \geq t\}.$$

We also study these problems for a wider class of α-subordinated semimartingales and establish a related estimate for smooth functions on Euclidean domains.

1. Introduction

The purpose of this paper is to provide a wide family of sharp estimates for certain class of Itô processes and, more generally, for the class of semimartingales satisfying the so-called α-subordination relation. To describe our motivation, it is convenient to start with the setting of Itô processes. Suppose that $(\Omega, \mathcal{F}, \mathbb{P})$ is a complete probability space, filtered by a nondecreasing right-continuous family $(\mathcal{F}_t)_{t\geq 0}$ of sub-σ-fields of $\mathcal{F}$. As usual, we assume that $\mathcal{F}_0$ contains all the sets A satisfying $\mathbb{P}(A) = 0$. Let $B = (B_t)_{t\geq 0}$ be an adapted Brownian motion starting from 0 and let $X = (X_t)_{t\geq 0}$, $Y = (Y_t)_{t\geq 0}$ be Itô processes with respect to B (cf. Ikeda and Watanabe [15]):

$$\begin{aligned} X_t &= X_0 + \int_{0+}^t \phi_s \,\mathrm{d}B_s + \int_{0+}^t \psi_s \,\mathrm{d}s, \\ Y_t &= Y_0 + \int_{0+}^t \zeta_s \,\mathrm{d}B_s + \int_{0+}^t \xi_s \,\mathrm{d}s, \qquad t \geq 0. \end{aligned} \tag{1.1}$$

Here $(\phi_t)_{t\geq 0}$, $(\psi_t)_{t\geq 0}$, $(\zeta_t)_{t\geq 0}$, $(\xi_t)_{t\geq 0}$ are predictable processes which satisfy the usual assumptions

$$\mathbb{P}\left(\int_{0+}^t |\phi_s|^2 \,\mathrm{d}s < \infty \;\text{ and }\; \int_{0+}^t |\psi_s| \,\mathrm{d}s < \infty \text{ for all } t > 0\right) = 1,$$

$$\mathbb{P}\left(\int_{0+}^t |\zeta_s|^2 \,\mathrm{d}s < \infty \;\text{ and }\; \int_{0+}^t |\xi_s| \,\mathrm{d}s < \infty \text{ for all } t > 0\right) = 1.$$

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Roughly speaking, our problem can be formulated as follows: suppose that X_0 dominates Y_0, ϕ dominates ζ and ψ dominates ξ; what can be said about the sizes of X and Y?

This question is very general and its various versions and modifications have been investigated intensively in the literature. For instance, the whole class of the so-called comparison theorems falls into scope of this subject: see Yamada [23], Ikeda and Watanabe [14], [15], Le Gall [17] and references therein. The direction of our research is closely related to the problem which appeared for the first time in Burkholder's paper [6]. He showed that if X is a nonnegative submartingale and we have the domination $X_0 \geq |Y_0|$, $|\phi_s| \geq |\zeta_s|$ and $\psi_s \geq |\xi_s|$ for all s, then

$$\lambda \mathbb{P}(|Y|^* \geq \lambda) \leq 3||X||_1, \qquad \lambda > 0,$$

and

$$||Y||_p \leq \max\{(p-1)^{-1}, 2p-1\}||X||_p, \quad 1 < p < \infty.$$

Here we have used the notation $|X|^* = \sup_{t\geq 0} |X_t|$ for the two-sided maximal function of X and $||X||_p = \sup_t ||X_t||_p$ for the p-th moment of X, $p \geq 1$. Furthermore, Burkholder proved that both inequalities above are sharp. These results were generalized by C. Choi in [9] and [10], who showed that if $\alpha \in [0,1]$ is a fixed number, X is a nonnegative submartingale and, in addition,

$$|X_0| \geq |Y_0|, \quad |\phi_s| \geq |\zeta_s| \text{ and } \alpha\psi_s \geq |\xi_s| \text{ for all } s, \tag{1.2}$$

then we have the weak-type bound

$$\lambda \mathbb{P}(|Y|^* \geq \lambda) \leq (\alpha+2)||X||_1, \qquad \lambda > 0,$$

and the moment estimate

$$||Y||_p \leq \max\{(p-1)^{-1}, (\alpha+1)p-1\}||X||_p, \quad 1 < p < \infty.$$

Again, the constants $\alpha + 2$ and $\max\{(p-1)^{-1}, (\alpha+1)p-1\}$ are optimal. A related problem, in which there is no assumption on the sign of X, was studied by Hammack [13]. He proved that if (1.2) holds for some $\alpha \in [0,1]$, then we have the sharp estimate

$$\lambda \mathbb{P}(|Y|^* \geq \lambda) \leq 4||X^+||_1 - 2\mathbb{E}X_0 \leq 6||X||_1, \qquad \lambda > 0, \tag{1.3}$$

and that the norm inequalities fail to hold (here, as above, $||X^+||_1 = \sup_{t\geq 0} ||X_t^+||_1$). The case $\alpha > 1$ was studied by the author in [19].

We will deal with a version of the weak-type estimate for one-sided maximal function $Y^* = \sup_{t\geq 0} Y_t$, which is very close to the problem of the so-called optimal control of semimartingales (see Section 4 below for the detailed description and references to related results). In fact, we will work in a much wider setting. For any real-valued semimartingales X and Y, we say that Y is *differentially subordinate* to X, if the process $([X,X]_t - [Y,Y]_t)_{t\geq 0}$ is nondecreasing and nonnegative as a function of t (see Bañuelos and Wang [1] or Wang [22]). Here $[X,X]$ denotes the quadratic variance process of X, see e.g. Dellacherie and Meyer [12]. This type of domination implies many interesting inequalities if X and Y are martingales or local martingales, see [22]. In fact, throughout the paper we use a slightly different notion of differential subordination: namely, this domination will only mean that the process $([X,X]_t - [Y,Y]_t)_{t\geq 0}$ is nondecreasing (i.e., it may take negative values).

In the semimartingale setting, the domination must be strengthened so that it imposes some control on the finite variation parts. We will work under the assumption of *α-strong differential subordination* (α-subordination in short), introduced by Wang in [22] in the particular case $\alpha = 1$, and by the author in [18] for general $\alpha \geq 0$. The definition is the following. Let X be an adapted submartingale, Y be adapted semimartingale and write Doob-Meyer decompositions

$$X = X_0 + M + A, \qquad Y = Y_0 + N + B, \tag{1.4}$$

where M, N are local martingale parts, and A, B are finite variation processes (M, N, A and B are assumed to vanish at 0). In general the decompositions may not be unique; however, we assume that A is predictable and this determines the first of them. Let α be a fixed nonnegative number. We say that Y is α-subordinate to X, if Y is differentially subordinate to X and there is a decomposition (1.4) for Y such that the process $(\alpha A_t - |B|_t)_{t\geq 0}$ is nondecreasing as a function of t. Here $|B|_t$ denotes the total variation of B on the interval $[0,t]$. Two observations are in order: first in the setting of Itô processes described in (1.1), if $|\phi_s| \geq |\zeta_s|$ and $\alpha\psi_s \geq |\xi_s|$ for all s, then, obviously, Y is α-subordinate to X. Second, the above domination extends to the case when Y takes values in a certain separable Hilbert space $\mathcal{H}$ (which can be assumed to be ℓ^2): one applies the Doob-Meyer decomposition for each coordinate of Y and then rewrites the definition of α-subordination with $[Y,Y] = \sum_{j=1}^\infty [Y^j, Y^j]$ and $|B| = \sum_{j=1}^\infty |B^j|$.

We turn to the formulation of our main result. Let U and $(U_\alpha^+)_{\alpha\geq 0}$ be the functions introduced in Sections 2 and 3.

Theorem 1.1. *Suppose that X is a submartingale starting from x and Y is a semimartingale starting from y. Moreover, assume that Y is α-subordinate to X.*

(i) If $\alpha \in [0,1]$, then

$$\mathbb{P}(Y^* \geq 0) \leq ||X^+||_1 + U_\alpha(x,y). \tag{1.5}$$

(ii) If $\alpha \geq 0$ and X is nonnegative, then

$$\mathbb{P}(Y^* \geq 0) \leq ||X^+||_1 + U_\alpha^+(x,y). \tag{1.6}$$

The inequalities (1.5) *and* (1.6) *are sharp, even for the class of Itô processes* (1.1). *More precisely, for any $(x,y) \in \mathbb{R}^2$ (respectively, $(x,y) \in [0,\infty) \times \mathbb{R}$) and any $c < U(x,y)$ (resp., $c < U_\alpha^+(x,y)$), there is an appropriate pair of Itô processes X, Y starting from x, y, satisfying* (1.2), *and for which*

$$\mathbb{P}(Y^* \geq 0) > ||X^+||_1 + c.$$

A few words about the organization of the paper are in order. In Section 2, we show the first part of the above theorem: that is, we establish the inequality (1.5) and prove its sharpness. Section 3 is devoted to the proof of Theorem 1.1 for nonnegative X. The final part of the paper contains applications of our main result. In particular, we study there the optimal control problem as well as certain related estimates for smooth functions given on connected domains of $\mathbb{R}^n$.

2. Proof of Theorem 1.1 for general submartingales

2.1. **Special functions.** We start our analysis by defining the function U appearing in the statement of Theorem 1.1. Consider the following subsets of $\mathbb{R}^2$:

$$\begin{aligned} D_1 &= \{(x,y) : x \leq 0,\, y \geq x\}, \\ D_2 &= \{(x,y) : x > 0,\, y \geq -x\}, \\ D_3 &= \{(x,y) : y < x \leq 0,\, x+y+4 \leq 0\}, \\ D_4 &= \{(x,y) : y < -x < 0,\, y-x+4 \leq 0\}, \\ D_5 &= \mathbb{R}^2 \setminus (D_1 \cup D_2 \cup D_3 \cup D_4) \end{aligned}$$

(see Figure 1 below). Let $U : \mathbb{R}^2 \to \mathbb{R}$ be given by

$$U(x,y) = \begin{cases} 1 & \text{if } (x,y) \in D_1, \\ 1-x & \text{if } (x,y) \in D_2, \\ 2x/(x+y) & \text{if } (x,y) \in D_3, \\ 2x/(x-y) - x & \text{if } (x,y) \in D_4, \\ \frac{1}{16}(y^2-x^2) + \frac{1}{2}(y-x) + 1 & \text{if } (x,y) \in D_5. \end{cases}$$

Let us study the properties of this function. We begin with the following straightforward lemma.

Lemma 2.1. *(i) For any $x \in \mathbb{R}$, the function $U(x,\cdot) : y \mapsto U(x,y)$ is convex and increasing on the interval $I_x = \{y : (x,y) \in D_3 \cup D_4 \cup D_5\}$.*

(ii) For any $(x,y) \in \mathbb{R}^2$ we have the double inequality

$$1 - x^+ \geq U(x,y) \geq 1_{D_1 \cup D_2}(x,y) - x^+. \tag{2.1}$$

Proof. (i) It is easy to check that the partial derivative U_y is continuous on $D_3 \cup D_4 \cup D_5$ and hence the function $U(x,\cdot)$ is of class C^1 on I_x. Furthermore, it is not difficult to see that $U_{yy}(x,y) \geq 0$ whenever (x,y) lies in the interior of D_3, D_4 or D_5. It remains to observe that

$$\lim_{y \to -\infty} U(x,y) = -x^+, \tag{2.2}$$

which is also an immediate consequence of the definition of U.

(ii) The majorization is obvious for $(x,y) \in D_1 \cup D_2$, and for the other domains, it follows at once from continuity of U, the monotonicity of $U(x,\cdot)$ and (2.2). □

The key property of U is described in the following lemma.

Lemma 2.2. *For any x, y, $k \in \mathbb{R}$, introduce the function $G_{x,y,k} : \mathbb{R} \to \mathbb{R}$ by $G_{x,y,k}(t) = U(x+t, y+kt)$. Then, for any $|k| \leq 1$, the function $G_{x,y,k}$ is concave and nonincreasing on $\mathbb{R}$.*

Proof. It suffices to verify the concavity property. Indeed, if we establish it, the monotonicity of $G_{x,y,k}$ follows directly from the equality $\lim_{t \to -\infty} G'_{x,y,k}(t+) = 0$ (which is evident from the very definition). Therefore, we will be done if we prove that $G''_{x,y,k}(t) \leq 0$ whenever $(x+t, y+kt)$ lies in the interior of one of the sets D_i, and that $G'_{x,y,k}(t-) \geq G'_{x,y,k}(t+)$ for remaining t. Because of the equality $G_{x,y,k}(t+s) = G_{x+t,y+kt,k}(s)$, it is enough to check the above conditions for $t = 0$.

Now, when (x, y) lies in the interior of D_1 or D_2, then $G''_{x,y,k}(0) = 0$. If (x, y) belongs to D_3^o, the interior of D_3, then

$$G''_{x,y,k}(0) = \frac{4(k+1)(k-1)x}{(x+y)^3} + \frac{4(k+1)(x-y)}{(x+y)^3} \leq 0.$$

Similarly, if $(x, y) \in D_4^o$, we derive that

$$G''_{x,y,k}(0) = \frac{4(1-k)(x+y)}{(x-y)^3} + \frac{4(k-1)(k+1)x}{(x-y)^3} \leq 0. \tag{2.3}$$

Finally, when (x, y) lies in the interior of D_5, then $G''_{x,y,k}(0) = (1-k^2)/8 \leq 0$. Next, we turn to the verification of the inequalities for the one-sided derivatives at the points lying on the boundaries of D_i. First, if $(x, y) \in \partial D_1$, then the inequality $G'_{x,y,k}(0-) = 0$ and $G'_{x,y,k}(0+) \leq 0$ (where the latter follows, for example, from (2.1)). A similar argument works for the boundary of D_2: there we have $G'_{x,y,k}(0-) \geq -1 = G'_{x,y,k}(0+)$ (again, see (2.1)). If $(x, y) \in \partial D_3 \cap \partial D_5$ or $(x, y) \in \partial D_4 \cap \partial D_5$, then the derivatives match: $G'_{x,y,k}(0-) = G'_{x,y,k}(0+)$. Finally, when $(x, y) \in \partial D_3 \cap \partial D_4$, then $x = 0$ and $y \leq -4$; we compute that

$$G'_{x,y,k}(0-) = \frac{2}{y}, \qquad G'_{x,y,k}(0+) = -\frac{2}{y} - 1,$$

so the inequality $G'_{x,y,k}(0-) \geq G'_{x,y,k}(0+)$ holds true. The proof is complete. $\square$

2.2. **Proof of** (1.5)**.** Since α-subordination implies α'-subordination for $\alpha < \alpha'$, it suffices to establish the inequality in the less stringent case $\alpha = 1$. We start with the following well-known fact, see e.g. [12]. For any semimartingale X there exists a unique continuous local martingale part X^c of X satisfying

$$[X, X]_t = |X_0|^2 + [X^c, X^c]_t + \sum_{0<s\leq t} |\Delta X_s|^2$$

for all $t \geq 0$ (here $\Delta X_s = X_s - X_{s-}$ is the jump of X at time $s > 0$). Furthermore, we have $[X^c, X^c] = [X, X]^c$, the pathwise continuous part of $[X, X]$. In our further considerations, we will need Lemma 1 from [22], which can be stated as follows.

Lemma 2.3. *If X and Y are semimartingales, then Y is differentially subordinate to X if and only if Y^c is differentially subordinate to X^c and for any $s > 0$ we have $|\Delta Y_s| \leq |\Delta X_s|$.*

Now it is convenient to split the reasoning into a few parts.

Step 1. Convergence of X and Y. Of course, we may restrict ourselves to X satisfying $||X^+||_1 < \infty$, since otherwise there is nothing to prove. By Doob's convergence theorem, this assumption implies that the limit $X_\infty = \lim_{t\to\infty} X_t$ exists almost surely. In addition, it turns out that the semimartingale Y also converges with probability 1 to a certain random variable, say Y_∞. To show this, we need some auxiliary notation. Suppose that $f : [0, \infty) \to \mathbb{R}$ is a given right-continuous function with limits from the left. By Cauchy's criterion,

the limit $\lim_{t\to\infty} f(t)$ exists if and only if $C_\varepsilon(f) < \infty$ for all $\varepsilon > 0$,

where $C_\varepsilon(f)$ is the number of ε-escapes of f. The counting function $C_\varepsilon(\cdot)$ is given as follows: put $C_\varepsilon(f) = 0$ and $\nu_0(f) = \infty$ if the set $\{t \geq 0 : |f(t) - f(0)| \geq \varepsilon\}$ is empty. If this is not the case, let $\nu_0(f) = \inf\{t \geq 0 : |f(t) - f(0)| \geq \varepsilon\}$. Now, if the set $\{t > \nu_0(f) : |f(t) - f(\nu_0(f))| \geq \varepsilon\}$ is empty, put $C_\varepsilon(f) = 1$ and $\nu_1(f) = \infty$. If nonempty, continue as above. If $\nu_j(f)$ is not defined by this induction, i.e., there is a nonnegative $i < j$ such that $\nu_i(f) = \infty$, set $\nu_j(f) = \infty$. Then $C_\varepsilon(f) \leq j$ if and only if $\nu_j(f) = \infty$.

Consider an ℓ^2-valued process $\mathbb{Y}$, which for $t \in [\nu_n(Y), \nu_{n+1}(Y))$ is given by

$$\mathbb{Y}_t = (Y_{\nu_0(Y)} - Y_0, Y_{\nu_1(Y)} - Y_{\nu_0(Y)}, Y_{\nu_2(Y)} - Y_{\nu_1(Y)}, \ldots, Y_t - Y_{\nu_n(Y)}, 0, 0, \ldots).$$

Obviously, since Y is α-subordinate to X, so is $\mathbb{Y}$. In addition, we have $|\mathbb{Y}_0| = 0 \leq |X_0|$. Therefore, by (1.3) (which holds true also for vector valued processes Y, see [13] and [19]),

$$\mathbb{P}(\nu_{n-1}(Y) < \infty) \leq \mathbb{P}(|\mathbb{Y}|^* \geq \sqrt{n}\varepsilon) \leq \big(4||X^+||_1 - 2\mathbb{E}X_0\big)/(\sqrt{n}\varepsilon).$$

Since the right-hand side converges to 0 as $n \to \infty$, we obtain

$$\mathbb{P}(C_\varepsilon(Y) = \infty) = \mathbb{P}(\nu_n(Y) < \infty \text{ for all } n) = 0,$$

as desired.

Step 2. A reduction. We will replace the inequality (1.5) by its certain non-maximal version. Introduce the stopping time $\tau = \inf\{t : |X_t| + Y_t \geq 0\}$, where, as usual, $\inf \emptyset = \infty$. Since

$$\{Y^* \geq 0\} \subseteq \{(X_\tau, Y_\tau) \in D_1 \cup D_2\},$$

it suffices to establish the bound

$$\mathbb{P}((X_\tau, Y_\tau) \in D_1 \cup D_2) \leq ||X^+||_1 + U(x, y). \tag{2.4}$$

Note that here we have used the previous step: the random variable Y_τ makes sense on the set $\{\tau = \infty\}$.

Step 3. A mollification argument. Let $\varepsilon, \delta \in (0, 1)$ be fixed numbers and suppose that $g : \mathbb{R}^2 \to [0, \infty)$ is a C^∞ function, supported on the unit ball of $\mathbb{R}^2$ and satisfying $\int_{\mathbb{R}^2} g = 1$. For any $\alpha \geq 0$, we introduce the function $U^\delta : \mathbb{R}^2 \to \mathbb{R}$ by the convolution

$$U^\delta(x, y) = \int_{[-1,1]^2} U(x + \delta u, y - 2\delta + \delta v) g(u, v) \mathrm{d}u\mathrm{d}v$$

(note that we subtract 2δ on the second coordinate: this will guarantee the inequality (2.8) below). Of course, this function is of class C^∞ and inherits the properties studied in Lemma 2.2: it is nonincreasing along the lines of slope $k \in [-1, 1]$:

$$U_x^\delta(x, y) + k|U_y^\delta(x, y)| \leq 0, \qquad (x, y) \in \mathbb{R}^2, \tag{2.5}$$

and concave along the lines of slope $k \in [-1, 1]$:

$$U_{xx}^\delta(x, y) \pm 2U_{xy}^\delta(x, y)k + U_{yy}^\delta(x, y)k^2 \leq 0, \qquad (x, y) \in \mathbb{R}^2. \tag{2.6}$$

The second condition has two important consequences. First, it implies that U^δ is superharmonic (simply put $k = \pm 1$ and add both estimates). Furthermore, it yields that for all $(x, y) \in \mathbb{R}^2$ and any h, $k \in \mathbb{R}$, we have

$$U_{xx}^\delta(x, y)h^2 + 2U_{xy}^\delta(x, y)hk + U_{yy}^\delta(x, y)k^2 \leq c(x, y)(h^2 - k^2), \tag{2.7}$$

where

$$c(x, y) = \frac{U_{xx}^\delta(x, y) - U_{yy}^\delta(x, y)}{2}.$$

Indeed, (2.7) can be rewritten in the form

$$\big[U_{xx}^\delta(x, y) + U_{yy}^\delta(x, y)\big](h^2 + k^2) + 4U_{xy}^\delta(x, y)hk \leq 0,$$

which is true in view of (2.6):

$$4U^\delta_{xy}(x,y)hk \leq 2|U^\delta_{xy}(x,y)|(h^2+k^2) \leq -\big[U^\delta_{xx}(x,y)+U^\delta_{yy}(x,y)\big](h^2+k^2).$$

On the other hand, as we have already observed in the proof of Lemma 2.1, the function U_y is continuous on $D_3\cup D_4\cup D_5$. Therefore, the integration by parts and Lemma 2.1 (i) yield

$$U^\delta_{yy}(x,y) = \int_{[-1,1]^2} U_{yy}(x+\delta u, y-2\delta+\delta v)g(u,v)\mathrm{d}u\mathrm{d}v \geq 0 \tag{2.8}$$

for $(x,y)\in D_3\cup D_4\cup D_5$. Thus, for all such (x,y),

$$c(x,y) = \frac{U^\delta_{xx}(x,y)+U^\delta_{yy}(x,y)}{2} - U^\delta_{yy}(x,y) \leq 0. \tag{2.9}$$

Step 4. The proof of (2.4). Let M, N, A, B be the local martingale and finite variation parts of X and Y, coming from the Doob-Meyer decomposition (1.4). It follows from the general theory of stochastic integration that the process

$$\left(\int_{0+}^t U^\delta_x(X_{s-},Y_{s-})\mathrm{d}M_s + \int_{0+}^t U^\delta_y(X_{s-},Y_{s-})\mathrm{d}N_s\right)_{t\geq 0}$$

is a local martingale. Let $(\sigma_n)_{n\geq 0}$ denote the corresponding localizing sequence of stopping times. Since the function U^δ is of class C^∞, we are allowed to apply Itô's formula to $\big(U^\delta(X_{\sigma_n\wedge\tau\wedge t}, Y_{\sigma_n\wedge\tau\wedge t})\big)_{t\geq 0}$ (recall the stopping time τ given in Step 2 above). We obtain

$$U^\delta(X_{\sigma_n\wedge\tau\wedge t}, Y_{\sigma_n\wedge\tau\wedge t}) = U^\delta(x,y) + I_1 + I_2 + I_3/2 + I_4, \tag{2.10}$$

where

$$\begin{aligned}
I_1 &= \int_{0+}^{\sigma_n\wedge\tau\wedge t} U^\delta_x(X_{s-},Y_{s-})\mathrm{d}M_s + \int_{0+}^{\sigma_n\wedge\tau\wedge t} U^\delta_y(X_{s-},Y_{s-})\mathrm{d}N_s,\\
I_2 &= \int_{0+}^{\sigma_n\wedge\tau\wedge t} U^\delta_x(X_{s-},Y_{s-})\mathrm{d}A_s + \int_{0+}^{\sigma_n\wedge\tau\wedge t} U^\delta_y(X_{s-},Y_{s-})\mathrm{d}B_s,\\
I_3 &= \int_{0+}^{\sigma_n\wedge\tau\wedge t} U^\delta_{xx}(X_{s-},Y_{s-})\mathrm{d}[X,X]^c_s\\
&\quad + 2\int_{0+}^{\sigma_n\wedge\tau\wedge t} U^\delta_{xy}(X_{s-},Y_{s-})\mathrm{d}[X,Y]^c_s + \int_{0+}^{\sigma_n\wedge\tau\wedge t} U^\delta_{yy}(X_{s-},Y_{s-})\mathrm{d}[Y,Y]^c_s,\\
I_4 &= \sum_{0<s\leq\sigma_n\wedge\tau\wedge t}\Big[U^\delta(X_s,Y_s) - U^\delta(X_{s-},Y_{s-}) - \big\langle \nabla U^\delta(X_{s-},Y_{s-}), (\Delta X_s, \Delta Y_s)\big\rangle\Big]
\end{aligned}$$

The term I_1 is a martingale (as a function of t), so $\mathbb{E}I_1 = 0$. By the 1-subordination of Y to X and (2.5), we have

$$\begin{aligned}
I_2 &\leq \int_{0+}^{\sigma_n\wedge\tau\wedge t} U^\delta_x(X_{s-},Y_{s-})\mathrm{d}A_s + \int_{0+}^{\sigma_n\wedge\tau\wedge t} \big|U^\delta_y(X_{s-},Y_{s-})\big|\,\mathrm{d}|B|_s\\
&\leq \int_{0+}^{\sigma_n\wedge\tau\wedge t}\Big[U^\delta_x(X_{s-},Y_{s-}) + \big|U^\delta_y(X_{s-},Y_{s-})\big|\Big]\mathrm{d}A_s \leq 0.
\end{aligned}$$

The term I_3 is also nonpositive, which is a consequence of (2.7). To see this, let $0\leq s_0 < s_1 \leq t$. For any $j\geq 0$, let $(\eta^j_i)_{1\leq i\leq i_j}$ be a sequence of nondecreasing finite stopping times with $\eta^j_0 = s_0$, $\eta^j_{i_j} = s_1$ such that $\lim_{j\to\infty}\max_{1\leq i\leq i_j-1}|\eta^j_{i+1}-\eta^j_i| = 0$.

Keeping j fixed, we apply, for each $i = 0, 1, 2, \ldots, i_j$, the inequality (2.7) to $x = X_{s_0-}$, $y = Y_{s_0-}$ and $h = h_i^j = X^c_{\eta^j_{i+1}\wedge\tau\wedge\sigma_n} - X^c_{\eta^j_i\wedge\tau\wedge\sigma_n}$, $k = k_i^j = Y^c_{\eta^j_{i+1}\wedge\tau\wedge\sigma_n} - Y^c_{\eta^j_i\wedge\tau\wedge\sigma_n}$. Summing the obtained i_j+1 inequalities and letting $j\to\infty$ yields

$$\begin{aligned} U^\delta_{xx}(X_{s_0-},Y_{s_0-})[X^c,X^c]^{\sigma_n\wedge\tau\wedge s_1}_{\sigma_n\wedge\tau\wedge s_0} &+ 2U^\delta_{xy}(X_{s_0-},Y_{s_0-})[X^c,Y^c]^{\sigma_n\wedge\tau\wedge s_1}_{\sigma_n\wedge\tau\wedge s_0}\\ &+ U^\delta_{yy}(X_{s_0-},Y_{s_0-})[Y^c,Y^c]^{\sigma_n\wedge\tau\wedge s_1}_{\sigma_n\wedge\tau\wedge s_0}\\ \le c(X_{s_0-},Y_{s_0-})&\Big([X^c,X^c]^{\sigma_n\wedge\tau\wedge s_1}_{\sigma_n\wedge\tau\wedge s_0} - [Y^c,Y^c]^{\sigma_n\wedge\tau\wedge s_1}_{\sigma_n\wedge\tau\wedge s_0}\Big), \end{aligned}$$

where we have used the notation $[S,T]^{s_1}_{s_0} = [S,T]_{s_1} - [S,T]_{s_0}$. The latter expression is nonpositive: if $s_0 > \tau\wedge\sigma_n$, then it is equal to 0, while for $s_0 \le \tau\wedge\sigma_n$ we have $c(X_{s_0-},Y_{s_0-}) \le 0$ in view of (2.9), and it suffices to use the differential subordination of Y^c to X^c. Finally, $I_4 \le 0$ because of the concavity of the function U^δ along the lines of slope $k\in[-1,1]$ and the fact that $|\Delta Y_s| \le |\Delta X_s|$, in virtue of the differential subordination. Consequently, combining all the above facts with (2.10) and taking expectation of both sides yields

$$\mathbb{E}U^\delta(X_{\sigma_n\wedge\tau\wedge t},Y_{\sigma_n\wedge\tau\wedge t}) \le \mathbb{E}U^\delta(x,y) = U^\delta(x,y).$$

However, the process $(X_t^+)_{t\ge0}$ is a submartingale, and thus, by Doob's optional sampling theorem, $\mathbb{E}X^+_{\sigma_n\wedge\tau\wedge t} \le \mathbb{E}X_t^+ \le ||X^+||_1$. Therefore, adding $\mathbb{E}X^+_{\sigma_n\wedge\tau\wedge t}$ to both sides of the preceding estimate gives

$$\mathbb{E}\Big[U^\delta(X_{\sigma_n\wedge\tau\wedge t},Y_{\sigma_n\wedge\tau\wedge t}) + X^+_{\sigma_n\wedge\tau\wedge t}\Big] \le ||X^+||_1 + U^\delta(x,y).$$

It is clear from (2.1) and the definition of U^δ, that the expression in the square brackets above is bounded from below by a certain universal constant. Furthermore, $U^\delta \to U$ pointwise as $\delta\to0$, since U is continuous. Consequently, letting $\delta\to0$ gives, by Fatou's lemma,

$$\mathbb{E}\Big[U(X_{\sigma_n\wedge\tau\wedge t},Y_{\sigma_n\wedge\tau\wedge t}) + X^+_{\sigma_n\wedge\tau\wedge t}\Big] \le ||X^+||_1 + U(x,y).$$

Next, we let $n\to\infty$ and then $t\to\infty$; applying Fatou's lemma again, we obtain

$$\mathbb{E}\Big[U(X_\tau,Y_\tau) + X_\tau^+\Big] \le ||X^+||_1 + U(x,y)$$

and hence, by (2.1), we obtain (2.4). This completes the proof.

2.3. **Sharpness for discrete-time processes.** We will use the following procedure. First we will exhibit appropriate examples in the discrete-time setting, and then, using a simple embedding into the Brownian motion, we will prove that the bound is optimal for the class of Itô processes (1.1).

Since α-subordination implies α'-subordination for $\alpha' > \alpha$ and the bound (1.5) does not depend on α, it suffices to show its sharpness for $\alpha = 0$. Consider the discrete-time Markov family $(f_n,g_n)_{n\ge0}$ with the transition function uniquely determined by the following conditions.

(a) The state $(x,y)\in D_1$, $x<0$, leads to $(0,y-x)$ or to $(2x,x+y)$; each possibility has probability 1/2. The states of the form $(0,y)$, $y\ge0$, are absorbing.

(b) The state $(x,y)\in D_2$ leads to $(0,x+y)$ or to $(2x,y-x)$; each possibility has probability 1/2.

(c) The state $(x,y)\in D_3$ leads to $(0,y+x)$ with probability $(x-y)/(x+y)$ or to $((x+y)/2,(x+y)/2)$ with probability $-2x/(x+y)$.

(d) The state $(x,y) \in D_4$ leads to $(0, y-x)$ with probability $(-x-y)/(x-y)$ or to $((x-y)/2, (y-x)/2)$ with probability $2x/(x-y)$.

(e) The state $(x,y) \in D_5$ leads to $((x-y)/2, (y-x)/2)$ with probability $1+(x+y)/4$ or to $((x-y)/2-2, (y-x)/2-2)$ with probability $-(x+y)/4$.

For the reader's convenience, we have decided to illustrate the transition function on Figure 1 below.

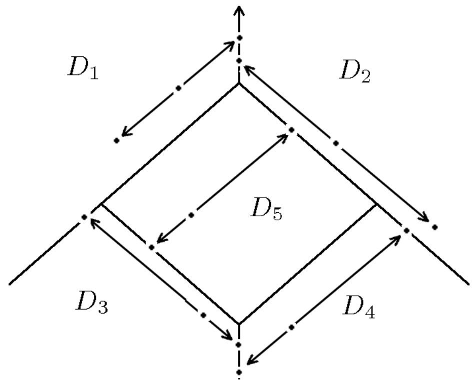

FIGURE 1. The evolution of the above Markov family, $\alpha \in [0,1]$: the behavior of the points from the domains $D_1 - D_5$. The halflines $\partial D_1 \cap \partial D_2$ and $\partial D_3 \cap \partial D_4$ are absorbing.

Now suppose that the pair (f,g) satisfies $\mathbb{P}((f_0,g_0) = (x,y)) = 1$ for some given $(x,y) \in \mathbb{R}^2$. It is easy to check that in fact, both processes f, g are martingales and for any $n \geq 1$ we have $df_n = \varepsilon_n dg_n$, where $\varepsilon_n \in \{-1,1\}$ is a predictable sign (that is, ε_n is measurable with respect to the σ-algebra generated by f and g up to time $n-1$). In addition, it is obvious that both f, g converge almost surely; let f_∞, g_∞ denote the corresponding limits. Consider two functions P, $M : \mathbb{R}^2 \to \mathbb{R}$, defined by

$$P(x,y) = \mathbb{P}(g_\infty \geq 0 | (f_0,g_0) = (x,y)) \quad \text{and} \quad M(x,y) = \sup_{n \geq 0} \mathbb{E}(f_n^+ | (f_0,g_0) = (x,y)).$$

We will prove that $P(x,y) - M(x,y) = U(x,y)$; this will clearly give the sharpness in the discrete-time case. Let us consider several cases separately.

Case 1. Suppose that (f,g) starts from $(x,y) \in D_1$. Then, by (a), the martingale f is nonpositive and g converges almost surely to $y - x \geq 0$. In consequence, we have $P(x,y) = 1$, $M(x,y) = 0$ and hence $P(x,y) - M(x,y) = U(x,y)$.

Case 2. If $(f_0,g_0) \in D_2$, the analysis is similar to that from the previous case. The condition (b) implies that f is a nonnegative martingale and g converges almost surely to $x + y \geq 0$. Therefore, we have $P(x,y) = 1$, $M(x,y) = x$ and $P(x,y) - M(x,y) = U(x,y)$.

Case 3. Assume that the starting point (x,y) lies in D_3. By (c), at the first step the pair either moves to $(0, x+y)$ and stays there forever, or jumps to $((x+y)/2, (x+y)/2)$ and we are in the position studied in Case 1. Consequently, we derive that g reaches 0 with probability $P(x,y) = -2x/(x+y)$; furthermore, $M(x,y) = 0$, since f is nonpositive. This yields $P(x,y) - M(x,y) = U(x,y)$.

Case 4. If (f,g) starts from D_4, we argue as in the previous case. By (d), at the first step the pair (f,g) goes to $(0,y-x)$ (with probability $(-x-y)/(x-y)$) and stops, or moves to the half-line $\{(x,-x):x>0\}$ (with probability $2x/(x-y)$) and then it evolves as in the Case 2 above. Therefore, we have $P(x,y)=2x/(x-y)$ and $M(x,y)=x$, so $P(x,y)-M(x,y)=U(x,y)$.

Case 5. Finally, assume that $(f_0,g_0)\equiv(x,y)\in D_5$. Then, by (e), the pair (f,g) goes either to $((x-y)/2,(y-x)/2)$ (and then we apply Case 2) or to the point $((x-y)/2-2,(y-x)/2-2)$ and then we use Case 3. Hence,

$$P(x,y)=\left(1+\frac{x+y}{4}\right)\cdot 1-\frac{x+y}{4}\cdot\frac{2\cdot\left(\frac{x-y}{2}-2\right)}{\left(\frac{x-y}{2}-2\right)+\left(\frac{y-x}{2}-2\right)}=1+\frac{x^2-y^2}{16}.$$

On the other hand, we have

$$\mathbb{E}f_0^+\leq\mathbb{E}f_1^+=\left(1+\frac{x+y}{4}\right)\cdot\frac{x-y}{2}=\frac{x-y}{2}+\frac{x^2-y^2}{8}.$$

However, we easily check that for $n\geq 2$, the variable f_n has the same sign as f_1. This implies that the above expression is equal to $M(x,y)$ and thus the difference $P(x,y)-M(x,y)$ is equal to $U(x,y)$.

2.4. **Sharpness of** (1.5) **for Itô processes of the form** (1.1)**.** Fix the starting point $(x,y)\in\mathbb{R}^2$. We know that the corresponding pair (f,g), constructed in the previous subsection, is a martingale. Thus, using the classical embedding theorems, there is a nondecreasing sequence $(\tau_n)_{n\geq 0}$ of stopping times converging to τ_∞ (with $\tau_\infty<\infty$ almost surely) satisfying $\tau_0\equiv 0$ and such that the sequence $(x+B_{\tau_n})_{n\geq 0}$ has the same distribution as $(f_n)_{n\geq 0}$. Define

$$X_t=x+\int_0^t 1_{[0,\tau_\infty)}(s)\mathrm{d}B_s.$$

Furthermore, as we have observed above, for any $n\geq 1$ we have $dg_n=\varepsilon_n df_n$, where $\varepsilon_n\in\{-1,1\}$ is predictable. Therefore, if we put

$$Y_t=y+\int_0^t\zeta_s\mathrm{d}B_s,$$

where $\zeta_s=\varepsilon_n$ for $s\in[\tau_{n-1},\tau_n)$ and $\zeta_s=0$ for $s\in[\tau_\infty,\infty)$, then $(X_{\tau_n},Y_{\tau_n})_{n\geq 0}$ has the same distribution as $(f_n,g_n)_{n\geq 0}$. The processes X, Y are of the form (1.1) and the integrands satisfy the required conditions. It remains to observe that

$$\mathbb{P}(Y^*\geq 0)-||X^+||_1=\mathbb{P}(g_\infty\geq 0)-||f^+||_1=U(x,y).$$

The sharpness follows.

3. Proof of Theorem 1.1 for nonnegative submartingales

3.1. **Special functions.** Here the situation is much more complicated. For a positive α, introduce the auxiliary parameter

$$\gamma=\gamma(\alpha)=\frac{2(1+\alpha)^2}{2\alpha+1} \tag{3.1}$$

and consider the following subsets of $[0,\infty)\times\mathbb{R}$:

$$\begin{aligned}
D_1 &= \{(x,y): x+y\geq 0\},\\
D_2 &= \{(x,y): x+y<0,\ y+\gamma>x/(2\alpha+1)\},\\
D_3 &= \{(x,y): y+\gamma\leq x/(2\alpha+1),\ y>x-2(1+\alpha)\},\\
D_4 &= ([0,\infty)\times\mathbb{R})\setminus(D_1\cup D_2\cup D_3)
\end{aligned}$$

(see Figure 2 below). Define

$$U_\alpha^+(x,y)=\begin{cases} 1-x & \text{if } (x,y)\in D_1,\\ \frac{1}{\gamma}((1-\gamma)x+y+\gamma)\left(\frac{x+y}{\gamma}+1\right)^{1/(2\alpha+1)} & \text{if } (x,y)\in D_2,\\ -\left(\frac{\alpha}{\alpha+1}\right)^{2\alpha/(2\alpha+1)} x\left(\frac{x-y}{\gamma}-1\right)^{1/(2\alpha+1)} & \text{if } (x,y)\in D_3,\\ 2x/(x-y)-x & \text{if } (x,y)\in D_4. \end{cases}$$

Here is the analogue of Lemma 2.1.

Lemma 3.1. *(i) For any $x\in\mathbb{R}$, the function $U_\alpha^+(x,\cdot): y\mapsto U_\alpha^+(x,y)$ is convex and increasing on the interval $I_x=(-\infty,-x]$.*

(ii) For any $(x,y)\in\mathbb{R}^2$ we have the double inequality

$$1-x\geq U_\alpha^+(x,y)\geq 1_{D_1}(x,y)-x. \tag{3.2}$$

The proof of this lemma goes along the same lines as that of Lemma 2.1, so we leave the details to the reader. The next statement concerns the key monotonicity and concavity properties of U_α^+.

Lemma 3.2. *For any x, y, $k\in\mathbb{R}$ and $\alpha\geq 0$, let $G=G_{x,y,\alpha,k}:[-x,\infty)\to\mathbb{R}$ be given by $G(t)=U_\alpha^+(x+t,y+kt)$.*

(i) For any $(x,y)\in[0,\infty)\times\mathbb{R}$ and $|k|\leq 1$, the function $G_{x,y,\alpha,k}$ is concave.

(ii) For any $(x,y)\in[0,\infty)\times\mathbb{R}$ and $|k|\leq\alpha$, the function $G_{x,y,\alpha,k}$ is nonincreasing.

Proof. (i) We argue as in the proof of Lemma 2.2. The inequality $G''_{x,y,\alpha,k}(0)\leq 0$ is obvious in the interior of D_1, and has already been established in the interior of D_4 (see (2.3)). If (x,y) belongs to D_2^o, we derive that

$$\begin{aligned}
G''_{x,y,\alpha,k}(0)=&\frac{2(k+1)}{\gamma^3(2\alpha+1)}\left(\frac{x+y}{\gamma}+1\right)^{1/(2\alpha+1)-2}\times\\
&\times\left[(1-\gamma+k)(x+y+\gamma)-\frac{\alpha(k+1)}{2\alpha+1}\big((1-\gamma)x+y+\gamma\big)\right]
\end{aligned}$$

and it suffices to show that the expression in the square brackets is nonpositive. However, we have $y+\gamma\geq x/(2\alpha+1)$ (by the definition of D_2) and $1-\gamma+k\leq\alpha(k+1)/(2\alpha+1)$, so the expression is not larger than

$$\begin{aligned}
&(1-\gamma+k)\left(x+\frac{x}{2\alpha+1}\right)-\frac{\alpha(k+1)}{2\alpha+1}\left((1-\gamma)x+\frac{x}{2\alpha+1}\right)\\
&\quad=\frac{2\alpha+2}{2\alpha+1}x\left[1-\gamma+k+\frac{\alpha^2(k+1)}{2\alpha+1}\right].
\end{aligned}$$

It remains to observe that the above term, considered as a function of k, attains its maximum for $k=1$, and the maximum equals 0. Finally, when $(x,y)\in D_3^o$, we compute that

$$G''_{x,y,\alpha,k}(0) = \frac{2}{2\alpha+1}\left(\frac{\alpha}{\alpha+1}\right)^{2\alpha/(2\alpha+1)}\left(\frac{1-k}{\gamma^2}\right)\left(\frac{x-y}{\gamma}-1\right)^{1/(2\alpha+1)-2}\times$$
$$\times\left[-x+y+\gamma+\frac{\alpha(1-k)x}{2\alpha+1}\right]$$

and the expression in the square brackets is nonpositive. Indeed, $y+\gamma \leq x/(2\alpha+1)$, so the expression does not exceed

$$-\frac{\alpha x(k+1)}{2\alpha+1} \leq 0.$$

It remains to establish the appropriate inequalities between one-sided derivatives of $G_{x,y,\alpha,\pm 1}$. If $(x,y) \in \partial D_1$, $x>0$, then the estimate $G'_{x,y,\alpha,k}(0-) \geq G'_{x,y,\alpha,k}(0+)$ follows directly from (3.2) (the right derivative is -1, and the left is at least -1). For remaining points at the common boundaries, the derivatives match.

(ii) Since U_α is continuous, we must show $G'_{x,y,\alpha,k}(0) \leq 0$ for (x,y) lying in the interior of D_1, D_2, D_3 or D_4. This is obvious if $(x,y) \in D_1^o$. For remaining (x,y), we note that $U^+_{\alpha y} \geq 0$ and hence

$$G'_{x,y,\alpha,k}(0) \leq G'_{x,y,\alpha,\alpha}(0).$$

The derivative on the right-hand side equals

$$-\frac{\alpha+1}{2\alpha+1}\gamma^{-1/(2\alpha+1)}(x+y+\gamma)^{1/(2\alpha+1)-1}x \leq 0$$

if $(x,y) \in D_2^o$, and

$$-\gamma^{-1/(2\alpha+1)}\left(\frac{\alpha+1}{\alpha}(x-y-\gamma)\right)^{1/(2\alpha+1)-1}\left[\frac{\alpha+1}{2\alpha+1}x+\left(\frac{x}{2\alpha+1}-(y+\gamma)\right)\right] \leq 0$$

for $(x,y) \in D_3^o$. Finally, if (x,y) belongs to the interior of D_4, we compute that

$$G'_{x,y,\alpha,\alpha}(0) = \frac{2x(\alpha-1)+2(x-y)}{(x-y)^2} - 1.$$

By the definition of D_4, we have $x-y \in (2(1+\alpha),\infty]$. We keep this difference fixed, and maximize the above expression over x (which lies in $(0,(x-y)/2)$, again by the definition of D_4). If $\alpha \leq 1$, then

$$\frac{2x(\alpha-1)+2(x-y)}{(x-y)^2} - 1 \leq \frac{2(x-y)}{(x-y)^2} - 1 \leq \frac{1}{1+\alpha} - 1 \leq 0.$$

On the other hand, for $\alpha > 1$, we may write

$$\frac{2x(\alpha-1)+2(x-y)}{(x-y)^2} - 1 \leq \frac{(x-y)(\alpha-1)+2(x-y)}{(x-y)^2} - 1 = \frac{1+\alpha}{x-y} - 1 \leq -\frac{1}{2}.$$

This completes the proof. □

3.2. **Proof of** (1.6)**.** The reasoning is the same as in the general case, so we will be brief. We restrict ourselves to X satisfying $||X||_1 < \infty$, which implies the almost sure convergence of X and Y. Introduce the stopping time $\tau = \inf\{t \geq 0 : X_t + Y_t \geq 0\}$ and note that it suffices to prove that

$$\mathbb{P}(X_\tau + Y_\tau \geq 0) \leq ||X||_1 + U_\alpha^+(x, y).$$

To accomplish this, we make use of the mollified function

$$U_\alpha^{\delta+}(x, y) = \int_{[-1,1]^2} U_\alpha^+(x + \delta + \delta u, y - 3\delta + \delta v) g(u, v) \mathrm{d}u\mathrm{d}v,$$

where $g : \mathbb{R}^2 \to [0, \infty)$ is a smoothing kernel. Now we repeat, essentially word by word, the argumentation appearing in the proof of (1.5).

3.3. **Sharpness, the discrete-time case.** Here the calculations will be much more involved. In contrast with the previous section, the optimality of U_α^+ will be obtained asymptotically, i.e., in the limit. Let δ be a small positive number and consider the Markov family on the state space $[0, \infty) \times \mathbb{R}$, determined by the following conditions.

(a) The state $(x, y) \in D_1$, $x > 0$, leads to $(0, x + y)$ or to $(2x, y - x)$; each possibility has probability $1/2$. The states $(0, y)$, $y \geq 0$, are absorbing.

(b) The state $(0, y)$ with $y \in (-\gamma, 0)$ leads to $(2\delta/(\alpha + 1), y + 2\alpha\delta/(\alpha + 1))$.

(c) The state $(x, y) \in D_2$, $x > 0$, leads to $(0, y + x)$ with probability

$$\frac{(2\alpha + 1)y - x + 2(\alpha + 1)^2}{(2\alpha + 1)(x + y) + 2(\alpha + 1)^2}$$

or to the point

$$\left(\frac{2\alpha + 1}{2(\alpha + 1)}(x + y) + \alpha + 1, \frac{x + y}{2(\alpha + 1)} - \alpha - 1\right) \in \partial D_2 \cap \partial D_3$$

with probability

$$\frac{2(\alpha + 1)x}{(2\alpha + 1)(x + y) + 2(\alpha + 1)^2}.$$

(d) The state $(x, y) \in D_3$ leads to $(0, y - x)$ with probability

$$\frac{x - (2\alpha + 1)y - 2(1 + \alpha)^2 + 2\alpha\delta}{(2\alpha + 1)(x - y) - 2(1 + \alpha)^2 + 2\alpha\delta},$$

or to the point

$$\left(\frac{(2\alpha + 1)(x - y) - 2(1 + \alpha)^2 + 2\alpha\delta}{2\alpha}, \frac{x - y - 2(1 + \alpha)^2 + 2\alpha\delta}{2\alpha}\right)$$

with probability

$$\frac{2\alpha x}{(2\alpha + 1)(x - y) - 2(1 + \alpha)^2 + 2\alpha\delta}.$$

(e) The state $(x, y) \in D_4$ leads to $(0, y - x)$ with probability $(-x - y)/(x - y)$ or to $((x - y)/2, (y - x)/2)$ with probability $2x/(x - y)$.

We would like to stress here that f, g are not martingales: this is due to (b) (however, the remaining moves *are* of martingale type). In analogy with the previous section, we introduce the functions P, M by

$$P(x, y) = \mathbb{P}(g_\infty \geq 0 | (f_0, g_0) = (x, y)), \qquad M(x, y) = \sup_n \mathbb{E}\big[f_n | (f_0, g_0) = (x, y)\big].$$

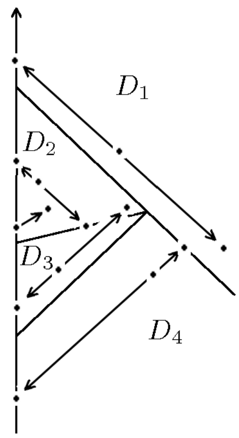

FIGURE 2. The transity function of the above Markov process. The part of the y-axis which lies outside D_2 is absorbing. A state lying in the interior of D_2 leads either to the y-axis, or to the common boundary of D_2 and D_3. Hovever, a state from D_3 leads either to the y-axis, or slightly above the set $\partial D_2 \cap \partial D_3$.

Observe that the functions P and M do depend on δ; for notational simplicity, we will not indicate this dependence. Consider two cases:

Case 1: $(x,y) \in D_1 \cup D_4$. We repeat the analysis presented in the general setting. For these starting points, the process (f,g) is a martingale, so $M(x,y)=x$; in addition, $P(x,y)=1$ for $(x,y)\in D_1$ and $P(x,y)=2x/(x-y)$ for $(x,y)\in D_4$. Thus, $P(x,y)-M(x,y)=U_\alpha^+(x,y)$.

Case 2: $(x,y)\in D_2\cup D_3$. This is the main technical part. For $x\geq 0$ and $y\in\mathbb{R}$, let

$$B(y)=P(0,y) \quad \text{and} \quad C(x)=P\left(x,\frac{x}{2\alpha+1}-\gamma\right).$$

Note that the point, to which P is applied above, belongs to $\partial D_2\cap\partial D_3$ (at least when $x\leq\alpha+1$). Suppose that $x\in(0,\alpha+1)$. By (d) and the fact that $P(0,y)=0$ for $y\leq-\gamma$,

$$\begin{aligned} C(x) &= \frac{\delta}{\delta+x}\cdot P\left(0,\frac{x}{2\alpha+1}-\gamma-x\right)+\frac{x}{x+\delta}P\left(x+\delta,\frac{x}{2\alpha+1}-\gamma+\delta\right) \\ &= \frac{x}{x+\delta}P\left(x+\delta,\frac{x}{2\alpha+1}-\gamma+\delta\right). \end{aligned}$$

Now, if $\left(x+\delta,\frac{x}{2\alpha+1}-\gamma+\delta\right)\in D_1$, then

$$C(x)=\frac{x}{x+\delta}. \tag{3.3}$$

On the other hand, if the point belongs to D_2, we apply (c):

$$P\left(x+\delta, \frac{x}{2\alpha+1}-\gamma+\delta\right) = \frac{\alpha\delta}{(\alpha+1)x+(2\alpha+1)\delta}B\left(\frac{(2\alpha+2)x}{2\alpha+1}-\gamma+2\delta\right) + \frac{(\alpha+1)(x+\delta)}{(\alpha+1)x+(2\alpha+1)\delta)}C\left(x+\frac{2\alpha+1}{\alpha+1}\delta\right)$$

and combine it with the preceding identity to obtain

$$\begin{aligned} C(x) = {} & \frac{\alpha\delta x}{\big((\alpha+1)x+(2\alpha+1)\delta\big)(x+\delta)}B\left(\frac{(2\alpha+2)x}{2\alpha+1}-\gamma+2\delta\right) \\ & + \frac{(\alpha+1)x}{(\alpha+1)x+(2\alpha+1)\delta)}C\left(x+\frac{2\alpha+1}{\alpha+1}\delta\right). \end{aligned} \tag{3.4}$$

Similarly, since $(2\alpha+2)x/(2\alpha+1)-\gamma \in (-\gamma, 0)$, the condition (b) implies

$$B\left(\frac{(2\alpha+2)x}{2\alpha+1}-\gamma\right) = P\left(\frac{2\delta}{\alpha+1}, \frac{(2\alpha+2)x}{2\alpha+1}-\gamma+\frac{2\alpha\delta}{\alpha+1}\right).$$

If the point at which P is evaluated belongs to D_1, then

$$B\left(\frac{(2\alpha+2)x}{2\alpha+1}-\gamma\right) = 1. \tag{3.5}$$

On the other hand, if this point belongs to D_2 (it is easy to see that this is equivalent to $\left(x+\delta, \frac{x}{2\alpha+1}-\gamma+\delta\right) \in D_2$), we may use (c) to get

$$\begin{aligned} B\left(\frac{(2\alpha+2)x}{2\alpha+1}-\gamma\right) = {} & \frac{(\alpha+1)x+(2\alpha-1)\delta}{(\alpha+1)x+(2\alpha+1)\delta}B\left(\frac{(2\alpha+2)x}{2\alpha+1}-\gamma+2\delta\right) \\ & \frac{2\delta}{(\alpha+1)x+(2\alpha+1)\delta}C\left(x+\frac{2\alpha+1}{\alpha+1}\delta\right). \end{aligned} \tag{3.6}$$

Now multiply (3.6) by α and add it to (3.4). After some tedious, but straightforward calculations, we obtain

$$\begin{aligned} & C(x) + \alpha B\left(\frac{2\alpha+2}{2\alpha+1}x-\gamma\right) \\ & = \frac{(\alpha+1)x+2\alpha\delta}{(\alpha+1)x+(2\alpha+1)\delta}\left[C\left(x+\frac{2\alpha+1}{\alpha+1}\delta\right)+\alpha B\left(\frac{2\alpha+2}{2\alpha+1}x-\gamma+2\delta\right)\right] \\ & \quad - \frac{\alpha\delta^2}{((\alpha+1)x+(2\alpha+1)\delta)(x+\delta)}B\left(\frac{2\alpha+2}{2\alpha+1}x-\gamma+2\delta\right). \end{aligned}$$

The function B is bounded by 1, so the second term above is of order $O(\delta^2)$. Now we will use induction. Let n be a positive integer such that

$$x+\frac{(2\alpha+1)(n-1)\delta}{\alpha+1} < \alpha+1 \leq x+\frac{(2\alpha+1)n\delta}{\alpha+1}. \tag{3.7}$$

Let us apply the preceding estimate for x, $x+(2\alpha+1)\delta/(\alpha+1)$, $\ldots$, $x+(2\alpha+1)(n-2)\delta/(\alpha+1)$. We cannot use this estimate for $x+(2\alpha+1)(n-1)\delta/(\alpha+1)$, since the corresponding points arising from (b) and (d) do not belong to D_2, but to D_1; nevertheless, then we obtain (compare the expressions below to (3.3) and

(3.5))

$$
\begin{aligned}
&C\left(x+\frac{(2\alpha+1)(n-1)\delta}{\alpha+1}\right)+\alpha B\left(\frac{2\alpha+2}{2\alpha+1}x+2(n-1)\delta\right)\\
&=\frac{x+\frac{(2\alpha+1)(n-1)\delta}{\alpha+1}}{x+\frac{(2\alpha+1)(n-1)\delta}{\alpha+1}+\delta}+\alpha\\
&=\frac{(\alpha+1)((\alpha+1)x+(2\alpha+1)(n-1)\delta+2\alpha\delta}{(\alpha+1)x+(2\alpha+1)n\delta}+O(\delta^2).
\end{aligned}
\tag{3.8}
$$

Combining the above n inequalities, we get

$$
\begin{aligned}
&C(x)+\alpha B\left(\frac{2\alpha+2}{2\alpha+1}x-\gamma\right)\\
&=\eta_{n,\delta}+(\alpha+1)\prod_{k=0}^{n-1}\left(\frac{(\alpha+1)x+(2\alpha+1)k\delta+2\alpha\delta}{(\alpha+1)x+(2\alpha+1)(k+1)\delta}\right),
\end{aligned}
\tag{3.9}
$$

where $\eta_{n,\delta}$ denotes the corresponding error term of order $O(n\delta^2)$. Next, we let $\delta \to 0$ (and hence $n \to \infty$, in view of (3.7)). Then $\eta_{n,\delta} \to 0$ and

$$
\begin{aligned}
\prod_{k=0}^{n-1}\left(\frac{(\alpha+1)x+(2\alpha+1)k\delta+2\alpha\delta}{(\alpha+1)x+(2\alpha+1)(k+1)\delta}\right)&=\prod_{k=0}^{n-1}\left(1-\frac{\delta}{(\alpha+1)x+(2\alpha+1)(k+1)\delta}\right)\\
&\sim \exp\left[-\sum_{k=0}^{n-1}\frac{\delta}{(\alpha+1)x+(2\alpha+1)(k+1)\delta}\right]\\
&\to \exp\left[-\int_0^b \frac{1}{(\alpha+1)x+(2\alpha+1)t}\mathrm{d}t\right],
\end{aligned}
$$

where $b=(\alpha+1-x)(\alpha+1)/(2\alpha+1)$ and the notation $A \sim B$ means that the ratio A/B tends to 1 as $\delta \to 0$. Computing the above integral, we obtain that the product converges to $(x/(\alpha+1))^{1/(2\alpha+1)}$. Plugging all the above facts to (3.9), we see that

$$
C(x)+\alpha B\left(\frac{2\alpha+2}{2\alpha+1}x-\gamma\right)=\left(\frac{x}{\alpha+1}\right)^{1/(2\alpha+1)}(1+\alpha)+O(\delta).
\tag{3.10}
$$

Similarly, if we multiply (3.6) by $-1/2$ and add it to (3.4), and combine it with the corresponding version of (3.8), we get

$$
\begin{aligned}
&C(x)-\frac{1}{2}B\left(\frac{2\alpha+2}{2\alpha+1}x-\gamma\right)\\
&=\frac{(\alpha+1)x-\delta}{(\alpha+1)x+(2\alpha+1)\delta}\left[C\left(x+\frac{2\alpha+1}{\alpha+1}\delta\right)-\frac{1}{2}B\left(\frac{2\alpha+2}{2\alpha+1}x-\gamma+2\delta\right)\right]\\
&\quad-\frac{\alpha\delta^2}{((\alpha+1)x+(2\alpha+1)\delta)(x+\delta)}B\left(\frac{2\alpha+2}{2\alpha+1}x-\gamma+2\delta\right).
\end{aligned}
$$

By induction, for n satisfying (3.7), we obtain

$$\begin{aligned} C(x)-\frac{1}{2}B\left(\frac{2\alpha+2}{2\alpha+1}x-\gamma\right) &= \overline{\eta}_{n,\delta}+\prod_{k=0}^{n-1}\left(\frac{(\alpha+1)x+(2\alpha+1)k\delta-\delta}{(\alpha+1)x+(2\alpha+1)(k+1)\delta}\right) \\ &\quad \times\left[C\left(x+\frac{2\alpha+1}{\alpha+1}n\delta\right)+\alpha B\left(\frac{2\alpha+2}{2\alpha+1}x-\gamma+2n\delta\right)\right] \\ &= \overline{\eta}_{n,\delta}+\frac{1}{2}\prod_{k=0}^{n-1}\left(\frac{(\alpha+1)x+(2\alpha+1)k\delta-\delta}{(\alpha+1)x+(2\alpha+1)(k+1)\delta}\right). \end{aligned}$$

Carrying out similar calculations, we verify that for $\delta \to 0$, the product above converges to $(x/(\alpha+1))^{(2\alpha+2)/(2\alpha+1)}$ and hence

$$C(x)-\frac{1}{2}B\left(\frac{2\alpha+2}{2\alpha+1}x-\gamma\right)=\frac{1}{2}\left(\frac{x}{\alpha+1}\right)^{(2\alpha+2)/(2\alpha+1)}+O(\delta). \tag{3.11}$$

Combining this identity with (3.10), we finally obtain that for $x \in (0, \alpha+1)$,

$$\begin{aligned} P\left(0,\frac{2\alpha+2}{2\alpha+1}x-\gamma\right) &= B\left(\frac{2\alpha+2}{2\alpha+1}x-\gamma\right) \\ &= \left(\frac{x}{\alpha+1}\right)^{1/(2\alpha+1)}\frac{2(1+\alpha)^2-x}{(2\alpha+1)(\alpha+1)}+O(\delta) \end{aligned} \tag{3.12}$$

and

$$P\left(x,\frac{x}{2\alpha+1}-\gamma\right)=C(x)=\left(\frac{x}{\alpha+1}\right)^{1/(2\alpha+1)}\frac{(1+\alpha)^2+\alpha x}{(2\alpha+1)(\alpha+1)}+O(\delta). \tag{3.13}$$

To compute the corresponding values of the function M, we repeat, word by word, the above reasoning. We let

$$B(y)=M(0,y) \qquad \text{and} \qquad C(x)=M\left(x,\frac{x}{2\alpha+1}-\gamma\right).$$

The only difference lies in the equations (3.3) and (3.5): this time, if n satisfies (3.7), the corresponding equalities read

$$C\left(x+\frac{(2\alpha+1)n\delta}{\alpha+1}\right)=x+\frac{(2\alpha+1)n\delta}{\alpha+1}, \qquad B\left(\frac{2\alpha+2}{2\alpha+1}x-\gamma+2n\delta\right)=0.$$

Therefore, the analogues of (3.10) and (3.11) are

$$C(x)+\alpha B\left(\frac{2\alpha+2}{2\alpha+1}x-\gamma\right)=\left(\frac{x}{\alpha+1}\right)^{1/(2\alpha+1)}(1+\alpha)+O(\delta).$$

and

$$C(x)-\frac{1}{2}B\left(\frac{2\alpha+2}{2\alpha+1}x-\gamma\right)=\left(\frac{x}{\alpha+1}\right)^{(2\alpha+2)/(2\alpha+1)}(1+\alpha)+O(\delta).$$

Using these identities, we derive the corresponding values of M and obtain that

$$P\left(x,\frac{x}{2\alpha+1}-\gamma\right)-M\left(x,\frac{x}{2\alpha+1}-\gamma\right)=U_\alpha\left(x,\frac{x}{2\alpha+1}-\gamma\right)+O(\delta)$$

and

$$P\left(0, \frac{2\alpha+2}{2\alpha+1}x-\gamma\right) - M\left(0, \frac{2\alpha+2}{2\alpha+1}x-\gamma\right) = U_\alpha\left(0, \frac{2\alpha+2}{2\alpha+1}x-\gamma\right) + O(\delta).$$

This gives the optimality of the constant U_α on $D_2 \cap \{y=0\}$ and on the common boundary of D_2 and D_3. For the remaining states from $D_2 \cup D_3$, we use (c) and (d). Indeed, suppose that $(x, y) \in D_2$. Then, using the optimality we have just established,

$$\begin{aligned}
P(x,y) - M(x,y) &= \frac{(2\alpha+1)y - x + 2(\alpha+1)^2}{(2\alpha+1)(x+y)+2(\alpha+1)^2}\left(U_\alpha(0, x+y) + O(\delta)\right) \\
&\quad + \frac{2(\alpha+1)x}{(2\alpha+1)(x+y)+2(\alpha+1)^2} \times \\
&\quad \times \left[U_\alpha\left(\frac{2\alpha+1}{2(\alpha+1)}(x+y)+\alpha+1, \frac{x+y}{2(\alpha+1)} - \alpha - 1\right) + O(\delta)\right] \\
&= U_\alpha(x,y) + O(\delta).
\end{aligned}$$

The points from D_3 are dealt with in a similar manner.

3.4. **Sharpness, the continuous-time case.** Let (x, y) be a fixed starting point. To show the optimality of the constant $U_\alpha^+(x, y)$, one can embed the examples from the previous subsection into a pair of appropriate submartingales. It is also possible to define the corresponding pair (X, Y) directly. Let us briefly explain this. Fix $\delta > 0$ and let B be a standard Brownian motion, starting from 0. Suppose, for example, that $(x, y) \in D_2$, $x > 0$. Let us present the continuous-time version of the pair (f, g). Put $\tau_0 \equiv 0$. The first move is described by (c), so define

$$\tau_1 = \inf\left\{t > 0 : x + B_t \in \left\{0, \frac{2\alpha+1}{2(\alpha+1)}(x+y)+\alpha+1\right\}\right\} \tag{3.14}$$

and $(\phi_s, \psi_s, \zeta_s, \xi_s) = (1, 0, -1, 0)$ for $s \in [0, \tau_1)$.

Next, we deal with the second step. If $x + B_{\tau_1} = 0$, we follow (b): to embed this move, we put $\tau_2 = \tau_1 + 2\delta/(\alpha+1)$ and $(\phi_s, \psi_s, \zeta_s, \xi_s) = (0, 1, 0, \alpha)$ for $s \in [\tau_1, \tau_2)$. If $x+B_{\tau_1}$ is equal to the second expression in (3.14), we must construct the continuous version of the step described in (d). This is straightforward: we set

$$\tau_2 = \inf\left\{t > \tau_1 : x + B_t \in \left\{0, \frac{(2\alpha+1)(x-y) - 2(1+\alpha)^2 + 2\alpha\delta}{2\alpha}\right\}\right\}$$

and $(\phi_s, \psi_s, \zeta_s, \xi_s) = (1, 0, 1, 0)$ for $s \in [\tau_1, \tau_2)$. Proceeding in such a way, we obtain the predictable quadruple (ϕ, ψ, ζ, ξ) and the sequence $(\tau_n)_{n\geq 0}$ of stopping times such that the corresponding Itô processes (1.1) satisfy

$$\big((X_{\tau_n}, Y_{\tau_n})\big)_{n\geq 0} \quad \text{has the same distribution as} \quad (f, g).$$

All that is left is to observe that

$$\mathbb{P}(Y^* \geq 0) - ||X||_1 = \mathbb{P}(g^* \geq 0) - ||f^+||_1,$$

and the right-hand side can be made arbitrarily close to $U_\alpha^+(x, y)$, by the appropriate choice of δ.

4. Applications

4.1. **One-sided weak-type inequalities for α-subordinated processes.** As the first application, we present the proof of the one-sided version of the inequality (1.3).

Theorem 4.1. *Assume that X is a submartingale and Y is a semimartingale which α-subordinate to X, $\alpha \in [0,1]$. If $|Y_0| \leq |X_0|$, then for any $\lambda > 0$ we have*

$$\lambda \mathbb{P}(Y^* \geq \lambda) \leq 4||X^+||_1 - 2\mathbb{E}X_0 \leq 6||X||_1. \tag{4.1}$$

The first inequality is sharp.

Proof. Of course, the claim follows immediately from Hammack's result (1.3), but it is instructive to see how this bound can be deduced from (1.5). Fix a positive number λ and observe that $4Y/\lambda$ is α-subordinate to $4X/\lambda$, so

$$\begin{aligned}\mathbb{P}(Y^* \geq \lambda) &= \mathbb{P}\left(\left(\frac{4Y}{\lambda} - 4\right)^* \geq 0\right)\\ &\leq \frac{4}{\lambda}||X^+||_1 + \mathbb{E}U\left(\frac{4X_0}{\lambda}, \frac{4(Y_0-\lambda)}{\lambda}\right).\end{aligned}$$

By Lemma 2.1 (i) and Lemma 2.2 (i), we have

$$U_\alpha\left(\frac{4X_0}{\lambda}, \frac{4(Y_0-\lambda)}{\lambda}\right) \leq U_\alpha\left(\frac{4X_0}{\lambda}, \frac{4(|X_0|-\lambda)}{\lambda}\right) = -\frac{2X_0}{\lambda}$$

This yields (4.1). To see that the bound is sharp, let X be a Brownian motion started at $-\lambda/2$ and stopped at 0. Put $Y_t = \lambda + X_t$ for $t \geq 0$; then Y is α-subordinate to X, $|Y_0| = |X_0|$, $\mathbb{P}(Y^* \geq \lambda) = 1$, $||X^+||_1 = 0$ and $-2\mathbb{E}X_0 = \lambda$, so both sides of (4.1) are equal. □

When X is assumed to be nonnegative, we have an even nicer statement. Recall the parameter $\gamma(\alpha)$ given by (3.1).

Theorem 4.2. *Assume that X is a nonnegative submartingale and Y is α-subordinate to X. If $|Y_0| \leq |X_0|$, then for any $\lambda > 0$ we have*

$$\lambda \mathbb{P}(Y^* \geq \lambda) \leq \gamma(\alpha)||X||_1. \tag{4.2}$$

The inequality is sharp.

Proof of Theorem 4.2. We argue as previously. Fix $\lambda > 0$ and note that by (1.6),

$$\begin{aligned}\mathbb{P}(Y^* \geq \lambda) &= \mathbb{P}\left(\left(\frac{\gamma(\alpha)Y}{\lambda} - \gamma(\alpha)\right)^* \geq 0\right)\\ &\leq \frac{\gamma(\alpha)||X||_1}{\lambda} + \mathbb{E}U_\alpha^+\left(\frac{\gamma(\alpha)X_0}{\lambda}, \frac{\gamma(\alpha)Y_0}{\lambda} - \gamma(\alpha)\right).\end{aligned}$$

It suffices to observe that by Lemma 3.2 and the inequality $|Y_0| \leq |X_0|$, we have

$$U_\alpha^+\left(\frac{\gamma(\alpha)X_0}{\lambda}, \frac{\gamma(\alpha)Y_0}{\lambda} - \gamma(\alpha)\right) \leq U_\alpha^+(0, -\gamma(\alpha)) = 0.$$

Hence, (4.2) follows. To prove that this bound is sharp, we will use the examples from the previous section. Fix $\beta \in (0,\gamma)$ and $\varepsilon \in (0, U_\alpha^+(0,-\beta))$. There is a pair (X, Y) of Itô processes of the form (1.1) for which (1.2) holds, $X_0 = 0$, $Y_0 = -\beta$ and

$$\mathbb{P}(Y^* \geq 0) - ||X||_1 \geq U_\alpha^+(0,-\beta) - \varepsilon \geq 0.$$

Then $Y+\beta$ is α-subordinate to X and

$$\beta\mathbb{P}((Y+\beta)^* \geq \beta) \geq \beta||X||_1,$$

which implies that the best constant must be at least β. Letting $\beta \to \gamma(\alpha)$ yields the sharpness. $\square$

4.2. **On optimal control of semimartingales.** The second application we discuss here concerns the so-called optimal control of semimartingales. Let $X = X_0 + M + A$ be a real-valued semimartingale and let H, K be real predictable processes. Assume that H, K are bounded or, more generally, that $\int_0^t |H_s|^2 \mathrm{d}[M,M]_s$, $\int_0^t K_s \mathrm{d}|A|_s$ are finite almost surely for all $t \geq 0$. Controlling X by the pair (H,K) gives the right-continuous semimartingale Y given by the stochastic integral

$$Y_t = H_0 X_0 + \int_{0+}^t H_s \,\mathrm{d}M_s + \int_{0+}^t K_s \,\mathrm{d}A_s, \qquad t \geq 0.$$

Let β be a fixed real number and suppose that the goal is to find a pair (H,K) in some given class of predictable processes such that

$$\mathbb{P}(Y_s \geq \beta \quad \text{for some } s \geq 0) = 1. \tag{4.3}$$

The example $X \equiv 0$ shows that this is not always possible. The following theorem gives a necessary condition for the existence of such a pair in the case when X is a martingale (see Burkholder [3] and [5]).

Theorem 4.3. *Let a, b, $x, \beta \in \mathbb{R}$ with $a \leq 0$ and $b \geq 1$. Suppose that X is a martingale with $X_0 \equiv x$ and H is a predictable process taking values in $[a,b]$ and such that $H_0 \equiv 1$. If* (4.3) *holds, then*

$$||X||_1 \geq |x| \vee \left[\frac{(a+b-2)x+2\beta}{b-a}\right].$$

The inequality is sharp.

Next, fix $t \in [0,1]$ and suppose that (4.3) is replaced by the less stringent condition

$$\mathbb{P}(Y_s \geq \beta \quad \text{for some } s \geq 0) \geq t. \tag{4.4}$$

Then we have the following version of Theorem 4.3, proved by K. P. Choi in [11].

Theorem 4.4. *Let x, $\beta \in \mathbb{R}$, $t \in [0,1]$ and assume that X is a martingale starting from x. Let H be a predictable process satisfying $-1 \leq H(\omega) \leq 1$ and $H_0 \equiv 1$. If* (4.4) *holds true, then*

$$||X||_1 \geq |x| \vee \left\{\beta - x - [\beta^+(\beta-2x)^+(1-t)]^{1/2}\right\}.$$

The equality can hold.

Our contribution in this direction concerns the case when X is a submartingale. Furthermore, we allow H_0 to be any number from $[-1,1]$. We start with the result for general submartingales.

Theorem 4.5. *Let $x, \beta \in \mathbb{R}$, $\alpha \in [0,1]$, $t \in [0,1]$ and assume that X is a submartingale starting from x. Let H, K be predictable processes taking values in $[-1,1]$ and $[-\alpha,\alpha]$, respectively. If (4.4) holds true, then*

$$||X^+||_1 \geq x^+ \vee \left\{ \frac{\beta - 2x^- - \sqrt{\beta^+(\beta-2|x|)^+(1-t)}}{2} \right\}. \tag{4.5}$$

The inequality is sharp.

Proof. First let us exclude the trivial cases. If $\beta \leq 2|x|$ or $t \leq 2|x|/\beta$, then the inequality (4.5) is equivalent to $||X^+||_1 \geq x^+$, which is obvious. Hence we assume that the two estimates are not true. Pick $C \in (0, 4/\beta]$ and apply (1.5) to the pair CX, $C(Y-\beta)$. We obtain

$$\begin{aligned} ||X^+||_1 = \frac{||CX^+||_1}{C} &\geq \frac{\mathbb{P}(C(Y-\beta)^* \geq 0) - U(Cx, C(H_0x-\beta))}{C} \\ &\geq \frac{\mathbb{P}(Y^* \geq \beta) - U(Cx, C(|x|-\beta))}{C} \end{aligned}$$

since $H_0x \leq |x|$ and the function $U(x,\cdot)$ is nondecreasing. Because $\beta > 2|x|$ and $C < 4/\beta$, we have $(Cx, C(|x|-\beta)) \in D_5$ and hence, using the bound $\mathbb{P}(Y^* \geq \beta) \geq t$, we get

$$||X^+||_1 \geq \frac{t-1}{C} - \frac{C}{16}\left((|x|-\beta)^2 - x^2\right) + \frac{1}{2}\left(x - |x| + \beta\right).$$

The right-hand side, as a function of C, attains its maximum for

$$C = 4\sqrt{\frac{1-t}{\beta(\beta-2|x|)}} \in (0, 4/\beta).$$

The maximum is equal to the expression in the parentheses appearing in (4.5).

To show the sharpness, we will exploit Theorem 4.4 above. Fix the parameters $x, \beta \in \mathbb{R}$, $t \in [0,1]$. By Choi's result, there is a martingale pair (X, Y), starting from $(|x|, |x|)$, such that Y is the integral, with respect to X, of a certain predictable process taking values in $[-1,1]$, for which (4.4) holds and

$$||X||_1 = |x| \vee \left\{ \beta - |x| - [\beta^+(\beta-2|x|)^+(1-t)]^{1/2} \right\}.$$

It remains to observe that

$$||X^+||_1 = \frac{1}{2}(x + ||X||_1) = x^+ \vee \left\{ \frac{\beta - 2x^- - \sqrt{\beta^+(\beta-2|x|)^+(1-t)}}{2} \right\}$$

and we are done. □

In the nonnegative case, we have the following statement, which is proved exactly in the same manner as above. We omit the details.

Theorem 4.6. *Let x, $\alpha \geq 0$, $\beta \in \mathbb{R}$, $t \in [0,1]$ and assume that X is a nonnegative submartingale starting from x. Let H, K be predictable processes taking values in $[-1,1]$ and $[-\alpha,\alpha]$, respectively. If (4.4) holds true, then*

$$||X||_1 \geq \sup_{C>0} \frac{t - U_\alpha^+(Cx, C(x-\beta))}{C}. \tag{4.6}$$

Unfortunately, there does not seem to be an explicit formula for the right hand side. Nonetheless, if $\beta t \leq 2x$, then it is easy to check that the supremum equals x.

4.3. **Inequalities for smooth functions.** As another application of Theorem 1.1, we present a weak-type estimate for α-subordinate smooth functions on Euclidean domains. Suppose that Ω is an open, connected subset of $\mathbb{R}^n$, n being a fixed positive integer, such that $0 \in \Omega$. Let $\overline{\Omega}$ be a connected bounded subdomain of Ω with $0 \in \overline{\Omega}$ and $\partial\overline{\Omega} \subset \Omega$. Denote by μ the harmonic measure on $\partial\overline{\Omega}$ with respect to 0. Consider two real-valued C^2 functions u, v on Ω. Following [4], we say that v is differentially subordinate to u if

$$|\nabla v(x)| \leq |\nabla u(x)| \quad \text{for } x \in \Omega.$$

For example, if u, v are harmonic functions satisfying Cauchy-Riemann equations, then this condition is satisfied. Furthermore, for $\alpha \geq 0$, the function v is α-subordinate to u if it is differentially subordinate to u and, in addition,

$$|\Delta v(x)| \leq \alpha |\Delta u(x)| \quad \text{for } x \in \Omega$$

(see [7] and [8]). The inequalities comparing the sizes of u and v under the assumption of this type of subordination were studied by a number of authors, see e.g. [1], [2], [4], [7], [8], [16], [18], [19] and [20]. Our contribution in this direction is described in the following result.

Theorem 4.7. *Suppose that u is subharmonic and v is α-subordinate to u.*

(i) If $\alpha \in [0,1]$, then

$$\sup_{\lambda>0} \lambda\mu(v(x) \geq \lambda) \leq \int_{\partial\overline{\Omega}} u(x)^+ d\mu(x) + U(u(0), v(0)). \tag{4.7}$$

(ii) If $\alpha \geq 0$ and u is nonnegative, then

$$\sup_{\lambda>0} \lambda\mu(v(x) \geq \lambda) \leq \int_{\partial\overline{\Omega}} u(x) d\mu(x) + U_\alpha^+(u(0), v(0)). \tag{4.8}$$

Proof. Consider n-dimensional Brownian motion W starting from 0 and let τ denote the exit time of $\overline{\Omega}$: $\tau = \inf\{t : W_t \notin \overline{\Omega}\}$. Introduce the processes

$$X = (X_t)_{t\geq 0} = (u(W_{\tau\wedge t}))_{t\geq 0}, \quad Y = (Y_t)_{t\geq 0} = (v(W_{\tau\wedge t}))_{t\geq 0}.$$

By Itô's formula, for any $t \geq 0$ we have

$$X_t = u(0) + \int_0^t \nabla u(W_{\tau\wedge s})\mathrm{d}W_s + \frac{1}{2}\int_0^t \Delta u(W_{\tau\wedge s})\mathrm{d}s = X_0 + M_t + A_t,$$

$$Y_t = v(0) + \int_0^t \nabla v(W_{\tau\wedge s})\mathrm{d}W_s + \frac{1}{2}\int_0^t \Delta v(W_{\tau\wedge s})\mathrm{d}s = Y_0 + N_t + B_t.$$

Since

$$[M,M]_t - [N,N]_t = |u(0)|^2 - |v(0)|^2 + \int_0^t \left(|\nabla u(W_{\tau\wedge s})|^2 - |\nabla v(W_{\tau\wedge s})|^2\right) ds$$

and

$$\alpha A_t - |B|_t = \frac{1}{2}\int_0^t \left(\alpha\Delta u(W_{\tau\wedge s}) - |\Delta v(W_{\tau\wedge s})|\right) ds,$$

we see that α-subordination of the functions u and v implies that Y is α-subordinate to X. Furthermore, X is a submartingale, we have $\mu(v(x) \geq \lambda) \leq \mathbb{P}(Y^* \geq \lambda)$ and $||X^+||_1 = \int_{\partial\overline{\Omega}} u(x)^+ d\mu(x)$. Therefore, (1.5) implies (4.7) and (1.6) implies (4.8). This completes the proof. □

In a similar manner, one can establish the appropriate versions of Theorems 4.1, 4.2, 4.5 and 4.6; of course, the analogue of the condition (4.4) is

$$\mu(\{x \in \partial\overline{\Omega} : v(x) \geq \beta\}) \geq t.$$

The details are left to the reader.

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