SHARP INEQUALITIES FOR GEOMETRIC MAXIMAL OPERATORS ASSOCIATED WITH GENERAL MEASURES

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Abstract. We determine the best constants in the weak-type (p, p) and L^p estimates for geometric maximal operator on $(\mathbb{R}, \mu)$. It is also shown that in higher dimensions such inequalities fail to hold.

1. Introduction

Let μ be a nonnegative Borel measure on $\mathbb{R}^n$. The maximal geometric mean operator associated with μ is an operator acting on measurable functions $f : \mathbb{R}^n \to \mathbb{R}$ by the formula

$$G_f(x) = \sup \exp \left\{ \frac{1}{\mu(B)} \int_B \log(|f(y)|) \mathrm{d}\mu(y) \right\}, \qquad x \in \mathbb{R}^n,$$

where the supremum is taken over all balls B in $\mathbb{R}^n$ containing x such that the integral is defined and $0 < \mu(B) < \infty$; if no such B exists, we put $G_f(x) = 0$. This operator was introduced in the one-dimensional setting by X. Shi, who studied weighted L^p estimates for this object. More precisely, it was proved in [14] that if $0 < p < \infty$ is fixed and μ is Lebesgue measure on $\mathbb{R}$, then the inequality

$$\int_{\mathbb{R}} G_f(x)^p w(x) \mathrm{d}\mu(x) \leq C_1 \int_{\mathbb{R}} |f(x)|^p w(x) \mathrm{d}\mu(x)$$

holds for all $f \in L^p(w\mathrm{d}\mu)$ if and only if w satisfies Muckenhoupt's (A_∞) condition

$$\sup_I \left(\frac{1}{|I|} \int_I w(x) \mathrm{d}\mu(x) \right) \left(\exp \left\{ \frac{1}{|I|} \int_I \log(1/w(x)) \mathrm{d}\mu(x) \right\} \right) \leq C_2.$$

Here the supremum is taken over all intervals I (see [8], [11]). Hu et. al. [9] showed that for each $0 < p < \infty$, the latter requirement is equivalent to the validity of the weak-type estimate

$$\sup_{\lambda > 0} \lambda \left\{ \int_{\{x : G_f(x) \geq \lambda\}} w(x) \mathrm{d}\mu(x) \right\}^{1/p} \leq C_3 \left\{ \int_{\mathbb{R}} |f(x)|^p w(x) \mathrm{d}\mu(x) \right\}^{1/p}$$

for all $f \in L^p(w\mathrm{d}\mu)$. These results have been extended in many directions, including two-weight case (see Cruz-Uribe [3], Cruz-Uribe and Neugebauer [4], Hu et. al. [9], Yin and Muckenhoupt [12]) and one-sided case (Luor [10], Ortega and Ramírez [13]). The concept of maximal geometric mean operator has been also transferred to probability theory: see [1], [2] and [15].

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The purpose of this paper is to determine best constants in the weak- and strong-type inequalities for maximal geometric operators associated to general Borel measures in the one-dimensional setting. For a given measure μ on $\mathbb{R}$ and a measurable $f:\mathbb{R}\to\mathbb{R}$, put

$$||f||_{L^p(\mathbb{R},\mu)}=\left\{\int_{\mathbb{R}}|f(x)|^p\mathrm{d}\mu(x)\right\}^{1/p}$$

and

$$||f||_{L^{p,\infty}(\mathbb{R},\mu)}=\sup_{\lambda>0}\left\{\lambda^p\mu(\{x:G_f(x)>\lambda\})\right\}^{1/p}.$$

Our main results can be stated as follows.

Theorem 1.1. *For any $0<p<\infty$ and any measurable function f,*

$$||G_f||_{L^{p,\infty}(\mathbb{R},\mu)}\leq\left(\frac{2}{e\log 2}\right)^{1/p}||f||_{L^p(\mathbb{R},\mu)}. \tag{1.1}$$

If μ is Lebesgue measure, then the inequality is sharp for each p.

Theorem 1.2. *For any $0<p<\infty$ and any measurable function f,*

$$||G_f||_{L^p(\mathbb{R},\mu)}\leq c^{1/p}||f||_{L^p(\mathbb{R},\mu)}, \tag{1.2}$$

where $c=3.5911\ldots$ is the unique solution to the equation

$$\log c=\frac{c+1}{c}. \tag{1.3}$$

If μ is Lebesgue measure, then the inequality is sharp for each p.

We have organized this note as follows. In the next section we prove Theorem 1.1, as well as its extension concerning the more general class of weak-Φ estimates. In Section 3 we deal with the strong-type estimate. The final part of the paper concerns the following negative, higher-dimensional result: neither weak- nor strong-type estimate for G_f holds in $\mathbb{R}^n$, $n\geq 2$.

2. Weak-type estimate

We start with the following auxiliary fact.

Lemma 2.1. *For any $x\in\mathbb{R}$ we have*

$$e^x\geq\frac{e}{4}x+\frac{e\log 2}{2} \tag{2.1}$$

and

$$e^x\geq\frac{e}{2}x+\frac{e\log 2}{2}. \tag{2.2}$$

Proof. This follows from a straightforward calculus. We leave the details to the reader. □

Proof of (1.1). We must prove that for any p, $\lambda>0$ and any function $f\in L^p(\mathbb{R},\mu)$ we have

$$\lambda^p\mu(\{x\in\mathbb{R}:G_f(x)>\lambda\})\leq\frac{2}{e\log 2}\int_{\mathbb{R}}|f(x)|^p\mathrm{d}\mu(x).$$

Replacing f with λf if necessary, we may assume that $\lambda=1$. Similarly, plugging $|f|^{1/p}$ in the place of f, we see that it suffices to establish the above bound for $p=1$ and nonnegative f.

By the definition of G_f, if $G_f(x) > 1$, then there is a non-degenerate interval I_x containing x such that

$$\frac{1}{\mu(I_x)} \int_{I_x} \log f(y) \mathrm{d}\mu(y) > 0 \tag{2.3}$$

and we have $G_f > 1$ on I_x. By Lindelöf's theorem, we may pick a countable subcollection $\{I_j\}_{j\geq 1}$ such that

$$\bigcup_{j=1}^{\infty} I_j = \bigcup \{I_x : G_f(x) > 1\}.$$

For a fixed integer N, put $\mathcal{I}_N = \{I_1, I_2, \ldots, I_N\}$ and let

$$F^N = \bigcup_{I \in \mathcal{I}^N} I.$$

By Lemma 4.4 in [6], there are two subcollections $\mathcal{I}_1$ and $\mathcal{I}_2$ of $\mathcal{I}^N$ such that the intervals in each of these are pairwise disjoint and

$$F^N = \bigcup_{i=1}^{2} \bigcup_{I \in \mathcal{I}_i} I$$

(this splitting is possible only in the one-dimensional case; the argument breaks down in $\mathbb{R}^n$, $n \geq 2$). Therefore, we can write F^N as the sum $A \cup B$, where A is the set of those $x \in F^N$, which belong to exactly one element of $\mathcal{I}_1 \cup \mathcal{I}_2$ and B is the set of those x, which lie in one element of $\mathcal{I}_1$ and one element of $\mathcal{I}_2$. By (2.1),

$$\int_A f(x) \mathrm{d}\mu(x) = \int_A e^{\log f(x)} \mathrm{d}\mu(x) \geq \frac{e}{4} \int_A \log f(x) \mathrm{d}\mu(x) + \frac{e \log 2}{2} \mu(A)$$

and, similarly, by (2.2),

$$\int_B f(x) \mathrm{d}\mu(x) = \int_B e^{\log f(x)} \mathrm{d}\mu(x) \geq \frac{e}{2} \int_B \log f(x) \mathrm{d}\mu(x) + \frac{e \log 2}{2} \mu(B).$$

Summing these two estimates, we get

$$\int_{F^N} f(x) \mathrm{d}\mu(x) \geq \frac{e}{4} \sum_{I \in \mathcal{I}_1 \cup \mathcal{I}_2} \int_I \log f(x) \mathrm{d}\mu(x) + \frac{e \log 2}{2} \mu(F^N).$$

However, by (2.3), each term under the above sum is positive. Therefore,

$$\int_{F^N} f(x) \mathrm{d}\mu(x) \geq \frac{e \log 2}{2} \mu(F^N)$$

and letting $N \to \infty$ yields

$$\begin{aligned} \mu(\{x : G_f(x) > 1\}) = \mu\left(\bigcup_{j=1}^{\infty} I_j\right) &\leq \frac{2}{e \log 2} \int_{\{G_f > 1\}} f(x) \mathrm{d}\mu(x) \\ &\leq \frac{2}{e \log 2} \int_{\mathbb{R}} f(x) \mathrm{d}\mu(x), \end{aligned}$$

which is the desired estimate. We turn to the sharpness of this bound: assume that μ is the Lebesgue measure. First note that by the arguments presented at the beginning of the proof, it suffices to focus on the case $p = 1$. Consider the function $f : \mathbb{R} \to \mathbb{R}$ given by

$$f(x) = \frac{e}{2}\chi_{\{|x|\leq 1\}}(x) + \frac{e}{4}\chi_{\{1<|x|\leq 1/(2\log 2-1)\}}.$$

We compute that

$$\int_{-1/(2\log 2-1)}^{1} \log f(x)\mathrm{d}\mu(x) = \left(\frac{1}{2\log 2-1} - 1\right)\log\frac{e}{4} + 2\log\frac{e}{2} = 0$$

and, similarly,

$$\int_{-1}^{1/(2\log 2-1)} \log f(x)\mathrm{d}\mu(x) = 0.$$

This implies $G_f > 1$ on the interval $(-1/(2\log 2-1), 1/(2\log 2-1))$ and hence

$$||G_f||_{L^{1,\infty}(\mathbb{R},\mu)} \geq \mu(G_f > 1) \geq \frac{2}{2\log 2-1}.$$

On the other hand, we easily check that

$$\int_{\mathbb{R}} f(x)\mathrm{d}\mu(x) = \frac{e\log 2}{2}\cdot\frac{2}{2\log 2-1}.$$

This shows the optimality and completes the proof of Theorem 1.1. □

Remark 2.2. The reasoning presented above can be used to obtain much wider class of weak Φ-estimates. Suppose that $\Phi : [0,\infty) \to [0,\infty)$ is a C^1 function satisfying $\Phi(0) = 0$, $\lim_{x\downarrow 0} x\Phi'(x) = 0$, $\lim_{x\to\infty}\Phi'(x)x = \infty$ and such that the composition $x \mapsto \Phi(e^x)$ is strictly convex on $\mathbb{R}$. We will show how to determine the best constant C_Φ in the inequality

$$\Phi(1)\,\mu(\{x\in\mathbb{R} : G_f(x) > 1\}) \leq C_\Phi \int_{\mathbb{R}} \Phi(|f(x)|)\mathrm{d}\mu(x).$$

For any $\beta \in [0,\Phi(1)]$, let $\alpha_-(\beta)$, $\alpha_+(\beta)$ ($\alpha_-(\beta) < \alpha_+(\beta)$) be the slopes of two lines passing through $(0,\beta)$, tangent to the graph of the function $x \mapsto \Phi(e^x)$. By Darboux property, there is a unique β_0 for which $\alpha_+(\beta_0) = 2\alpha_-(\beta_0)$ (indeed: we have $\alpha_+(0) > 0 = 2\alpha_-(0)$ and $\alpha_-(\Phi(1)) = \alpha_+(\Phi(1)) > 0$). Replace (2.1) and (2.2) with the estimates

$$\Phi(e^x) \geq \alpha_-(\beta_0)x + \beta_0, \qquad \Phi(e^x) \geq \alpha_+(\beta_0)x + \beta_0$$

and repeat all the above arguments to obtain the inequality

$$\Phi(1)\,\mu(\{x\in\mathbb{R} : G_f(x) > 1\}) \leq \frac{\Phi(1)}{\beta_0}\int_{\mathbb{R}} \Phi(|f(x)|)\mathrm{d}\mu(x). \tag{2.4}$$

To see that this bound is sharp, let μ be the Lebesgue measure and let $x_\pm$ be numbers satisfying $\Phi(e^{x_\pm}) = \alpha_\pm(\beta_0)x_\pm + \beta_0$. Define $f : \mathbb{R} \to \mathbb{R}$ by the formula

$$f(x) = e^{x_+}\chi_{[-a,a]}(x) + e^{x_-}\chi_{[-1,1]\setminus[-a,a]}(x),$$

where the number $a \in (0,1)$ is chosen to satisfy $\int_{-a}^{1}\log f\mathrm{d}\mu = 0$ (then, by symmetry, we also have $\int_{-1}^{a}\log f\mathrm{d}\mu = 0$). Directly from this property of a and the definition of the geometric maximal operator, we have $G_f > 1$ on $(-1,1)$; thus, $|\{G_f > 1\}| \geq 2$. On the other hand,

$$\begin{aligned}\int_{\mathbb{R}} \Phi(f)\mathrm{d}\mu &= \int_{[-a,a]} \Phi(e^{x_+})\mathrm{d}\mu + \int_{[-1,1]\setminus[-a,a]} \Phi(e^{x_-})\mathrm{d}\mu \\ &= \int_{[-a,a]} \big[\alpha_+(\beta_0)x_+ + \beta_0\big]\mathrm{d}\mu + \int_{[-1,1]\setminus[-a,a]} \big[\alpha_-(\beta_0)x_- + \beta_0\big]\mathrm{d}\mu \\ &= \alpha_-(\beta_0)\left[\int_{[-a,1]} \log f\mathrm{d}\mu + \int_{[-1,a]} \log f\mathrm{d}\mu\right] + 2\beta_0 \\ &= 2\beta_0\end{aligned}$$

and hence both sides of (2.4) are equal.

3. L^p-estimate

We start with two auxiliary results.

Lemma 3.1. *For any nonnegative function f on $\mathbb{R}$ such that the integral $\int_{\mathbb{R}} \log f\, d\mu$ is well-defined, we have*

$$\int_{\{G_f>1\}} \log f\, d\mu + \int_{\{f>1\}} \log f\, d\mu \geq 0. \tag{3.1}$$

Proof. Let N be an arbitrary integer and let F^N, $\{I_j\}_{j\geq 1}$, $\mathcal{I}_i$, A and B be as in the proof of (1.1) above. Write

$$\int_{F^N} \log f\mathrm{d}\mu = \int_A \log f\mathrm{d}\mu + \int_B \log f\mathrm{d}\mu = \sum_{I\in\mathcal{I}_1\cup\mathcal{I}_2} \int_I \log f\mathrm{d}\mu - \int_B \log f\mathrm{d}\mu.$$

However,

$$\int_B \log f\mathrm{d}\mu \leq \int_{B\cap\{f>1\}} \log f\mathrm{d}\mu \leq \int_{\{f>1\}} \log f\mathrm{d}\mu,$$

which combined with the previous identity (and the fact that $\int_I \log f\mathrm{d}\mu \geq 0$ for any $I \in \mathcal{I}_1 \cup \mathcal{I}_2$) gives

$$\int_{F^N} \log f\mathrm{d}\mu + \int_{\{f>1\}} \log f\mathrm{d}\mu \geq 0.$$

It remains to let $N \to \infty$ to get the claim. □

We will also need the following technical fact. Recall the number c given by the equation (1.3).

Lemma 3.2. *For any x, $y \geq 0$ we have*

$$y\log x - y\log y + x + y \leq (cx - y)\log c. \tag{3.2}$$

Proof. By continuity of both sides, we may assume that $y > 0$. Divide throughout by y to transform the inequality into

$$\log s + 1 + s \leq (cs - 1)\log c,$$

which follows by a straightforward analysis: the left hand side is a concave function of s, the right-hand side is a linear function of s, and both these functions agree, along with their derivatives, at the point $s = 1/c$ (see (1.3)). □

Proof of (1.2). Arguing as in the previous section, it suffices to establish the estimate for $p = 1$. Fix a function $f : \mathbb{R} \to \mathbb{R}$; we may and do assume that f is integrable, since otherwise the right-hand side of (1.2) is infinite and there is nothing to prove. Pick $\lambda > 0$, apply the inequality (3.1) to $|f|/\lambda$ and integrate the obtained bound over λ. We get

$$\int_0^\infty \int_{\mathbb{R}} \log(|f|/\lambda)\big\{\chi_{\{G_f>\lambda\}} + \chi_{\{|f|>\lambda\}}\big\}\mathrm{d}\mu\mathrm{d}\lambda \geq 0.$$

Since f is integrable, the use of Fubini's theorem is allowed and we obtain

$$\int_{\mathbb{R}} \big[G_f \log|f| - G_f \log G_f + G_f + |f|\big]\mathrm{d}\mu \geq 0.$$

This, by (3.2), implies $\int_{\mathbb{R}} c|f| - G_f \mathrm{d}\mu \geq 0$, and we are done. □

Sharpness. Again, we may restrict ourselves to $p = 1$. Let μ be the Lebesgue measure on $\mathbb{R}$. Fix $\varepsilon \in (0, 1/2)$ and consider the function f on $\mathbb{R}$ given by

$$f_\varepsilon(x) = \varepsilon^{-1}\chi_{\{|x|\leq\varepsilon\}} + |x|^{-1}\chi_{\{\varepsilon<|x|\leq\varepsilon^{-1}\}}.$$

We have $||f_\varepsilon||_{L^1(\mathbb{R},\mu)} = 2(1 - 2\log\varepsilon)$. Furthermore, for any $x \in (c\varepsilon, \varepsilon^{-1})$, we easily derive that

$$\begin{aligned}\frac{1}{|[-c^{-1}x, x]|}\int_{-c^{-1}x}^{x} \log f_\varepsilon \mathrm{d}\mu &= 1 - \log x + \frac{1}{c+1}\log c - \frac{2\varepsilon}{(1+c^{-1})x}\\ &= \log c - \log x - \frac{2\varepsilon}{(1+c^{-1})x}\end{aligned}$$

and consequently, $G_{f_\varepsilon}(x) \geq \frac{c}{x}\exp\left(-\frac{2\varepsilon}{(1+c^{-1})x}\right)$. By symmetry, we have $G_{f_\varepsilon}(x) \geq \frac{c}{|x|}\exp\left(-\frac{2\varepsilon}{(1+c^{-1})|x|}\right)$ whenever $x \in (-\varepsilon^{-1}, -c\varepsilon)$, and therefore,

$$\begin{aligned}\frac{\int_{\mathbb{R}} G_{f_\varepsilon}\mathrm{d}\mu}{||f_\varepsilon||_{L^1(\mathbb{R},\mu)}} &\geq c(1-2\log\varepsilon)^{-1}\int_{c\varepsilon}^{\varepsilon^{-1}} \frac{1}{x}\exp\left(-\frac{2\varepsilon}{(1+c^{-1})x}\right)\mathrm{d}x\\ &= c(1-2\log\varepsilon)^{-1}\int_{c}^{\varepsilon^{-2}} \frac{1}{x}\exp\left(-\frac{2}{(1+c^{-1})x}\right)\mathrm{d}x.\end{aligned}$$

It remains to note that, by l'Hospital rule, the latter expression converges to c as $\varepsilon \to 0$. This proves the desired sharpness. □

4. Higher-dimensional setting

There is a natural question about the analogues of the previous results in the higher-dimensional case. Unfortunately, the answer is negative: if $n \geq 2$, then the geometric maximal operator is in general not bounded on L^p for any $p > 0$. In fact, as we will prove now, it does not even map $L^p(\mathbb{R}^n, \mu)$ into $L^{p,\infty}(\mathbb{R}^n, \mu)$ for general measures μ. Consider the following example (similar constructions can be found in [5] and [7]). Let B_1, B_2, ... be closed balls in $\mathbb{R}^n$ such that the origin lies on the boundary of each ball and such that for every B_j there is a point $x_j \in B_j \setminus \bigcup_{i\neq j} B_i$. In addition, put $x_0 = 0$ and define the measure μ on $\mathbb{R}^n$ by

$$\mu = \sum_{j=0}^{\infty} \delta_{x_j},$$

where δ_x stands for Dirac measure concentrated on x. Let f be given by

$$f(x) = \sum_{j=0}^{\infty} (j+1)^{-3/(2p)} \chi_{\{x_j\}}(x).$$

Then

$$||f||_{L^p(\mathbb{R},\mu)} = \left(\sum_{j=0}^{\infty} (j+1)^{-3/2} \right)^{1/p} < \infty,$$

but, on the other hand,

$$\begin{aligned} G_f(x_j) &\geq \exp\left(\frac{1}{\mu(B_j)} \int_{B_j} \log f \mathrm{d}\mu \right) \\ &= \exp\left(\frac{\log 1 + \log(j+1)^{-3/(2p)}}{2} \right) = (j+1)^{-3/(4p)}, \qquad j = 0,\, 1,\, 2,\, \ldots. \end{aligned}$$

Consequently,

$$\begin{aligned} ||G_f||_{L^{p,\infty}(\mathbb{R},\mu)} &\geq \sup_j \left\{ (j+1)^{-3/(4p)} \left[\mu(G_f > (j+1)^{-3/(4p)}) \right]^{1/p} \right\} \\ &= \sup_j \left\{ j^{1/p} (j+1)^{-3/(4p)} \right\} = \infty \end{aligned}$$

and hence the weak-type bound does not hold for G.

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