A PROBABILISTIC APPROACH TO HILBERT TRANSFORMS ON FREE GROUP VON NEUMANN ALGEBRAS

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ABSTRACT. The paper contains a probabilistic proof of the L^p-boundedness of the Hilbert transform in the context of a free group von Neumann algebra $VN(\mathbb{F}_q)$. The argument rests on noncommutative version of good-λ inequalities and yields a tight $L\log L$ order of the L^p norms as $p \to 1^+$ and $p \to \infty$.

1. INTRODUCTION

Let $f(z) = \sum_{n\in\mathbb{Z}} \hat{f}(n)z^n$ be a trigonometric polynomial on the unit circle $\mathbb{T}$, equipped with the normalized Haar (Lebesgue) measure. Then $\mathbb{H}f$, the Hilbert transform of f, is given by

$$\mathbb{H}f(z) = -i\sum_{n\in\mathbb{Z}} \operatorname{sgn}(n)\hat{f}(n)z^n. \tag{1.1}$$

This is a fundamental object in harmonic analysis, a prototypical example of Calderón-Zygmund singular integral operators, and its numerous applications range from convergence of Fourier series to signal processing. In particular, the boundedness properties of $\mathbb{H}$ in various function spaces are of fundamental importance. Directly from the definition, one easily checks that $\|\mathbb{H}\|_{L^2(\mathbb{T})\to L^2(\mathbb{T})} = 1$. A classical result of M. Riesz [27] asserts that $\mathbb{H}$ is bounded as an operator on $L^p(\mathbb{T})$ for $1 < p < \infty$; it can be shown that the corresponding norm is of order $O(p)$ as $p \to \infty$, and $O((p-1)^{-1})$ as $p \to 1$. At the endpoint cases $p \in \{1,\infty\}$, the Hilbert transform is no longer L^p-bounded, but it satisfies the appropriate weak-type inequalities: consult Bennett, DeVore and Sharpley [4], Kolmogorov [17] and Osękowski [22]. Various deep structural connections of the Hilbert transform with the theory of harmonic functions enable the precise identification of the corresponding norms. This topic has a lot of interest in the literature, we refer the interested reader to the works of Bañuelos and Wang [1], Bennett [3], Davis [6], Janakiraman [9], Osękowski [20, 21, 22] and Pichorides [24].

The purpose of this paper is to investigate the L^p-boundedness of the Hilbert transform from the noncommutative perspective, in the context of free group von Neumann algebras. We need some additional notation, the precise definitions will be provided in Section 2 below. In what follows, for a given $q \in \mathbb{Z}_+ \cup \{\infty\}$, $(\mathbb{F}_q, e)$ will stand for the free group with q generators g_1, g_2, ... and the identity element e, and the symbol $VN(\mathbb{F}_q)$ will denote the associated free group von Neumann algebra. For a prescribed sequence ε_1, ε_2, ... with values in $\{-1, 1\}$, we define the 'sign' function $\varepsilon : \mathbb{F}_q \to \{-1, 0, 1\}$ as follows. If $g \in \mathbb{F}_q$ (in reduced form) starts with g_k for some k, then set $\varepsilon(g) = \varepsilon_k$; if it starts with g_k^{-1}, put $\varepsilon(g) = -\varepsilon_k$; finally, define $\varepsilon(e) = 0$. Then, for any $f \in VN(\mathbb{F}_q)$ with the finite Fourier expansion $f = \sum_{g\in\mathbb{F}_q} \hat{f}(g)\lambda(g)$, its Hilbert

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transform is given by the formula

$$\mathbb{H}f=-i\sum_{g\in\mathbb{F}_q}\varepsilon(g)\hat{f}(g)\lambda(g). \tag{1.2}$$

This definition is consistent with the equation (1.1), which corresponds to $q=1$, $\mathbb{F}_q=\mathbb{Z}$, the generator $g_1=1$ and the sign function $\varepsilon(g)=\operatorname{sgn}(g)$. Let us also introduce the adjoint operator, which acts on f as above by the identity

$$\mathbb{H}^{op}f=i\sum_{g\in\mathbb{F}_q}\varepsilon(g^{-1})\hat{f}(g)\lambda(g).$$

One easily checks that $(\mathbb{H}f)^*=\mathbb{H}^{op}f^*$ (see Proposition 3.1 in [19]).

There is a natural question whether the free Hilbert transform defined above extends to the bounded operator on the associated L^p spaces, $1<p<\infty$. Various modifications and partial solutions to this problem appeared in the works of several mathematicians (cf. Junge and Mei [12], Junge, Parcet and Xu [14], Ozawa [23]) and was answered in the positive by Mei and Ricard [19]. Quantitatively, it was proved that the orders of the L^p norms do not exceed $O(p^\gamma)$ as $p\to\infty$ and $O((p-1)^{-\gamma})$ as $p\to 1$, where $\gamma=\ln(1+\sqrt{2})/\ln 2>1$. Our main result improves the constant to the order $L\log L$. Precisely, we will establish the following statement.

Theorem 1.1. *Let $1<p<\infty$ and $q\in\mathbb{Z}_+\cup\{\infty\}$. Then for any $f\in VN(\mathbb{F}_q)$ we have*

$$\|\mathbb{H}f\|_{\mathcal{L}^p(\mathbb{F}_q)}\leq C_p\,\|f\|_{\mathcal{L}^p(\mathbb{F}_q)}, \tag{1.3}$$

where $C_p=O(-(p-1)^{-1}\log(p-1))$ as $p\to 1$ and $C_p=O(p\log p)$ as $p\to\infty$. The same boundedness holds true for the adjoint transform $\mathbb{H}^{op}$.

The proof of the L^p estimate invented by Mei and Ricard rests on an appropriate version of the '$p\to 2p$ argument' and the free version of Cotlar's identity: for all f, $g\in VN(\mathbb{F}_q)$ we have

$$\mathbb{H}f^*\mathbb{H}^{op}g-(\widehat{\mathbb{H}f^*\mathbb{H}^{op}}g)(e)=\mathbb{H}(f^*\mathbb{H}^{op}g)+\mathbb{H}^{op}(\mathbb{H}(f^*)g)-\mathbb{H}^{op}\mathbb{H}(f^*g). \tag{1.4}$$

More precisely, one starts with the boundedness of $\mathbb{H}$ on $\mathcal{L}^2(\mathbb{F}_q)$, which is evident. Then one shows that if $\mathbb{H}$ is bounded on $\mathcal{L}^p(\mathbb{F}_q)$, then it is also bounded on $\mathcal{L}^{2p}(\mathbb{F}_q)$, with an appropriate recursive upper bound for $\|\mathbb{H}\|_{\mathcal{L}^{2p}(\mathbb{F}_q)\to\mathcal{L}^{2p}(\mathbb{F}_q)}$ in terms of $\|\mathbb{H}\|_{\mathcal{L}^p(\mathbb{F}_q)\to\mathcal{L}^p(\mathbb{F}_q)}$: this follows from (1.4) applied to $f=g$ and Schwarz' inequality. Solving the recursion and applying interpolation, one arrives at $\|\mathbb{H}\|_{\mathcal{L}^p(\mathbb{F}_q)\to\mathcal{L}^p(\mathbb{F}_q)}\leq O(p^\gamma)$ for $p\to\infty$, and the behavior of the norms for $p\to 1$ is handled by duality.

Our approach to the L^p-boundedness will exploit certain additional probabilistic arguments. The following reasoning in the classical setting turns out to be very effective, actually, it yields the optimal constants. Consider trigonometric polynomials $f(x)=\sum_n a_ne^{inx}$, $i\mathbb{H}f(x)=\sum_n\operatorname{sgn}(n)a_ne^{inx}$ on $\mathbb{T}$ and treat them as a 2π-periodic functions on $\mathbb{R}$. We extend these functions to the half-plane $\mathbb{R}^2_+=\mathbb{R}\times[0,\infty)$ by means of the Poisson integrals

$$U(x,y)=\frac{1}{\pi}\int_{\mathbb{R}}\frac{yf(z)\mathrm{d}z}{y^2+(x-z)^2}=\sum_n a_ne^{inx-ny}$$

and

$$V(x,y)=\frac{i}{\pi}\int_{\mathbb{R}}\frac{y\mathbb{H}f(z)\mathrm{d}z}{y^2+(x-z)^2}=\sum_n\operatorname{sgn}(n)a_ne^{inx-ny},$$

for $(x,y)\in\mathbb{R}\times(0,\infty)$. Now, fix a huge number $y>0$ and consider the two-dimensional Brownian motion $B=(\mathbb{X},\mathbb{Y})$ starting from $(0,y)$ and stopped when $\mathbb{Y}$ reaches the level 0. The functions U, V are harmonic on $\mathbb{R}^2_+$ and hence the compositions

$$\xi_t^y=U(\mathbb{X}_t,\mathbb{Y}_t),\quad \zeta_t^y=V(\mathbb{X}_t,\mathbb{Y}_t),\qquad t\geq 0,$$

are martingales. One can show that for each y, the process ζ^y is appropriately dominated by ξ^y, by means of the so-called differential subordination and orthogonality (we will not recall the definition here, we refer the reader to [1] for details). This domination implies that for any $2\leq p<\infty$ we have

$$\mathbb{E}|\zeta_\infty^y|^p\leq\big(\cot(\pi/2p)\big)^p\mathbb{E}|\xi_\infty^y|^p. \tag{1.5}$$

Letting $y\to\infty$, we obtain the desired L^p-boundedness of $\mathbb{H}$: it can be shown that the left-hand side above converges to $\|\mathbb{H}f\|^p_{L^p(\mathbb{T})}$, while the right tends to $\big(\cot(\pi/2p)\big)^p\|f\|^p_{L^p(\mathbb{T})}$.

It should be emphasized that the main ingredient of the above approach is hidden in the sharp martingale inequality (1.5). Actually, all the aforementioned sharp estimates for the classical Hilbert transform were obtained with the use of similar probabilistic arguments. This gives rise to the natural question whether the noncommutative martingale theory, which has been developed very intensively in the recent two decades, can be successfully applied in the study of free Hilbert transforms. We will answer this question in the affirmative, using the noncommutative analogue of the so-called good-λ inequalities, invented recently in [11]. Specifically, this probabilistic argument will allow us to obtain the estimate (1.3) for all f such that $\mathbb{H}f$ is self-adjoint. Combining this inequality with the argument of Mei and Ricard, we will get the claim in full generality.

The remaining part of the paper is organized as follows. The next section contains some preliminary material: we introduce there all the relevant notions and discuss their basic properties. The final part of the paper is devoted to the proof of the L^p estimate.

2. Background and notation

Noncommutative operators and L^p spaces. Let us present some basic facts from the operator theory, for more detailed information we refer the reader to [15], [16] and [29]. Although we will mainly deal with operators from free group von Neumann algebras and their embeddings into matrix spaces, it is convenient to discuss the general context, which allows a transparent introduction of martingale methods.

Let $\mathcal{M}$ be a von Neumann algebra acting on a Hilbert space $\mathcal{H}$, equipped with semifinite, faithful and normal trace τ. The identity element of $\mathcal{M}$ will be denoted by $I_\mathcal{M}$. For a densely defined self-adjoint operator $x\in\mathcal{M}$ and a sufficiently regular function f on $\mathbb{R}$, we define the operator $f(x)$ via the spectral resolution. An operator $x\in\mathcal{H}$ affiliated with $\mathcal{M}$ is said to be *τ-measurable*, if there exists $s\geq 0$ such that $\tau\left(\chi_{(s,\infty)}(|x|)\right)<\infty$, where $|x|=(x^*x)^{\frac{1}{2}}$ is the modulus of x. We denote the space of all τ-measurable operators by $L^0(\mathcal{M},\tau)$. For all $0<p<\infty$, we define the noncommutative L^p spaces associated with $(\mathcal{M},\tau)$ as $L^p(\mathcal{M},\tau)=\{x\in L^0(\mathcal{M},\tau):\|x\|_{L^p(\mathcal{M})}<\infty\}$, where the (semi-)norm is given by

$$\|x\|_{L^p(\mathcal{M})}=\left(\tau\left(|x|^p\right)\right)^{\frac{1}{p}}.$$

Furthermore, we let $L^\infty(\mathcal{M})=\mathcal{M}$, equipped with the operator norm.

Noncommutative martingales. Let us start with the notion of a filtration, which will be used below in two slightly different meanings. In the classical case, given a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, the term 'filtration' will refer to a nondecreasing sequence $(\mathcal{F}_n)_{n\in I}$ of sub-σ-fields of $\mathcal{F}$. Here I is a directed set, in our considerations below it will be either $\{0, 1, 2, \ldots, N\}$ or the interval $[0,T]$ for some fixed $T>0$. In the noncommutative setting, the word 'filtration' will be used to denote the finite nondecreasing sequence $(\mathcal{M}_n)_{n=0}^N$ of von Neumann subalgebras of a given von Neumann algebra $\mathcal{M}$. In such a case, for each $n = 0, 1, 2, \ldots, N$ there exists a conditional expectation $\mathcal{E}_n : \mathcal{M} \to \mathcal{M}_n$, defined as the dual map of natural inclusion $i : L^1(\mathcal{M}_n) \to L^1(\mathcal{M})$. It can be verified readily that $\mathcal{E}_n$ enjoys the following bi-module and trace-preservation properties:

(i) $\mathcal{E}_n(axb) = a\mathcal{E}_n(x)\, b$ for all $a, b \in \mathcal{M}_n$ and $x \in \mathcal{M}$;
(ii) $\tau \circ \mathcal{E}_n = \tau$.

In addition, the conditional expectations satisfy the tower property $\mathcal{E}_n\mathcal{E}_m = \mathcal{E}_m\mathcal{E}_n = \mathcal{E}_{\min(m,n)}$ and because of the trace-preserving condition (ii) above, each $\mathcal{E}_n$ extends to a contractive projection from $L^p(\mathcal{M}, \tau)$ onto $L^p\left(\mathcal{M}_n, \tau_{|\mathcal{M}_n}\right)$.

A finite sequence $x = (x_n)_{n=0}^N$ in $L^1(\mathcal{M})$ is called a noncommutative martingale with respect to $(\mathcal{M}_n)_{n=0}^N$, if we have the equality

$$\mathcal{E}_n(x_{n+1}) = x_n \qquad \text{for all } n.$$

The associated difference sequence $dx = (dx_n)_{n=0}^N$ is defined by $dx_0 = x_0$ and $dx_n = x_n - x_{n-1}$ for $n = 1, 2, \ldots, N$. Martingale inequalities play a prominent role in probability theory (both commutative and noncommutative) and apply in numerous contexts of harmonic analysis. Let us briefly discuss a single direction of this extensive subject, which will be important for our considerations below. Suppose that $x = (x_n)_{n=0}^N$ and $y = (y_n)_{n=0}^N$ are two martingales relative to the same filtration $(\mathcal{M}_n)_{n=0}^N$. In addition, impose a certain domination of x over y, expressed in terms of the corresponding difference sequences. For instance, we say that y is the transform of x by a sequence $\varepsilon = (\varepsilon_n)_{n\geq 0}$ of signs, if for any $n \geq 0$ we have the equality $dy_n = \varepsilon_n dx_n$. Another crucial and more general example concerns the so-called differential subordination (cf. [2, 10]): the martingale y is differentially subordinate to x if for any $n \geq 0$ we have the inequality $|dy_n|^2 \leq |dx_n|^2$. For an overview of other examples of martingale dominations, we refer the reader to [11, 18]. Now, assuming such type of subordination, there is a natural problem of proving various estimates (e.g., strong- or weak-type) between x and y. There is a method, invented recently in [11], which enables an efficient and unified study of a wide class of such problems. The method is the noncommutative generalization of the so-called extrapolation (or good-λ approach), which was introduced by Burkholder and Gundy [5] in the classical case. The precise formulation is the following.

Theorem 2.1. *Let $2 \leq p < \infty$. Suppose that $x = (x_n)_{n=0}^N$, $y = (y_n)_{n=0}^N$ are self-adjoint martingales in $L^p(\mathcal{M})$ such that*

$$\mathcal{E}_{n-1}(dy_n^2) \leq \mathcal{E}_{n-1}(dx_n^2) \qquad \text{and} \qquad \|dy_n\|_{L^p(\mathcal{M})} \leq \|dx_n\|_{L^p(\mathcal{M})}$$

for all $n = 0, 1, 2, \ldots, N$. Then we have the estimate

$$\|y_N\|_{L^p(\mathcal{M})} \leq c_p \|x_N\|_{L^p(\mathcal{M})},$$

for some universal constant c_p of order $O(p)$ as $p \to \infty$.

See [11] for more information on the subject and applications. We would like to emphasize that the result above concerns the self-adjoint processes. This restriction can be easily weakened to the case in which only the symmetry of y is required.

Lemma 2.2. *Let $2 \leq p < \infty$. Suppose that $x = (x_n)_{n=0}^N$, $y = (y_n)_{n=0}^N$ are martingales in $L^p(\mathcal{M})$ such that y is self-adjoint. If for any $n = 0, 1, 2, \ldots, N$ we have*

$$\mathcal{E}_{n-1}(dy_n^2) \leq \mathcal{E}_{n-1}(|dx_n|^2) \quad \text{and} \quad \|dy_n\|_{L^p(\mathcal{M})} \leq \|dx_n\|_{L^p(\mathcal{M})}, \tag{2.1}$$

then

$$\|y_N\|_{L^p(\mathcal{M})} \leq C_p \|x_N\|_{L^p(\mathcal{M})},$$

for some constant C_p of order $O(p)$ as $p \to \infty$.

Proof. Using a simple embedding into an appropriate space of 2×2 matrices, we reduce the context to the self-adjoint setting of the previous theorem. Consider the von Neumann algebra $\widetilde{\mathcal{M}} = \mathbb{M}_2 \overline{\otimes} \mathcal{M}$, equipped with the tensor trace $\mathrm{Tr} \otimes \tau$ and the filtration $(\widetilde{\mathcal{M}}_n)_{n=0}^N = (\mathbb{M}_2 \overline{\otimes} \mathcal{M}_n)_{n=0}^N$. The associated conditional expectations will be denoted by the symbols $\widetilde{\mathcal{E}}_n$, $n = 0, 1, 2, \ldots, N$. Next, consider the martingale differences

$$d\widetilde{y}_n = \begin{bmatrix} dy_n & 0 \\ 0 & 0 \end{bmatrix}, \qquad d\widetilde{x}_n = \begin{bmatrix} 0 & dx_n^* \\ dx_n & 0 \end{bmatrix}$$

for $n = 0, 1, 2, \ldots, N$. These differences are self-adjoint operators satisfying

$$\widetilde{\mathcal{E}}_{n-1}(d\widetilde{y}_n^2) = \begin{bmatrix} \mathcal{E}_{n-1}(dy_n^2) & 0 \\ 0 & 0 \end{bmatrix} \leq \begin{bmatrix} \mathcal{E}_{n-1}|dx_n|^2 & 0 \\ 0 & \mathcal{E}_{n-1}|dx_n^*|^2 \end{bmatrix} = \widetilde{\mathcal{E}}_{n-1}(d\widetilde{x}_n^2)$$

and similarly

$$\|d\widetilde{x}_n\|_{L^p(\widetilde{\mathcal{M}})}^p = \|dx_n\|_{L^p(\mathcal{M})}^p + \|dx_n^*\|_{L^p(\mathcal{M})}^p \geq \|dy_n\|_{L^p(\mathcal{M})}^p = \|d\widetilde{y}_n\|_{L^p(\widetilde{\mathcal{M}})}^p.$$

Consequently, the assumptions of Theorem 2.1 are valid, so we obtain

$$\|y_N\|_{L^p(\mathcal{M})} = \|\widetilde{y}_N\|_{L^p(\widetilde{\mathcal{M}})} \leq c_p \|\widetilde{x}_N\|_{L^p(\widetilde{\mathcal{M}})} = 2^{1/p} c_p \|x_N\|_{L^p(\mathcal{M})}. \qquad \square$$

Group von Neumann algebras and crossed products. Consider a discrete group (G, e) with the left regular representation $\lambda : G \to \mathcal{B}(\ell_2(G))$ given by $\lambda(g)\delta_h = \delta_{gh}$. Here $(\delta_g)_{g \in G}$ is the canonical unit vector basis of $\ell_2(G)$. Then the group von Neumann algebra $VN(G)$ is the von Neumann subalgebra of $\mathcal{B}(\ell_2(G))$ generated by $\{\lambda(g) : g \in G\}$. There is a standard trace $\tau = \tau_G$ on $VN(G)$, which is determined by $\tau(\lambda(g)) = 1$ if $g = e$ and $\tau(\lambda(g)) = 0$ otherwise. Any operator $f \in VN(G)$ has a Fourier series expansion of the form

$$f = \sum_{g \in G} \hat{f}(g)\lambda(g)$$

and we have $\tau(f) = \hat{f}(e)$. Any f with a finite number of nonzero Fourier coefficients will be called a trigonometric polynomial. For any $0 < p \leq \infty$, the associated noncommutative space $L^p(VN(G), \tau)$ will be denoted by $\mathcal{L}^p(G)$.

In our investigation below, we will assume that G is equipped with a differential structure, expressed in terms of the so-called conditionally negative length function $\psi : G \to \mathbb{R}_+$. Such a function satisfies the following properties: $\psi(e) = 0$, $\psi(g) = \psi(g^{-1})$ and $\sum_{g,h} \overline{\beta}_g \beta_h \psi(g^{-1}h) \leq 0$ for any sequence $(\beta_g)_{g \in G}$ of complex numbers summing up to zero. It is well-known (see [28], for example) that there is a one-to-one correspondence between conditionally negative length functions ψ and cocycles $(\mathcal{H}_\psi, \alpha_\psi, b_\psi)$, where $\mathcal{H}_\psi$ is a Hilbert space, $\alpha_\psi : G \to \mathrm{Aut}(\mathcal{H}_\psi)$ is an isometric action of G on $\mathcal{H}_\psi$ and $b_\psi : G \to \mathcal{H}_\psi$ is a mapping which satisfies the cocycle law

$$b_\psi(gh) = \alpha_{\psi,g}(b_\psi(h)) + b_\psi(g). \tag{2.2}$$

Let us specify our further considerations to the case of $G = \mathbb{F}_q$, the free group with q generators $g_1, g_2, \ldots$. Haagerup showed in [8] that the length function $\psi = |\cdot|$, i.e., the distance from g to

e in the Cayley graph of $\mathbb{F}_q$, is conditionally negative, and the associated cocycle $(\mathcal{H}_{|\cdot|}, \alpha_{|\cdot|}, b_{|\cdot|})$ can be described as follows. Let Π_0 be the class of trigonometric polynomials in $VN(\mathbb{F}_q)$, whose Fourier coefficients sum up to zero. Then the space $\mathcal{H}_{|\cdot|}$ is the closure of the pre-Hilbert space defined in Π_0 by means of the inner product

$$\langle f_1, f_2\rangle_{\mathcal{H}_{|\cdot|}} = -\frac{1}{2}\sum_{g,h\in\mathbb{F}_q} \overline{\hat{f}_1(g)}\hat{f}_2(h)|g^{-1}h|.$$

In addition, the action $\alpha_{|\cdot|,g}$ is $\lambda(g)$ and the cocycle mapping is defined as $b_{|\cdot|}(g) = \lambda(e) - \lambda(g)$. Since the length function is fixed, we will skip the index $|\cdot|$ and denote the cocycle by $(\mathcal{H}, \alpha, b)$.

To link the context of free Hilbert transforms with the theory of noncommutative martingales, we need to recall another operation on von Neumann algebras. Given a noncommutative measure space $(\mathcal{M}, \tau)$ with $\mathcal{M} \subset \mathcal{B}(\mathcal{H})$, suppose that there is a trace-preserving action $\alpha : G \to \mathrm{Aut}(\mathcal{M})$. We define the crossed product algebra $\mathcal{M} \rtimes_\alpha G$ as the weak operator closure of the $*$-algebra generated by $I_M \otimes \lambda(G)$ and $\rho(\mathcal{M})$ in $\mathcal{B}(\ell_2(G;\mathcal{H}))$. Here the $*$-representation $\rho : \mathcal{M} \to \mathcal{B}(\ell_2(G;\mathcal{H}))$ is defined by $\rho(f) = \sum_{g\in G} \alpha_{g^{-1}}(f) \otimes e_{g,g}$, where $e_{g,h}$ stands for the matrix unit in $\ell_2(G)$. A generic element of $\mathcal{M} \rtimes_\alpha G$ is of the form $\sum_{g\in G} f_g \rtimes_\alpha \lambda(g)$, where $f_g \in \mathcal{M}$, and we have

(i) $(f_g \rtimes_\alpha \lambda(g))^* = \alpha_{g^{-1}}(f_g^*) \rtimes_\alpha \lambda(g^{-1})$;
(ii) $(f_g \rtimes_\alpha \lambda(g))(f_h \rtimes_\alpha \lambda(h)) = f_g\alpha_g(f_h) \rtimes_\alpha \lambda(gh)$.

It is easy to see that $\mathcal{M} \rtimes_\alpha G$ forms a von Neumann subalgebra of $\mathcal{M}\bar{\otimes}\mathcal{B}(\ell^2(G))$. Furthermore, there is a canonical trace ν on $\mathcal{M}\rtimes_\alpha G$, uniquely determined by the requirement $\nu(f_g\rtimes_\alpha\lambda(g)) = \tau(f_g)\delta_{g=e}$ for all $f_g \in \mathcal{M}$ and $g \in G$. For the more detailed description of the construction, see [29].

Cylindrical Brownian motion and an action on $L^2(\Omega, \mathcal{F}, \mathbb{P})$. In our considerations below, we will need certain elements of the classical probability theory. We start with the notion of a cylindrical Brownian motion (see e.g. [7]) which, in a sense, is a Brownian motion taking values in an infinite-dimensional Hilbert space. The precise definition is the following.

Definition 2.3. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a classical probability space, let $\mathcal{H}$ be a separable Hilbert space and let $T > 0$ be a time horizon. A cylindrical $\mathcal{H}$-Brownian motion $\mathbb{X} = (\mathbb{X}_t)_{t\in[0,T]}$ is a family of bounded linear maps from $\mathcal{H}$ into $L^2(\Omega, \mathcal{F}, \mathbb{P})$ which satisfies the following two conditions:

(i) For all $e \in \mathcal{H}$ with $\|e\| = 1$, the process $(\mathbb{X}_t e)_{t\in[0,T]}$ is a standard Brownian motion in $\mathbb{R}$.
(ii) For all s, $t \in [0, T]$ and e, $e' \in \mathcal{H}$, we have $\mathbb{E}(\mathbb{X}_s e \cdot \mathbb{X}_t e') = (s \wedge t)\langle e, e'\rangle_{\mathcal{H}}$.

Next, given $y > 0$, let $\mathbb{Y} = (\mathbb{Y}_t)_{t\geq 0}$ be a one-dimensional Brownian motion independent of $\mathbb{X}$, started at y. Consider the stopping time

$$\eta = \inf\{t \geq 0 : \mathbb{Y}_t = 0\}. \tag{2.3}$$

Below, we will often work with the corresponding stopped versions $\mathbb{X}^\eta$, $\mathbb{Y}^\eta$ of $\mathbb{X}$ and $\mathbb{Y}$, defined by $\mathbb{X}^\eta_t e = \mathbb{X}_{\eta\wedge t} e$ and $\mathbb{Y}^\eta_t = \mathbb{Y}_{\eta\wedge t}$. For $t \in [0, T]$, let $\mathcal{F}_t = \mathcal{F}_t^{\mathbb{X},\mathbb{Y},\eta}$ stand for the σ-field generated by all the events of the form $\{\omega \in \Omega : (\mathbb{X}^\eta_s e)(\omega) \in B_1, \mathbb{Y}^\eta_u(\omega) \in B_2\}$ for some s, $u \in [0, t]$, $e \in \mathcal{H}$ and some Borel subsets B_1, B_2 of $\mathbb{R}$. So, in the language of classical probability, $(\mathcal{F}_t)_{t\in[0,T]}$ is the natural filtration of the stopped process $(\mathbb{X}^\eta, \mathbb{Y}^\eta)$. For technical reasons, we will assume that $\mathcal{F}_0$ is complete, i.e., contains all the events of probability 0 and all their subsets. The family $(\mathcal{F}_t)_{t\in[0,T]}$ gives rise to the corresponding filtration $(\mathcal{N}_t)_{t\in[0,T]}$, where $\mathcal{N}_t = L^\infty(\Omega, \mathcal{F}_t, \mathbb{P})$

is the von Neumann algebra of all bounded $\mathcal{F}_t$-measurable random variables. This algebra is equipped with the standard expectation with respect to $\mathbb{P}$ as the trace.

Suppose that α is an orthogonal representation of a group G over a real Hilbert space $\mathcal{H}$ and $t \in [0,T]$ is a fixed time parameter. Then α induces the expectation-preserving action on $L^2(\Omega, \mathcal{F}_T, \mathbb{P})$ by the following procedure. Given a random variable Z from the latter space, there exists a sequence $(Z_n)_{n\geq 0} \subset L^2(\Omega, \mathcal{F}_T, \mathbb{P})$ converging to Z, such that for each n,

$$Z_n = \varphi_n\big(\mathbb{X}^\eta_{t_1} e_1, \mathbb{X}^\eta_{t_2} e_2, \ldots, \mathbb{X}^\eta_{t_n} e_n, \mathbb{Y}^\eta_{u_1}, \mathbb{Y}^\eta_{u_2}, \ldots, \mathbb{Y}^\eta_{u_n}\big)$$

for some vectors $e_1, e_2, \ldots, e_n \in \mathcal{H}$, $t_1, t_2, \ldots, t_n$, $u_1, u_2, \ldots, u_n \in [0,T]$ and some Borel function $\varphi_n : \mathbb{R}^{2n} \to \mathbb{R}$. On such 'simple' functions Z_n, we consider the action $\tilde{\alpha}$ given by

$$\tilde{\alpha}_g Z_n = \varphi_n\big(\mathbb{X}^\eta_{t_1} \alpha_g(e_1), \mathbb{X}^\eta_{t_2} \alpha_g(e_2), \ldots, \mathbb{X}^\eta_{t_n} \alpha_g(e_n), \mathbb{Y}^\eta_{u_1}, \mathbb{Y}^\eta_{u_2}, \ldots, \mathbb{Y}^\eta_{u_n}\big).$$

Since α_g is an isometry and $\mathbb{X}$, $\mathbb{Y}$ are independent, we see that for each n and m the distributions of the pairs (Z_n, Z_m) and $(\tilde{\alpha}_g Z_n, \tilde{\alpha}_g Z_m)$ are the same. Thus, in particular, $(\tilde{\alpha}_g Z_n)_{n\geq 0}$ is also L^2-convergent and we define $\tilde{\alpha}_g Z$ to be the corresponding limit. It is straightforward to check that this gives a well-defined action on $L^2(\Omega, \mathcal{F}_T, \mathbb{P})$ which is invariant on $L^2(\Omega, \mathcal{F}^{\mathbb{Y}}_T, \mathbb{P})$, the class of square-integrable random variables measurable with respect to $\mathbb{Y}^\eta$. Furthermore, since the distributions of Z_n and $\tilde{\alpha}_g Z_n$ coincide for each n, the same is true for the pair Z, $\tilde{\alpha}_g Z$ and hence in particular $\mathbb{E}Z = \mathbb{E}\tilde{\alpha}_g Z$, i.e., the action preserves the expectation. It is also evident that for any $t \geq 0$, the subspace $L^2(\Omega, \mathcal{F}_t, \mathbb{P})$ is invariant under $\tilde{\alpha}$. Note that $\mathcal{N}_t \subset L^2(\Omega, \mathcal{F}_t, \mathbb{P})$, since $\mathbb{P}$ is a probability measure. Since $\tilde{\alpha}_g$ is distribution-preserving, as we observed above, we see that $\tilde{\alpha}$ can be restricted to the action on the von Neumann algebras $\mathcal{N}_t$, $t \in [0,T]$, and hence in particular one might speak of the crossed product algebras $\mathcal{N}_t \rtimes_{\tilde{\alpha}} G$ for each t.

3. L^p bounds for the free Hilbert transform

We return to the context of the free group $\mathbb{F}_q$ equipped with the length function $\psi(g) = |g|$ and the associated cocycle $(\mathcal{H}, \alpha, b)$. As before, $\mathbb{X}$ is the $\mathcal{H}$-cylindrical Brownian motion, $\mathbb{Y}$ is a one-dimensional Brownian motion independent from $\mathbb{X}$ and

$$\mathcal{N}_t = L^\infty\Big(\Omega, \mathcal{F}^{\mathbb{X},\mathbb{Y},\eta}_t, \mathbb{P}\Big) \rtimes_{\tilde{\alpha}} \mathbb{F}_q, \qquad t \in [0,T],$$

is the increasing family of von Neumann subalgebras of $\mathcal{N} = L^\infty\big(\Omega, \mathcal{F}^{\mathbb{X},\mathbb{Y},\eta}, \mathbb{P}\big) \rtimes_{\tilde{\alpha}} \mathbb{F}_q$, introduced above. Let f be an arbitrary trigonometric polynomial from $VN(\mathbb{F}_q)$. Consider the process $\xi = \xi^f = (\xi_t)_{t\in[0,T]}$ given by

$$\xi_t = \sum_{g\in\mathbb{F}_q} \varepsilon(g) \hat{f}(g) \exp(i\mathbb{X}^\eta_t b(g) - |g|^{1/2} \mathbb{Y}^\eta_t) \rtimes_{\tilde{\alpha}} \lambda(g), \qquad t \in [0,T],$$

where ε is the sign function associated with the Hilbert transform $\mathbb{H}$. For the sake of brevity, let us introduce the (commutative) exponential process

$$(\Lambda^g_t)_{t\in[0,T]} = (\exp(i\mathbb{X}^\eta_t b(g) - |g|^{1/2} \mathbb{Y}^\eta_t))_{t\in[0,T]}, \qquad g \in \mathbb{F}_q.$$

This process is a martingale with respect to $(\mathcal{F}_t)_{t\in[0,T]}$, which follows immediately from Itô's formula. Indeed, if we write $\Lambda^g_t = F(\mathbb{X}^\eta_t b(g), |g|^{1/2}\mathbb{Y}^\eta_t)$ with $F(x,y) = \exp(ix - y)$, then

$$F(\mathbb{X}^\eta_t b(g), |g|^{1/2}\mathbb{Y}^\eta_t) = F(\mathbb{X}^\eta_0 b(g), |g|^{1/2}\mathbb{Y}^\eta_0) + I_1 + I_2,$$

where

$$I_1 = \int_0^t F_x(\mathbb{X}_s^\eta b(g), |g|^{1/2}\mathbb{Y}_s^\eta)\mathrm{d}\mathbb{X}_s^\eta b(g) + \int_0^t F_y(\mathbb{X}_s^\eta b(g), |g|^{1/2}\mathbb{Y}_s^\eta)\mathrm{d}(|g|^{1/2}\mathbb{Y}_s^\eta)$$
$$= \int_0^t \Lambda_s^g \mathrm{d}\big(i\mathbb{X}_s^\eta b(g) - |g|^{1/2}\mathbb{Y}_s^\eta\big)$$

is the martingale part (a stochastic integral with respect to a martingale) and

$$I_2 = \frac{1}{2}\bigg[\int_0^t F_{xx}(\mathbb{X}_s^\eta b(g), |g|^{1/2}\mathbb{Y}_s^\eta)\mathrm{d}[\mathbb{X}^\eta b(g)]_s + 2\int_0^t F_{xy}(\mathbb{X}_s^\eta b(g), |g|^{1/2}\mathbb{Y}_s^\eta)\mathrm{d}[\mathbb{X}^\eta b(g), |g|^{1/2}\mathbb{Y}^\eta]_s$$
$$+ \int_0^t F_{yy}(\mathbb{X}_s^\eta b(g), |g|^{1/2}\mathbb{Y}_s^\eta)\mathrm{d}[|g|^{1/2}\mathbb{Y}^\eta]_s\bigg]$$

is the finite finite-variation part. But $\mathbb{X}^\eta b(g)$ and $|g|^{1/2}\mathbb{Y}^\eta$ are independent, so their square bracket is zero and the middle term in I_2 vanishes. Furthermore, both these processes are constant multiples of a (stopped) Brownian motion, so if we compute the partial derivatives, we obtain that the finite variation part I_2 is actually equal to

$$-\frac{1}{2}\int_0^t \Lambda_s^g \mathrm{d}[\mathbb{X}^\eta b(g)]_s + \frac{1}{2}\int_0^t \Lambda_s^g \mathrm{d}[|g|^{1/2}\mathbb{Y}^\eta]_s = -\frac{1}{2}\int_0^{\eta\wedge t} \Lambda_s^g\big(\|b(g)\|_{\mathcal{H}}^2 - |g|\big)\mathrm{d}s = 0.$$

Thus in particular we get the following alternative definition of ξ:

$$\xi_t = \xi_0 + \sum_{g\in\mathbb{F}_q} \varepsilon(g)\hat{f}(g)\int_0^t \Lambda_s^g \mathrm{d}\big(i\mathbb{X}_s^\eta b(g) - |g|^{1/2}\mathbb{Y}_s^\eta\big) \rtimes_{\tilde{\alpha}} \lambda(g)$$

for $t \geq 0$. Consider another process $\zeta = \zeta^f = (\zeta_t)_{t\in[0,T]}$, which is defined by a similar stochastic integral:

$$\zeta_t = \sum_{g\in\mathbb{F}_q} \hat{f}(g)\int_0^t \Lambda_s^g \mathrm{d}\big(i\mathbb{X}_s^\eta b(g)\big) \rtimes_{\tilde{\alpha}} \lambda(g).$$

Remark 3.1. Note that in comparison to the above formula for ξ, the term $-|g|^{1/2}\mathbb{Y}_s^\eta$ was removed from the integrator in ζ. If we kept this term, then an appropriate domination of ζ by ξ (which will be shown in a moment), would not be true. We will comment on this later.

At some places, we will assume additionally that f is self-adjoint: we have $\overline{\hat{f}(g)} = \hat{f}(g^{-1})$ for each $g \in \mathbb{F}_q$. In such a case, by the form of the adjoint operation and the identities $\alpha_{g^{-1}}b(g) = -b(g^{-1})$, $\tilde{\alpha}_{g^{-1}}(\overline{\Lambda_s^g}) = \Lambda_s^{g^{-1}}$, we obtain

$$\zeta_t^* = \sum_{g\in\mathbb{F}_q} \overline{\hat{f}(g)}\alpha_{g^{-1}}\left(\int_0^t \overline{\Lambda_s^g}\mathrm{d}\big(-i\mathbb{X}_s^\eta b(g)\big)\right) \rtimes_{\tilde{\alpha}} \lambda(g^{-1})$$
$$= \sum_{g\in\mathbb{F}_q} \hat{f}(g^{-1})\int_0^t \Lambda_s^{g^{-1}}\mathrm{d}\big(i\mathbb{X}_s^\eta b(g^{-1})\big) \rtimes_{\tilde{\alpha}} \lambda(g^{-1}) = \zeta_t,$$

i.e., ζ is also self-adjoint. Using the good-λ approach, we will prove the following L^p estimate between ξ and ζ.

Theorem 3.2. *Suppose that f is self-adjoint. Then for any $2 \leq p < \infty$, we have*

$$\|\zeta_T\|_{L^p(\mathcal{N})} \leq c_p\|\xi_T\|_{L^p(\mathcal{N})}, \tag{3.1}$$

for some universal constant c_p of order $O(p)$ as $p \to \infty$.

Proof. For $p = 2$ the claim is trivial (the estimate holds with the constant 1), so from now on, we assume that $p > 2$. Fix an arbitrary partition $0 = t_0 < t_1 < t_2 < \ldots < t_N = T$ and consider the discretized versions of ξ and ζ, given by $x_n = \xi_{t_n}$ and $y_n = \zeta_{t_n}$, $n = 0, 1, 2, \ldots, N$. Both these sequences are martingales with respect to the filtration $(\mathcal{N}_{t_n})_{n=0}^N$, since so are both stochastic integrals appearing in the definitions of ξ and ζ. We will check the assumptions of Lemma 2.2. First, note that y is self-adjoint, because the same is true for ζ. We have $dy_0 = 0$, so both estimates involving dx_0 and dy_0 are satisfied. Next, fix $1 \leq n \leq N$ and note that

$$dx_n = \sum_{g \in \mathbb{F}_q} \varepsilon(g) \hat{f}(g) \int_{t_{n-1}}^{t_n} \Lambda_s^g \mathrm{d}\big(i\mathbb{X}_s^\eta b(g) - |g|^{1/2} \mathbb{Y}_s^\eta\big) \rtimes_{\tilde{\alpha}} \lambda(g).$$

As we have already observed above, we have the identity $\tilde{\alpha}_{g^{-1}}(\overline{\Lambda_s^g}) = \Lambda_s^{g^{-1}}$ for each $g \in \mathbb{F}_q$. So, by the form of the adjoint operation in crossed product algebras,

$$\begin{aligned}\left(\int_{t_{n-1}}^{t_n} \Lambda_s^g \mathrm{d}\big(i\mathbb{X}_s^\eta b(g) - |g|^{1/2}\mathbb{Y}_s^\eta\big) \rtimes_{\tilde{\alpha}} \lambda(g)\right)^* &= \int_{t_{n-1}}^{t_n} \Lambda_s^{g^{-1}} \mathrm{d}\big(i\mathbb{X}_s^\eta b(g^{-1}) - |g|^{1/2}\mathbb{Y}_s^\eta\big) \rtimes_{\tilde{\alpha}} \lambda(g^{-1}) \\ &= \int_{t_{n-1}}^{t_n} \Lambda_s^{g^{-1}} \mathrm{d}\big(i\mathbb{X}_s^\eta b(g^{-1}) - |g^{-1}|^{1/2}\mathbb{Y}_s^\eta\big) \rtimes_{\tilde{\alpha}} \lambda(g^{-1}).\end{aligned}$$

Next, for any g, $h \in \mathbb{F}_q$ we have

$$\begin{aligned}&\left(\int_{t_{n-1}}^{t_n} \Lambda_s^{g^{-1}} \mathrm{d}\big(i\mathbb{X}_s^\eta b(g^{-1}) - |g^{-1}|^{1/2}\mathbb{Y}_s^\eta\big) \rtimes_{\tilde{\alpha}} \lambda(g^{-1})\right)\left(\int_{t_{n-1}}^{t_n} \Lambda_s^h \mathrm{d}\big(i\mathbb{X}_s^\eta b(h) - |h|^{1/2}\mathbb{Y}_s^\eta\big) \rtimes_{\tilde{\alpha}} \lambda(h)\right) \\ &= \left(\int_{t_{n-1}}^{t_n} \Lambda_s^{g^{-1}} \mathrm{d}\big(i\mathbb{X}_s^\eta b(g^{-1}) - |g^{-1}|^{1/2}\mathbb{Y}_s^\eta\big)\right) \tilde{\alpha}_{g^{-1}}\left(\int_{t_{n-1}}^{t_n} \Lambda_s^h \mathrm{d}\big(i\mathbb{X}_s^\eta b(h) - |h|^{1/2}\mathbb{Y}_s^\eta\big)\right) \rtimes_{\tilde{\alpha}} \lambda(g^{-1}h).\end{aligned}$$

Since $\tilde{\alpha}_{g^{-1}}\big(i\mathbb{X}_s^\eta b(h) - |h|^{1/2}\mathbb{Y}_s^\eta\big) = i\mathbb{X}_s^\eta \alpha_{g^{-1}} b(h) - |h|^{1/2}\mathbb{Y}_s^\eta$, we compute the classical conditional expectation

$$\begin{aligned}&\mathbb{E}_{\mathcal{F}_{t_{n-1}}}\left\{\left(\int_{t_{n-1}}^{t_n} \Lambda_s^{g^{-1}} \mathrm{d}\big(i\mathbb{X}_s^\eta b(g^{-1}) - |g^{-1}|^{1/2}\mathbb{Y}_s^\eta\big)\right) \tilde{\alpha}_{g^{-1}}\left(\int_{t_{n-1}}^{t_n} \Lambda_s^h \mathrm{d}\big(i\mathbb{X}_s^\eta b(h) - |h|^{1/2}\mathbb{Y}_s^\eta\big)\right)\right\} \\ &= \int_{t_{n-1}}^{t_n} \Lambda_s^{g^{-1}} \tilde{\alpha}_{g^{-1}} \Lambda_s^h \mathrm{d}\Big([i\mathbb{X}_s^\eta b(g^{-1}), i\mathbb{X}_s^\eta \alpha_{g^{-1}} b(h)] + |g^{-1}|^{1/2}|h|^{1/2}[\mathbb{Y}^\eta]_s\Big) \\ &= \Big(\langle b(g), b(h)\rangle_{\mathcal{H}} + |g^{-1}|^{1/2}|h|^{1/2}\Big) \int_{t_{n-1}}^{t_n} \Lambda_s^{g^{-1}} \tilde{\alpha}_{g^{-1}} \Lambda_s^h \mathrm{d}s.\end{aligned}$$

Consequently, we obtain

$$\begin{aligned}\mathcal{E}_{n-1}(|dx_n|^2) &= \sum_{g,h \in \mathbb{F}_q} \left(\varepsilon(g)\varepsilon(h)\overline{\hat{f}(g)}\hat{f}(h)\Big(\langle b(g), b(h)\rangle_{\mathcal{H}} + |g^{-1}|^{1/2}|h|^{1/2}\Big) \int_{t_{n-1}}^{t_n} \Lambda_s^{g^{-1}} \tilde{\alpha}_{g^{-1}} \Lambda_s^h \mathrm{d}s\right) \\ &\qquad \rtimes_{\tilde{\alpha}} \lambda(g^{-1}h) \\ &\geq \sum_{g,h \in \mathbb{F}_q} \left(\varepsilon(g)\varepsilon(h)\overline{\hat{f}(g)}\hat{f}(h)\langle b(g), b(h)\rangle_{\mathcal{H}} \int_{t_{n-1}}^{t_n} \Lambda_s^{g^{-1}} \tilde{\alpha}_{g^{-1}} \Lambda_s^h \mathrm{d}s\right) \rtimes_{\tilde{\alpha}} \lambda(g^{-1}h),\end{aligned}$$

where the latter bound follows from the estimate

$$\sum_{g,h\in\mathbb{F}_q}\left(\varepsilon(g)\varepsilon(h)\overline{\hat{f}(g)}\hat{f}(h)|g^{-1}|^{1/2}|h|^{1/2}\Lambda_s^{g^{-1}}\tilde{\alpha}_{g^{-1}}\Lambda_s^h\right)\rtimes_{\tilde{\alpha}}\lambda(g^{-1}h)$$
$$=\left|\sum_{g\in\mathbb{F}_q}\varepsilon(g)\hat{f}(g)|g|^{1/2}\Lambda_s^g\rtimes_{\tilde{\alpha}}\lambda(g)\right|^2\geq 0.$$

The above lower bound for $\mathcal{E}_{n-1}(|dx_n|^2)$ can be simplified a little. Note that if g and h (in reduced form) start with a different generator, then $\langle b(g),b(h)\rangle_{\mathcal{H}}=(|g|+|h|-|g^{-1}h|)/2=0$; otherwise, if g and h start with the same generator, then $\varepsilon(g)=\varepsilon(h)$. This yields

$$\mathcal{E}_{n-1}(|dx_n|^2)\geq\sum_{g,h\in\mathbb{F}_q}\overline{\hat{f}(g)}\hat{f}(h)\langle b(g),b(h)\rangle_{\mathcal{H}}\int_{t_{n-1}}^{t_n}\Lambda_s^{g^{-1}}\tilde{\alpha}_{g^{-1}}\Lambda_s^h\mathrm{d}s\rtimes_{\tilde{\alpha}}\lambda(g^{-1}h).$$

However, the repetition of the above calculation shows that the expression on the right is precisely $\mathcal{E}_{n-1}(dy_n^2)$ and thus the first assumption in (2.1) is satisfied. Here we refer the reader to the Remark 3.1 above: without the modification mentioned there, the domination $\mathcal{E}_{n-1}(dy_n^2)\leq\mathcal{E}_{n-1}(|dx_n|^2)$ would break down, since some additional summands would appear in $\mathcal{E}_{n-1}(dy_n^2)$.

Unfortunately, the second inequality in (2.1) does not seem to be true and we need to apply additional embedding and limiting arguments to overcome this difficulty. Consider the larger von Neumann algebra $\widetilde{\mathcal{N}}=\mathbb{M}_{N+2}\overline{\otimes}\mathcal{N}$, equipped with the tensor trace $\mathrm{Tr}\otimes\nu$ and the filtration $(\widetilde{\mathcal{N}}_n)_{n=0}^N=(\mathbb{M}_{N+2}\overline{\otimes}\mathcal{N}_{t_n})_{n=0}^N$. We embed the above differences dx_n, dy_n into this larger context as follows. Let $e_{j,k}$ stand for the matrix in $\mathbb{M}_{N+2}$ whose all entries are zero, except for a single 1 which stands at the intersection of j-th row and k-th column. For any $n=0,1,2,\ldots,N$, put

$$\widetilde{dx}_n=e_{1,1}\otimes dx_n+e_{n+2,n+2}\otimes dy_n\qquad\text{and}\qquad\widetilde{dy}_n=e_{1,1}\otimes dy_n.$$

Note that the operator dy_n appears in the definition of $\widetilde{dx}_n$. Both $\widetilde{dx}$ and $\widetilde{dy}$ are martingale differences, furthermore, the differences $\widetilde{dy}_n$ are self-adjoint. Moreover, by the above calculations, we immediately obtain $\widetilde{\mathcal{E}}_{n-1}\widetilde{dy}_n^2\leq\widetilde{\mathcal{E}}_{n-1}|\widetilde{dx}_n|^2$ and

$$\|\widetilde{dy}_n\|_{L^p(\widetilde{\mathcal{N}})}=\|dy_n\|_{L^p(\mathcal{N})}\leq\|\widetilde{dx}_n\|_{L^p(\widetilde{\mathcal{N}})}.$$

That is, $\widetilde{dx}$ and $\widetilde{dy}$ satisfy the assumptions of Lemma 2.2 and hence we get

$$\|y_N\|_{L^p(\mathcal{N})}=\|\widetilde{y}_N\|_{L^p(\widetilde{\mathcal{N}})}\leq C_p\|\widetilde{x}_N\|_{L^p(\widetilde{\mathcal{N}})}=C_p\left(\|x_N\|_{L^p(\mathcal{N})}^p+\sum_{n=0}^N\|dy_n\|_{L^p(\mathcal{N})}^p\right)^{1/p}.\tag{3.2}$$

Now, the process $(\zeta_t)_{t\geq0}$ has continuous paths, so the assumption $p>2$ implies

$$\sum_{n=0}^N\|dy_n\|_{L^p(\mathcal{N})}^p=\nu\left(\sum_{n=1}^N|\zeta_{t_n}-\zeta_{t_{n-1}}|^p\right)\to0\tag{3.3}$$

as $N\to\infty$ and the mesh of the partition $(t_n)_{n=0}^N$ tends to zero. Indeed, since f is a trigonometric polynomial, we have

$$\nu\left(\sum_{n=1}^N|\zeta_{t_n}-\zeta_{t_{n-1}}|^p\right)\leq C_f\sum_{g\in\mathbb{F}_q}|\hat{f}(g)|^p\mathbb{E}\left\{\sum_{n=1}^N\left|\int_{t_{n-1}}^{t_n}\Lambda_s^g\mathrm{d}(i\mathbb{X}_s^\eta b(g))\right|^p\right\},$$

where C_f depends on the number of nonzero terms in f. But for any $g \in \mathbb{F}_q$, the process $\left(\int_0^t \Lambda_s^g \mathrm{d}(i\mathbb{X}_s^\eta b(g))\right)_{t\geq 0}$ is a commutative continuous-path martingale, so

$$\mathbb{E}\left\{\sum_{n=1}^N \left|\int_{t_{n-1}}^{t_n} \Lambda_s^g \mathrm{d}(i\mathbb{X}_s^\eta b(g))\right|^p\right\} \to 0,$$

by elementary properties of stochastic integrals. Thus (3.3) follows and hence (3.2) yields the desired estimate: it remains to note that $y_N = \zeta_T$ and $x_N = \xi_T$. □

The next lemma provides a connection between the Hilbert transform and the martingale transformation $\xi \to \zeta$ studied above. It will be convenient to indicate the dependence of the processes on the underlying functions: we will write ξ^f, ζ^f, etc.

Lemma 3.3. *For any $f, \varphi \in VN(\mathbb{F}_q)$ of a finite Fourier expansion, we have*

$$\tau(f\mathbb{H}\varphi) = -2i \lim_{y\to\infty} \lim_{T\to\infty} \nu(\zeta_T^f \xi_T^\varphi). \tag{3.4}$$

Proof. By linearity, it is enough to check the equality for $f = \lambda(h)$ and $\varphi = \lambda(g)$, for some $g, h \in \mathbb{F}_q$. Moreover, from the very definition of trace and multiplication in the crossed product, we only have to consider the case $gh = e$, otherwise both sides in (3.4) are equal to zero. Directly from (1.2), we check that $\tau(f\mathbb{H}\varphi) = -i\varepsilon(g)$. As for the right-hand side of (3.4), recall that

$$\zeta_T^f = \int_0^T \Lambda_s^h \mathrm{d}(i\mathbb{X}_s^\eta b(h)) \rtimes_{\tilde{\alpha}} \lambda(h) = \int_0^T \Lambda_s^{g^{-1}} \mathrm{d}(i\mathbb{X}_s^\eta b(g^{-1})) \rtimes_{\tilde{\alpha}} \lambda(g^{-1}),$$

$$\xi_T^\varphi = \xi_0^\varphi + \varepsilon(g) \int_0^T \Lambda_s^g \mathrm{d}(i\mathbb{X}_s^\eta b(g) - |g|^{1/2}\mathbb{Y}_s^\eta) \rtimes_{\tilde{\alpha}} \lambda(g).$$

Since

$$\mathbb{E}\left(\exp(-|g|^{1/2}y) \int_0^T \Lambda_s^{g^{-1}} \mathrm{d}(i\mathbb{X}_s b(g^{-1}))\right) = 0,$$

we check that

$$\begin{aligned}
\nu(\zeta_T^f \xi_T^\varphi) &= \varepsilon(g)\mathbb{E}\left\{\left(\int_0^T \Lambda_s^g \mathrm{d}(i\mathbb{X}_s^\eta b(g) - |g|^{1/2}\mathbb{Y}_s^\eta)\right) \tilde{\alpha}_g \left(\int_0^T \Lambda_s^{g^{-1}} \mathrm{d}(i\mathbb{X}_s^\eta b(g^{-1}))\right)\right\} \\
&= \varepsilon(g)\mathbb{E}\int_0^T \Lambda_s^g \tilde{\alpha}_g \Lambda_s^{g^{-1}} \|b(g)\|_{\mathcal{H}}^2 \mathrm{d}s \\
&= \varepsilon(g)\mathbb{E}\int_0^T |\Lambda_s^g|^2 |g| \mathrm{d}s \xrightarrow{T\to\infty} \varepsilon(g)|g|\mathbb{E}\int_0^\infty |\Lambda_s^g|^2 \mathrm{d}s,
\end{aligned}$$

by Lebesgue's monotone convergence theorem. To compute the latter integral, we rewrite it in the form

$$\int_0^\infty \mathbb{E}\exp\big(-2|g|^{1/2}(y - Y_s)\big) 1_{\{s<\mu\}} \mathrm{d}s, \tag{3.5}$$

where $Y = (Y_t)_{t\geq 0}$ is the standard one-dimensional Brownian motion started at zero and $\mu = \inf\{t : Y_t \geq y\}$. Now, we will exploit some elementary facts from stochastic analysis. First, note the obvious equality $\{s < \mu\} = \{\max_{0\leq u\leq s} Y_u < y\}$. Second, for each s, the density of the pair $(Y_s, \max_{0\leq u\leq s} Y_u)$ is given by (cf. Revuz and Yor [26], p. 110)

$$d_s(a, b) = \left(\frac{2}{\pi s^3}\right)^{1/2} (2b - a) \exp\left(-\frac{(2b-a)^2}{2s}\right) 1_{\{a\leq b,\, b\geq 0\}}.$$

Consequently, the integral in (3.5) equals

$$\int_0^y \int_{-\infty}^b \int_0^\infty \exp(-2|g|^{1/2}(y-a)) d_s(a,b) \mathrm{d}s \mathrm{d}a \mathrm{d}b = 2\int_0^y \int_{-\infty}^b \exp(-2|g|^{1/2}(y-a)) \mathrm{d}a \mathrm{d}b$$
$$= \frac{1}{2|g|}\big[1-\exp(-2|g|^{1/2}y)\big].$$

Putting all the above observations together, we get the claim. □

In the next lemma, we compare the distributions of $\mathbb{H}f$ and ξ_T^f.

Lemma 3.4. *Suppose that $f \in VN(\mathbb{F}_q)$ has a finite Fourier expansion. Then we have $\lim_{T\to\infty} \|\xi_T^f\|_{L^p(\mathcal{N})} = \|\mathbb{H}f\|_{\mathcal{L}^p(\mathbb{F}_q)}$.*

Proof. We will show a stronger statement, which yields the claim by a straightforward approximation (both ξ_T^f and $\mathbb{H}f$ are bounded operators). Namely, we will prove that for any polynomial P in variables x, x^*, we have the identity

$$\lim_{T\to\infty} \nu(P(-i\xi_T^f)) = \tau(P(\mathbb{H}f)). \tag{3.6}$$

To this end, recall first that

$$\xi_T^f = \sum_{g\in\mathbb{F}_q} \varepsilon(g)\hat{f}(g)\exp(i\mathbb{X}_T^\eta b(g) - |g|^{1/2}\mathbb{Y}_T^\eta) \rtimes_{\tilde{\alpha}} \lambda(g).$$

If $T\to\infty$, then $\mathbb{Y}_T^\eta \to 0$ and $\mathbb{X}_T^\eta b(g) \to \mathbb{X}_\eta b(g)$. Thus, by Lebesgue's dominated convergence theorem, it is enough to show that

$$\nu\left(P\left(-i\sum_{g\in\mathbb{F}_q}\varepsilon(g)\hat{f}(g)J_g\right)\right) = \tau(P(\mathbb{H}f)),$$

where J_g is the generic element $J_g = \exp(i\mathbb{X}_\eta b(g)) \rtimes_{\tilde{\alpha}} \lambda(g)$. Directly from the cocycle property (2.2) and the identity $\alpha_{g^{-1}}b(g) = -b(g^{-1})$, we infer that

$$J_g^* = J_{g^{-1}}, \qquad J_gJ_h = J_{gh}. \tag{3.7}$$

By the direct expansion, $P\left(-i\sum_{g\in\mathbb{F}_q}\varepsilon(g)\hat{f}(g)J_g\right)$ is the finite sum of the terms of the form

$$\prod_{j=1}^k \left(-i\varepsilon(g_j)\hat{f}(g_j^{\kappa_j})J_{g_j^{\kappa_j}}\right),$$

where for $1\le j\le k$, $\kappa_j = \pm1$ indicates whether the corresponding term was conjugated or not. The trace of such a single term vanishes unless $\prod_{j=1}^k g_j^{\kappa_j} = e$ and then

$$\nu\left(\prod_{j=1}^k \left(-i\varepsilon(g_j)\hat{f}(g_j^{\kappa_j})J_{g_j^{\kappa_j}}\right)\right) = \prod_{j=1}^k \left(-i\varepsilon(g_j)\hat{f}(g_j^{\kappa_j})\right).$$

The right-hand side of (3.6) is handled similarly: we only need to replace ν with τ and J_g with $\lambda(g)$. Since the identities (3.7) hold for $(\lambda(g))_{g\in\mathbb{F}_q}$ as well, the arguments carry over. □

We are ready for the proof of L^p-boundedness of the Hilbert transform. The first step is the following.

Lemma 3.5. *For $2 \le p < \infty$ and any self-adjoint $f \in \mathcal{L}^p(\mathbb{F}_q)$ with $\hat{f}(e) = 0$, we have*

$$\|f\|_{\mathcal{L}^p(\mathbb{F}_q)} \le 4c_p \|\mathbb{H}f\|_{\mathcal{L}^p(\mathbb{F}_q)},$$

where c_p is the constant appearing in (3.1).

This result is very close to the desired L^p bound for $\mathbb{H}$. Indeed, replacing f with $\mathbb{H}f$ we obtain (1.3), but under the additional assumption that $\mathbb{H}f$ is self-adjoint. Unfortunately, it seems that this assumption cannot be dropped by any decomposition arguments: the Hilbert transform is not symmetric. To handle this problem, we need some additional effort: we will exploit Cotlar's identity (1.4).

Proof of Lemma 3.5. Pick arbitrary trigonometric polynomials f and $\varphi \in VN(\mathbb{F}_q)$ such that f is self-adjoint and $\hat{f}(e) = 0$. By (3.4), Hölder's inequality and (3.1), we have

$$\begin{aligned} |\tau(f\mathbb{H}\varphi)| = 2\left|\lim_{y\to\infty}\lim_{T\to\infty}\nu(\zeta_T^f\xi_T^\varphi)\right| &\le 2\limsup_{y\to\infty}\limsup_{T\to\infty}\|\zeta_T^f\|_{L^p(\mathcal{N})}\|\xi_T^\varphi\|_{L^{p'}(\mathcal{N})} \\ &\le 2\limsup_{y\to\infty}\limsup_{T\to\infty} c_p\|\xi_T^f\|_{L^p(\mathcal{N})}\|\xi_T^\varphi\|_{L^{p'}(\mathcal{N})} \\ &= 2c_p\|\mathbb{H}f\|_{\mathcal{L}^p(\mathbb{F}_q)}\|\mathbb{H}\varphi\|_{\mathcal{L}^{p'}(\mathbb{F}_q)}, \end{aligned}$$

where the last passage follows from Lemma 3.4. Now replace φ with $\mathbb{H}\varphi$ and observe that $\mathbb{H}^2\varphi = \hat{\varphi}(e)\lambda(e) - \varphi$. Since $\hat{f}(e) = 0$, we obtain

$$\begin{aligned} |\tau(f\varphi)| = \big|\tau\big(f(\hat{\varphi}(e)\lambda(e) - \varphi\big)\big| &\le 2c_p\|\mathbb{H}f\|_{\mathcal{L}^p(\mathbb{F}_q)}\|\hat{\varphi}(e)\lambda(e) - \varphi\|_{\mathcal{L}^{p'}(\mathbb{F}_q)} \\ &\le 4c_p\|\mathbb{H}f\|_{\mathcal{L}^p(\mathbb{F}_q)}\|\varphi\|_{\mathcal{L}^{p'}(\mathbb{F}_q)}. \end{aligned}$$

Here in the last passage we have used Hölder's inequality again:

$$|\hat{\varphi}(e)| = |\tau(\varphi\lambda(e))| \le \|\varphi\|_{\mathcal{L}^p(\mathbb{F}_q)}\|\lambda(e)\|_{\mathcal{L}^{p'}(\mathbb{F}_q)} = \|\varphi\|_{\mathcal{L}^p(\mathbb{F}_q)}.$$

Thus, by duality, we get the claim for polynomials; the passage to general (self-adjoint) f follows by approximation. □

Proof of Theorem 1.1. For any $p \ge 2$, let $\alpha_p = \|\mathbb{H}\|_{\mathcal{L}^p(\mathbb{F}_q)\to\mathcal{L}^p(\mathbb{F}_q)} = \|\mathbb{H}^{op}\|_{\mathcal{L}^p(\mathbb{F}_q)\to\mathcal{L}^p(\mathbb{F}_q)}$. Pick an arbitrary $f \in \mathcal{L}^{2p}(\mathbb{F}_q)$ and apply Cotlar's identity (1.4) with $f = g$ to obtain

$$\begin{aligned} &\||\mathbb{H}^{op}f|^2\|_{\mathcal{L}^p(\mathbb{F}_q)} \\ &\le \|\mathbb{H}^{op}f\|^2_{\mathcal{L}^2(\mathbb{F}_q)} + \|\mathbb{H}(f^*\mathbb{H}^{op}f)\|_{\mathcal{L}^p(\mathbb{F}_q)} + \|\mathbb{H}^{op}(\mathbb{H}(f^*)f)\|_{\mathcal{L}^p(\mathbb{F}_q)} + \|\mathbb{H}^{op}\mathbb{H}(|f|^2)\|_{\mathcal{L}^p(\mathbb{F}_q)}. \end{aligned} \tag{3.8}$$

Note that $\|\mathbb{H}^{op}f\|^2_{\mathcal{L}^2(\mathbb{F}_q)} \le \|f\|^2_{\mathcal{L}^2(\mathbb{F}_q)} \le \|f\|^2_{\mathcal{L}^p(\mathbb{F}_q)}$, since the trace in $VN(\mathbb{F}_q)$ is normalized. Furthermore, by the definition of α_p and Schwarz' inequality, we get

$$\|\mathbb{H}(f^*\mathbb{H}^{op}f)\|_{\mathcal{L}^p(\mathbb{F}_q)} \le \alpha_p\|f^*\mathbb{H}^{op}f\|_{\mathcal{L}^p(\mathbb{F}_q)} \le \alpha_p\|f\|_{\mathcal{L}^{2p}(\mathbb{F}_q)}\|\mathbb{H}^{op}f\|_{\mathcal{L}^{2p}(\mathbb{F}_q)} \le \alpha_p\alpha_{2p}\|f\|^2_{\mathcal{L}^{2p}(\mathbb{F}_q)}$$

and similarly, $\|\mathbb{H}^{op}(\mathbb{H}(f^*)f)\|_{\mathcal{L}^p(\mathbb{F}_q)} \le \alpha_p\alpha_{2p}\|f\|^2_{\mathcal{L}^{2p}(\mathbb{F}_q)}$. Now we will apply Lemma 3.5 to the last term on the right of (3.8). Note that $|f|^2$, and hence also $\mathbb{H}^{op}\mathbb{H}(|f|^2)$, is self-adjoint. Consequently,

$$\|\mathbb{H}^{op}\mathbb{H}(|f|^2)\|_{\mathcal{L}^p(\mathbb{F}_q)} \le 4c_p\|\mathbb{H}^{op}(|f|^2)\|_{\mathcal{L}^p(\mathbb{F}_q)} \le 4\alpha_p c_p\|f\|^2_{\mathcal{L}^{2p}(\mathbb{F}_q)}.$$

Putting all the above facts together, we obtain

$$\|\mathbb{H}^{op}f\|^2_{\mathcal{L}^{2p}(\mathbb{F}_q)} \le (1 + 2\alpha_p\alpha_{2p} + 4\alpha_p c_p)\|f\|^2_{\mathcal{L}^{2p}(\mathbb{F}_q)},$$

which implies $\alpha_{2p} \leq \alpha_p + \sqrt{1+\alpha_p^2 + 4\alpha_p c_p}$. Now suppose that $p = 2^n$ for some positive integer n. Then we have $c_{2^n} \leq \beta 2^n$ for some universal constant β and the preceding estimate yields

$$\alpha_{2^{n+1}} \leq \alpha_{2^n} + \sqrt{1+\alpha_{2^n}^2 + 4\alpha_{2^n}\cdot\beta 2^n} \leq 2\alpha_{2^n} + 2\beta\cdot 2^n.$$

But $\alpha_2 = 1$, so by induction, we get $\alpha_{2^n} \leq \beta n 2^n$, as desired. The estimate for general $p \in [2,\infty)$ follows from interpolation, and the passage to $1 < p < 2$ is due to duality. □

Remark 3.6. As proved in [19], the free Hilbert transform is completely bounded on L^p for $1 < p < \infty$, that is, $\|\mathbb{H}\|_{\mathrm{cb}} := \sup_{N\geq 1} \|\mathrm{id}_{\mathbb{M}_N} \otimes \mathbb{H}\|_{L^p(\mathbb{M}_N \overline{\otimes} VN(\mathbb{F}_q)) \to L^p(\mathbb{M}_N \overline{\otimes} VN(\mathbb{F}_q))} < \infty$. Here $L^p(\mathbb{M}_N \overline{\otimes} VN(\mathbb{F}_q))$ denotes the L^p space on the tensor product algebra $\mathbb{M}_N \overline{\otimes} VN(\mathbb{F}_q)$ (cf. [25] for details). This property can be regarded as an extension of (1.3) to the context in which f is a matrix with entries belonging to $VN(\mathbb{F}_q)$, on which $\mathbb{H}$ acts entrywise. The approach used above works perfectly in this more general setting and yields the L^p constant of the same order as in (1.3). Let us describe this briefly, the arguments go along the same lines. Let $N \geq 1$ be a fixed integer, $T > 0$ be a given constant and ε be the sign function of the Hilbert transform $\mathbb{H}$. Fix an operator $f = \sum_{1\leq i,j\leq N} e_{i,j} \otimes f^{i,j}$, where each $f^{i,j}$ is a trigonometric polynomial in $VN(\mathbb{F}_q)$. Such an f gives rise to the martingales $(\xi_t)_{t\in[0,T]}$ and $(\zeta_t)_{t\in[0,T]}$ on $\mathbb{M}_N \overline{\otimes} \mathcal{N}$, given by

$$\xi_t = \sum_{i,j} e_{i,j} \otimes \left(\sum_{g\in\mathbb{F}_q} \varepsilon(g) \hat{f}^{i,j}(g) \exp(i\mathbb{X}_t^\eta b(g) - |g|^{1/2}\mathbb{Y}_t^\eta) \rtimes_{\tilde{\alpha}} \lambda(g) \right) = \sum_{i,j} e_{i,j} \otimes \xi_t^{f^{i,j}}$$

and $\zeta_t = \sum_{i,j} e_{i,j} \otimes \zeta_t^{f^{i,j}}$, where $\xi^{f^{i,j}}$ and $\zeta^{f^{i,j}}$ have been introduced above (recall that $\mathcal{N} = L^\infty\big(\Omega, \mathcal{F}^{\mathbb{X},\mathbb{Y},\eta}, \mathbb{P}\big) \rtimes_{\tilde{\alpha}} \mathbb{F}_q$). We define x_n and y_n by the same formula as previously, for a given partition of $[0,T]$, and check the conditions (2.1) for a self-adjoint f, using analogous calculations (there is an additional sum due to the matrix component, but no extra effort is needed to handle it). If we send the mesh of the partition to zero, we get $\|\zeta_T\|_{L^p(\mathbb{M}_N \overline{\otimes} \mathcal{N})} \leq c_p \|\xi_T\|_{L^p(\mathbb{M}_N \overline{\otimes} \mathcal{N})}$, which in turn, by letting $T \to \infty$, yields $\|f\|_{L^p(\mathbb{M}_N \overline{\otimes} VN(\mathbb{F}_q))} \leq 4c_p \|(\mathrm{id}_{\mathbb{M}_N} \otimes \mathbb{H})f\|_{L^p(\mathbb{M}_N \overline{\otimes} VN(\mathbb{F}_q))}$ for any self-adjoint f with $(\mathrm{Tr} \otimes \tau)(f) = 0$. Finally, we may apply the version of Cotlar's identity (1.4), in which $\mathbb{H}$ and $\mathbb{H}^{op}$ are replaced with $\mathrm{id}_{\mathbb{M}_N} \otimes \mathbb{H}$ and $\mathrm{id}_{\mathbb{M}_N} \otimes \mathbb{H}^{op}$, respectively, and f, g are matrix-valued operators as above. The repetition of the above proof of (1.3) yields the desired complete L^p-boundedness.

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