EXPLICIT COUNTEREXAMPLES TO THE WEAK MUCKENHOUPT-WHEEDEN CONJECTURE

ADAM OSĘKOWSKI

Abstract. We present an explicit construction of examples showing that the estimate $\|T^\epsilon\|_{L^1(w)\to L^{1,\infty}(w)} \lesssim [w]_{A_1}\log(1+[w]_{A_1})$ for Haar multipliers is sharp in terms of the characteristic $[w]_{A_1}$.

1. Introduction

Suppose that M is the Hardy-Littlewood maximal operator on $\mathbb{R}^d$, given by

$$Mf(x) = \sup \frac{1}{|Q|}\int_Q |f|\mathrm{d}y,$$

where the supremum is taken over all cubes Q containing x, which have their edges parallel to the axes. In 1971, Fefferman and Stein established the weak-type $(1,1)$ inequality

$$w\big(\{x\in\mathbb{R}^d : Mf(x)\geq 1\}\big) \leq C\int_{\mathbb{R}^d}|f|Mw\mathrm{d}x. \tag{1.1}$$

Here w is an arbitrary weight on $\mathbb{R}^d$ (i.e., a nonnegative, locally integrable function) and C is a constant depending only on d; furthermore, we use the standard notation $w(E)=\int_E w\mathrm{d}x$ for any Borel subset E of $\mathbb{R}^d$. The above inequality was exploited in [1] to establish tight estimates for vector-valued maximal operators, it also served as a motivation for the development of the weighted theory. A few years later, Muckenhoupt and Wheeden conjectured that the estimate (1.1) remains valid if one replaces the maximal operator on the left-hand side by an arbitrary Calderón-Zygmund operator T:

$$w\big(\{x\in\mathbb{R}^d : Tf(x)\geq 1\}\big) \leq C\int_{\mathbb{R}^d}|f|Mw\mathrm{d}x. \tag{1.2}$$

This problem remained open for almost forty years and was finally handled by Reguera and Thiele [7]: it turns out that (1.1) fails to hold even for the Hilbert transform. The earlier work of Reguera [6] shows that the dyadic version of the conjecture is not true either. To describe this result in detail, fix the measure space $([0,1),\mathcal{B}([0,1)),|\cdot|)$ and consider the associated dyadic lattice. Let $h=(h_n)_{n\geq 0}$ be the standard Haar system: $h_0=[0,1)$, $h_1=[0,1/2)-[1/2,1)$, $h_2=[0,1/4)-[1/4,1/2)$, $h_3=[1/2,3/4)-[3/4,1)$ and so on, where we have identified a set with its indicator function. For any sequence $\epsilon=(\epsilon_n)_{n\geq 0}$ of numbers belonging to $[-1,1]$, we define the associated Haar multiplier T^ϵ which acts on integrable

2010 *Mathematics Subject Classification.* Primary: 46B15. Secondary: 46A35.
Key words and phrases. Weight, martingale.
The research was supported by Narodowe Centrum Nauki (Poland) grant DEC-2014/14/E/ST1/00532.

function $f=\sum_{n=0}^\infty a_nh_n$ by $T^\epsilon f=\sum_{n=0}^\infty \epsilon_na_nh_n$. Furthermore, let $\mathcal{M}$ be the dyadic maximal operator in $[0,1)$, which is given by

$$\mathcal{M}f(x)=\sup \frac{1}{|I|}\int_I |f|\mathrm{d}y,$$

where the supremum is taken over all dyadic intervals $I\subseteq[0,1)$ containing x. The aforementioned result of Reguera asserts that for any constant C there are a weight w, a function f on $[0,1)$ and a sequence ϵ such that

$$w\big(\{x\in[0,1):\mathcal{M}f(x)\geq 1\}\big)>C\int_{[0,1)}|f|\mathcal{M}w\mathrm{d}x.$$

There is a weaker form of the Muckenhoupt-Wheeden conjecture, which also gained a lot of interest in the literature. A weight w on $\mathbb{R}^d$ satisfies Muckenhoupt's condition A_1 (or belongs to the class A_1), if there is a constant c such that $Mw\leq cw$ almost everywhere. The smallest c with this property is denoted by $[w]_{A_1}$ and called the A_1 characteristic of the weight. Observe that if (1.2) held true, it would imply

$$w\big(\{x\in\mathbb{R}^d:Tf(x)\geq 1\}\big)\leq C_{d,T}[w]_{A_1}\int_{\mathbb{R}^d}|f|w\mathrm{d}x$$

for any A_1 weight w, where the constant C depends only on the parameters indicated. This weaker statement also fails to hold even for the Hilbert transform [4, 5]. The same negative result holds for the dyadic version of the above weak conjecture,

$$w\big(\{x\in[0,1):T^\epsilon f(x)\geq 1\}\big)\leq C[w]_{A_1}\int_{[0,1)}|f|w\mathrm{d}x,$$

where C is a universal constant and w is the dyadic A_1 weight on $[0,1)$: $[w]_{A_1}=\|\mathcal{M}w/w\|_{L^\infty(0,1)}<\infty$ (cf. [4, 5]). On the other hand, the following positive result was shown by Lerner et al. [3]: for any Calderón-Zygmund operator T,

$$w\big(\{x\in\mathbb{R}^d:Tf(x)\geq 1\}\big)\leq C_{d,T}[w]_{A_1}\log([w]_{A_1}+1)\int_{\mathbb{R}^d}|f|w\mathrm{d}x. \tag{1.3}$$

The analogous estimate is valid in the above dyadic context. There is a very natural question whether the above $L\log L$ dependence on the characteristic is optimal. In [4, 5] it was proved that some logarithmic correction is necessary: the constant is at least $C_{d,T}[w]_{A_1}\log^{1/3}[w]_{A_1}$. Finally, Lerner et al. [2] removed the exponent $1/3$ from this lower bound, i.e., proved that (1.3) is sharp in terms of the characteristic of $[w]_{A_1}$.

In the aforementioned papers, the arguments showing that the investigated strong or weak conjecture fails to hold, are not direct. In [2, 6, 7] the approach is to assume that the conjecture is true and apply some sort of extrapolation. This yields an appropriate weighted L^2 bound, which is more tractable from the viewpoint of counterexamples. The works [4, 5] use Bellman function method: again, the (assumed) validity of the conjecture implies the existence of a certain special function which enjoys appropriate size and concavity conditions; a careful and intricate analysis of these properties lead to the desired lower bound for the constant.

The purpose of this paper is to give the explicit construction of counterexamples for the dyadic weak conjecture of Muckenhoupt and Wheeden, showing that the $L\log L$ dependence is optimal. Here is the precise statement.

Theorem 1.1. *Suppose that the estimate*

$$w\big(\{x\in[0,1): T^\varepsilon f(x)\geq 1\}\big)\leq \varphi\big([w]_{A_1}\big)\int_{[0,1)}|f|w\,dx \tag{1.4}$$

holds for some universal nondecreasing function $\varphi:[1,\infty)\to[0,\infty)$ *and any sequence* ϵ *of signs. Then*

$$\liminf_{c\to\infty}\frac{\varphi(c)}{c\log c}\geq\frac{1}{432}.$$

It will be helpful to use the martingale language in the construction. The space $([0,1),\mathcal{B}([0,1)),|\cdot|)$ is a probability space and the Haar system gives rise to the corresponding dyadic filtration $(\mathcal{F}_n)_{n\geq 0}$: $\mathcal{F}_n=\sigma(h_0,\,h_1,\,\ldots,\,h_n)$ for $n\geq 0$. Then any integrable functions f, w on $[0,1)$ can be identified with the associated L^1-bounded martingales $(f_n)_{n\geq 0}=(\mathbb{E}(f|\mathcal{F}_n))_{n\geq 0}$, $(w_n)_{n\geq 0}=(\mathbb{E}(w|\mathcal{F}_n))_{n\geq 0}$. The fact that the filtration is dyadic is reflected in the following structural property of these processes: at each step, any $\mathcal{F}_n$-martingale either stays in its current location, or moves to one of two possible states, each having the same chance $1/2$ to be chosen. The implication can be reversed: any martingale $(u_n)_{n\geq 0}$ enjoying this structural property can be modeled at the interval $[0,1]$ equipped with the dyadic filtration. That is, there is a sequence $(a_n)_{n\geq 0}$ such that the associated process $(\sum_{k=0}^n a_kh_k)_{k\geq 0}$ has the same distribution as $(u_n)_{n\geq 0}$.

2. A COUNTEREXAMPLE

Fix a big constant $c>1$ and the associated integers $N=\lceil c\rceil$, $n=\lceil\frac{1}{2}\log c\rceil$. Next, put $p=3(2^{2n+1}+1)^{-1}$ and consider the sequence a_0, a_1, …, a_{2n+1} given as follows: $a_0=p/2$ and inductively,

$$a_{2k+1}=2a_{2k},\qquad a_{2k}=2a_{2k-1}-p/2$$

for $k=1,\,2,\,\ldots,\,n$. It is easy to find the explicit expression for the terms: we have

$$a_m=\begin{cases}\dfrac{p}{3}(2^m+1) & \text{if } m \text{ is odd},\\[2mm] \dfrac{p}{3}\left(2^m+\dfrac{1}{2}\right) & \text{if } m \text{ is even},\end{cases}$$

however, the above recurrence will be helpful to understand the evolution of certain martingales. Note that $a_{2n+1}=1$.

2.1. **A building block.** Let $a>0$, $b>0$ be two auxiliary parameters. We split the construction and the analysis of the block into a few parts.

The construction of (f,w). Consider the two-dimensional martingale (f,w), whose distribution is given as follows (all the nontrivial jumps occur with probability $1/2$):

(i) The pair starts from $(a_1,2a)$.
(ii) The state of the form $(a_{2k-1},2a)$ (for some $k\neq n+1$) leads to $(a_0,a(3-c^{-1}))$ or to $(a_{2k},a(1+c^{-1}))$.
(iii) The state of the form $(a_{2k},a(1+c^{-1}))$ leads to $(0,2ac^{-1})$ or to $(a_{2k+1},2a)$.
(iv) The state $(a_0,a(3-c^{-1}))$ leads to $(0,2ac^{-1})$ or to $(a_1,2a(3-2c^{-1}))$.
(v) The state $(1,2a)$ leads to $(2,2ac^{-1})$ or to $(0,2a(2a-c^{-1}))$.
(vi) All other states are absorbing: the process stops there.

The above recurrent definition of the sequence $(a_k)_{k=0}^{2n+1}$ shows that (f, w) is indeed a martingale. To understand what happens to the pair at the steps (i), (ii),

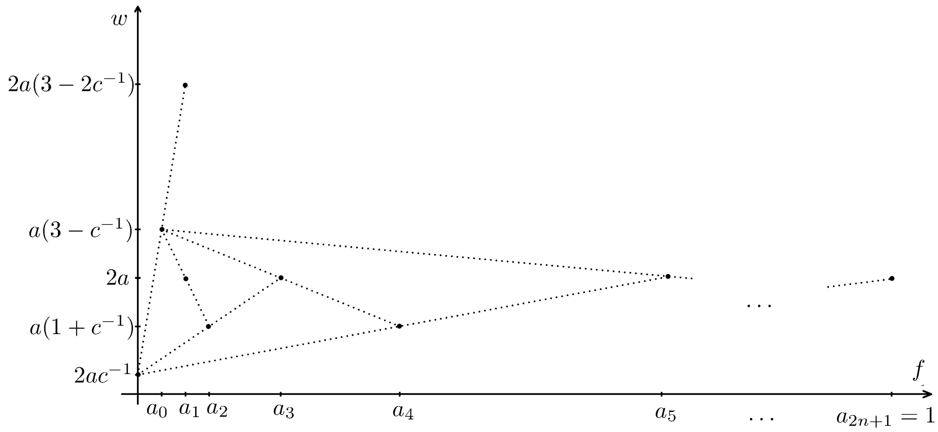

FIGURE 1. The behavior of (f, w). The pair starts at $(a_1, 2a)$ and then jumps over the endpoints of the dotted segments.

(iii) and (iv) above, it is helpful to look at Figure 1. The process starts at $(a_1, 2a)$; at each step we take the dotted line segment whose midpoint is the current position, and move along to one of the endpoints; each endpoint has the chance 1/2 to be reached. The rule (v) is not illustrated at the figure (we would have to shrink the picture significantly and it would become unreadable).

Terminal positions. There are four absorbing states of the pair (f, w), and they are reached at time $2n + 1$ at the latest: the points $(2, 2ac^{-1})$, $(0, 2a(2a - c^{-1}))$, $(a_1, 2a(3 - 2c^{-1}))$ and $(0, 2ac^{-1})$. To get to $(2, 2ac^{-1})$, f must make $2n + 1$ steps to the right and stop, so

$$\mathbb{P}\big((f_{2n+1}, w_{2n+1}) = (2, 2ac^{-1})\big) = 2^{-2n-1}.$$

Similarly, (f, w) reaches $(0, 2a(2a-c^{-1}))$ if f makes $2n$ steps to the right and then the final jump to the left, which implies

$$\mathbb{P}\Big((f_{2n+1}, w_{2n+1}) = \big(0, 2a(2a - c^{-1})\big)\Big) = 2^{-2n-1}.$$

Next, in order to visit $(a_1, 2a(3-2c^{-1}))$, the martingale f must make $k = 2m$ steps to the right (for some $m = 0, 1, 2, \ldots, n-1$), then jump to a_0 and finally move to a_1. Hence

$$\mathbb{P}\Big((f_{2n+1}, w_{2n+1}) = \big(a_1, 2a(3-2c^{-1})\big)\Big) = \sum_{m=0}^{n-1} \left(\frac{1}{2}\right)^{2m} \cdot \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{3}(1-2^{-2n}). \tag{2.1}$$

Consequently, the last, fourth terminal position $(0, 2ac^{-1})$ is attained with the probability

$$\mathbb{P}\big((f_{2n+1}, w_{2n+1}) = (0, 2ac^{-1})\big) = \frac{2}{3}(1 - 2^{-2n}).$$

A transform of f and the associated distribution. Let g be the transform of f, given as follows: $g_m = b + \sum_{k=1}^{m} (-1)^k df_k$ for $m \geq 0$. For our further considerations, it will be important to describe the distribution of $g_{2n+1} - g_0 = \sum_{k=1}^{m} (-1)^k df_k$ on the set $\Big\{(f_{2n+1}, w_{2n+1}) = \big(a_1, 2a(3-2c^{-1})\big)\Big\}$. Directly from the analysis preceding (2.1), we see that if f made exactly $2m$ steps to the right and then jumped to a_0 and a_1 (this scenario occurs with probability $(1/2)^{2m+2}$), then

$$\begin{aligned} g_{2n+1} - g_0 &= \sum_{k=1}^{2m} (-1)^k (a_{k+1} - a_k) + (a_{2m+1} - a_0) + a_0 \\ &= a_1 + 2(-a_2 + a_3 - a_4 + a_5 - \ldots - a_{2m} + a_{2m+1}) \\ &= a_1 + 2(a_2 + a_4 + \ldots + a_{2m}), \end{aligned}$$

since $-a_2 + a_3 = a_2$, $-a_4 + a_5 = a_4$, $\ldots$, $-a_{2m} + a_{2m+1} = a_{2m}$. Plugging the formula for a_m, we finally get

$$g_{2n+1} - g_0 = p + \frac{2p}{3}\left(\frac{4}{3}(4^m - 1) + \frac{m}{2}\right) =: b_m.$$

Introduce the corresponding conditional distribution

$$\begin{aligned} \mu(b_m) &= \mathbb{P}\Big(g_{2n+1} - g_0 = b_m \,\Big|\, (f_{2n+1}, w_{2n+1}) = \big(a_1, 2a(3 - 2c^{-1})\big)\Big) \\ &= \frac{(1/2)^{2m+2}}{(1 - 2^{-2n})/3} = \frac{3}{2^{2m+2}(1 - 2^{-2n})} \end{aligned} \tag{2.2}$$

(the probability in the denominator comes from (2.1)). Since $\frac{8p}{9}(1 - 2^{-2n}) \cdot 4^m \leq b_m \leq p \cdot 4^m$, the expectation and the variance of μ satisfy

$$\mathbb{E}(\mu) = \sum_{m=0}^{n-1} b_m \cdot \frac{3}{2^{2m+2}(1 - 2^{-2n})} \geq \sum_{m=0}^{n-1} \frac{2p}{3} = \frac{2np}{3} \tag{2.3}$$

and

$$\begin{aligned} \mathrm{Var}(\mu) \leq \int_{\mathbb{R}} x^2 \mathrm{d}\mu(x) = \sum_{m=0}^{n-1} b_m^2 \cdot \frac{3}{2^{2m+2}(1 - 2^{-2n})} &\leq \sum_{m=0}^{n-1} p^2 \cdot \frac{3 \cdot 2^{2m-2}}{1 - 2^{-2n}} \\ &= 2^{2n-2} p^2. \end{aligned} \tag{2.4}$$

2.2. **The counterexample.** Recall that $N = \lceil c \rceil$: this number will count the iterations of the construction from the previous subsection. The procedure is as follows: We take $a = 1/2$ and consider the above building block. If the pair (f, w) terminates at $(2, 2ac^{-1})$, $(0, 2a(2a - c^{-1}))$ or $(0, 2ac^{-1})$, it stops ultimately and the construction is over. On the other hand, if it gets to $(a_1, 2a(3 - 2c^{-1}))$, we denote this point by $(a_1, 2a')$ and iterate the building block above with $a = a'$. More precisely, we know from the above analysis that the set $\{(f_{2n+1}, w_{2n+1}) = (a_1, 2a(3 - 2c^{-1}))\}$ is the union of pairwise disjoint events $\{(f_{2n+1}, w_{2n+1}) = (a_1, 2a(3 - 2c^{-1})), g_{2n+1} = b_m\}$, $m = 0, 1, 2, \ldots, n-1$, each of which is a dyadic subinterval of $[0, 1)$. The construction from the previous subsection is used independently on each such subinterval. Now, if the martingale (f, w) gets to $(2, 2a'c^{-1})$, $(0, 2a'(2a' - c^{-1}))$ or $(0, 2a'c^{-1})$, it stops forever. Otherwise, if it visits $(a_1, 2a'(3 - 2c^{-1}))$, we denote this point by $(a_1, 2a')$ and iterate, and so on. We repeat this pattern N times, obtaining a martingale denoted by $(F, W) = (F_m, W_m)_{m=0}^{\infty}$ (actually, one can easily see that it is a finite martingales, i.e., has at most $2nN + 1$ nontrivial steps, but we will not need this). Let G be the transform of F by the alternating sequence $1, -1, 1, -1, \ldots$: that is, put $G_k = \sum_{m=0}^{k} (-1)^m dF_k$ for each $k \geq 0$.

2.3. **Calculations.** Observe that W is a dyadic A_1 weight with $[W]_{A_1} \leq \frac{3}{2}c$. To see this, we first note that W_∞ takes values in the set $\{(3-2c^{-1})^k c^{-1}, (3-2c^{-1})^k(1-c^{-1})\}$, where $k \in \{0, 1, 2, \ldots, N-1\}$. Observe that if W_∞ is equal to $(3-2c^{-1})^k c^{-1}$ or to $(3-2c^{-1})^k(1-c^{-1})$, then exactly $k+1$ iterations of the building block were needed, and hence the largest possible value of W obtained on the way does not exceed $(3-2c^{-1})^k \cdot (3-c^{-1})/2$. So, the dyadic maximal function satisfies

$$\mathcal{M}W \leq (3-2c^{-1})^k \cdot (3-c^{-1})/2 \leq 3(3-2c^{-1})^k/2 \leq \frac{3}{2}cW_\infty$$

and hence $[W]_{A_1} \leq 3c/2$. Next, let us establish a helpful upper bound for $\|F\|_{L^1(W)}$. To this end, we find the distribution of (F_∞, W_∞). First, note that $F_\infty \in \{0, a_1, 2\}$. The equality $F_\infty = a_1$ means that the process F returned N times to a_1 (and stopped). In such an event, we have $W_\infty = (3-2c^{-1})^N$, and hence by (2.1),

$$\mathbb{P}(F_\infty = a_1) = \mathbb{P}\Big((F_\infty, W_\infty) = (a_1, (3-2c^{-1})^N)\Big) = \left(\frac{1-2^{-2n}}{3}\right)^N.$$

Next, $F_\infty = 2$ holds if and only if the martingale F returned m times to the point a_1 (for some $m = 0, 1, 2, \ldots, N-1$), and then made $2n+1$ moves to the right. Note that in such a scenario we have $W_\infty = (3-2c^{-1})^m c^{-1}$ and

$$\mathbb{P}\Big((F_\infty, W_\infty) = (2, (3-2c^{-1})^m c^{-1})\Big) = \left(\frac{1-2^{-2n}}{3}\right)^m \cdot 2^{-2n-1} \leq 3^{-m} \cdot 2^{-2n-1}.$$

The final possibility is that $F_\infty = 0$, which can be skipped: this part of the distribution will not affect the integral $\int_\Omega F_\infty W_\infty \mathrm{d}x$. By the above observations,

$$\begin{aligned}
&\int_\Omega F_\infty W_\infty \mathrm{d}x \\
&\leq a_1(3-2c^{-1})^N \cdot \left(\frac{1-2^{-2n}}{3}\right)^N + \sum_{m=0}^{N-1} 2\left(\frac{3-2c^{-1}}{3}\right)^m c^{-1} \cdot 2^{-2n-1} \qquad (2.5)\\
&\leq a_1 + 2^{-2n}c^{-1}\sum_{m=0}^{\infty}\left(\frac{3-2c^{-1}}{3}\right)^m \leq a_1 + \frac{3}{2}\cdot 2^{-2n} \leq 3\cdot 2^{-2n}.
\end{aligned}$$

Now let us provide some information on the transform G. As we have noted above, the equality $(F_\infty, W_\infty) = (a_1, (3-2c^{-1})^N)$ means that the maximal number of N iterations was used. However, by the above construction, each iteration gives rise to an independent increment of G, which has the distribution μ given by (2.2). In the probabilistic language, the conditional distribution of G on the set $\{(F_\infty, W_\infty) = (a_1, (3-2c^{-1})^N)\}$ coincides with the distribution of $G_0 + X_1 + X_2 + \ldots + X_N$, where $X_1, X_2, \ldots, X_N$ are i.i.d. random variables such that $X_i \sim \mu$. Consequently,

$$\begin{aligned}
W(G_\infty \geq \lambda) &\geq W\Big(G_\infty \geq \lambda,\, (F_\infty, W_\infty) = \big(a_1, (3-2c^{-1})^N\big)\Big) \\
&= (3-2c^{-1})^N \cdot \mathbb{P}\Big(G_\infty \geq \lambda,\, (F_\infty, W_\infty) = \big(a_1, (3-2c^{-1})^N\big)\Big) \\
&= (3-2c^{-1})^N \cdot \mathbb{P}\Big(G_\infty \geq \lambda \,\Big|\, (F_\infty, W_\infty) = \big(a_1, (3-2c^{-1})^N\big)\Big) \\
&\qquad\qquad \cdot \mathbb{P}\big((F_\infty, W_\infty) = \big(a_1, (3-2c^{-1})^N\big)\big) \\
&\geq (3-2c^{-1})^N \cdot \left(\frac{1-2^{-2n}}{3}\right)^N \mathbb{P}(X_1 + X_2 + \ldots + X_N \geq \lambda)
\end{aligned}$$

(in the last line we omitted G_0, writing $X_1 + X_2 + \ldots + X_N$ instead of $G_0 + X_1 + X_2 + \ldots + X_N$). Setting $\lambda = Nnp/6$ we get, by (2.3), (2.4) and Chebyshev's bound,

$$\begin{aligned}\mathbb{P}\big(X_1 + X_2 + \ldots + X_N < \lambda\big) &\leq \mathbb{P}\big(X_1 + X_2 + \ldots + X_N - N\mathbb{E}(\mu) < \lambda - 2Nnp/3\big)\\ &\leq \mathbb{P}\Big(|X_1 + X_2 + \ldots + X_N - N\mathbb{E}(\mu)| > Nnp/2\Big)\\ &\leq \frac{N\,\mathrm{Var}(\mu)}{(Nnp/2)^2} \leq \frac{2^{2n}}{Nn^2} \leq \frac{1}{2}.\end{aligned}$$

Here the last inequality holds by our choice $N = \lceil c \rceil$ and $n = \lceil \frac{1}{2}\log c \rceil$, at least for sufficiently large c. Combining this with the previous estimate yields

$$W(G_\infty \geq \lambda) \geq (3 - 2c^{-1})^N \cdot \left(\frac{1 - 2^{-2n}}{3}\right)^N \cdot \frac{1}{2} = \left(1 - \frac{2}{3c}\right)^N \left(1 - 2^{-2n}\right)^N \cdot \frac{1}{2}.$$

However, if c is sufficiently large, then

$$\left(1 - \frac{2}{3c}\right)^N \geq \left(1 - \frac{2}{3c}\right)^c \geq 1/2, \qquad (1 - 2^{-2n})^N \geq \left(1 - \frac{1}{c}\right)^c \geq 1/3.$$

so the validity of (1.4) implies, by (2.5) and the bound $[W]_{A_1} \leq 3c/2$,

$$\frac{c\log c}{144} \cdot 2^{-2n} \leq \frac{Nnp}{6} \cdot \frac{1}{12} \leq \lambda W(G_\infty \geq \lambda) \leq \varphi\left(\frac{3}{2}c\right) \|F\|_{L^1(W)} \leq 3\varphi\left(\frac{3}{2}c\right) \cdot 2^{-2n}.$$

This yields the assertion of Theorem 1.1.

References

[1] C. Fefferman and E. M. Stein, *Some maximal inequalities*, Amer. J. Math. **93** (1971), 107–115.

[2] A. Lerner, F. Nazarov and S. Ombrosi, *On the sharp upper bound related to the weak Muckenhoupt-Wheeden conjecture*, available at arXiv:1710.07700.

[3] A. Lerner, S. Ombrosi, C. Pérez, *A_1 bounds for Calderón–Zygmund operators related to a problem of Muckenhoupt and Wheeden*, Math. Res. Lett. **16** (2009) no. 1, 149–156.

[4] F. Nazarov, A. Reznikov, V. Vasyunin, and A. Volberg, *A Bellman function counterexample to the A_1 conjecture: the blow-up of weak norm estimates of weighted singular operators*, arXiv:1506.04710v1, 1–23.

[5] F. Nazarov, A. Reznikov, V. Vasyunin and A. Volberg, *On weak weighted estimates of the martingale transform and a dyadic shift*, Anal. PDE **11**, Number 8 (2018), 2089–2109.

[6] M. C. Reguera, *On Muckenhoupt-Wheeden conjecture*, Adv. Math. **227** (2011), no. 4, 1436–1450.

[7] M. C. Reguera and C. Thiele, *The Hilbert transform does not map $L^1(Mw)$ to $L^{1,\infty}(w)$*, Math. Res. Lett. **19** (2012), no. 1, 1–7.

Faculty of Mathematics, Informatics and Mechanics, University of Warsaw, Banacha 2, 02-097 Warsaw, Poland

Email address: `ados@mimuw.edu.pl`