A WEIGHTED WEAK-TYPE BOUND FOR HAAR MULTIPLIERS

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Abstract. We study a weighted maximal weak type inequality for Haar multipliers which can be regarded as a dual problem of Muckenhoupt and Wheeden. More precisely, if T_ε is the Haar multiplier associated with the sequence ε with values in $[-1,1]$ and $\mathcal{M}_r$ is the r-maximal operator, then for any weight w and any function f on $[0,1)$ we have

$$w\left(\{x \in [0,1) : |T_\varepsilon f(x)| \geq \mathcal{M}_r w(x)\}\right) \leq C_r \int_0^1 |f| \mathrm{d}x,$$

for some constant C_r depending only on r. We also show that the analogous result does not hold if we replace $\mathcal{M}_r$ by the dyadic maximal operator M_d. The proof rests on Bellman function method: using this technique we establish related weighted L^p estimates for p close to 1, and then deduce the main result by extrapolation arguments.

1. Introduction

In 1971, Fefferman and Stein established the following weighted version of the weak-type (1,1) estimate for the Hardy-Littlewood maximal operator M on $\mathbb{R}^n$:

$$w\big(\{x \in \mathbb{R}^n : Mf(x) \geq 1\}\big) \leq c \int_{\mathbb{R}^n} |f| Mw \mathrm{d}x,$$

where w is a nonnegative locally integrable function, $w(E) = \int_E w \mathrm{d}x$ and c depends only on the dimension n. A few years after that Muckenhoupt and Wheeden conjectured that an analogous bound holds true if one replaces the maximal operator on the left-hand side by an arbitrary Calderón-Zygmund singular integral operator T; that is, there is a finite constant c depending only on n and T such that

$$w\big(\{x \in \mathbb{R}^n : Tf(x) \geq 1\}\big) \leq c \int_{\mathbb{R}^n} |f| Mw \mathrm{d}x. \tag{1.1}$$

There is a weaker statement involving A_1 weights. Recall that w satisfies Muckenhoupt's condition A_1 if there is a finite constant K such that $Mw \leq Kw$ almost everywhere; the smallest K enjoying this property is denoted by $[w]_{A_1}$ and called the A_1 characteristics of w. For A_1 weights w, (1.1) would imply

$$w\big(\{x \in \mathbb{R}^n : Tf(x) \geq 1\}\big) \leq c[w]_{A_1} \int_{\mathbb{R}^n} |f| \mathrm{d}x, \tag{1.2}$$

which is called the weak conjecture of Muckenhoupt and Wheeden. Both problems remained open for a long time, and there were many attempts which led to some partial results in this direction. Chanillo and Wheeden showed in [3] that (1.1) is

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true if T is replaced by the Paley-Littlewood square function. Buckley [1] showed that both conjectures are true for weights of the form $w_\delta(x)=|x|^{-d(1-\delta)}$, $0<\delta<1$. As far as we know, the best positive result in this direction is that of Pérez, who showed that if M^2 denotes the second iteration of M, then

$$\lambda w\left(\left\{x\in\mathbb{R}^d : Tf(x)\geq\lambda\right\}\right)\leq c||f||_{L^1(M^2w)}.$$

In fact, the paper [10] contains a stronger statement in which the operator M^2 is replaced by the smaller object $M_{L(\log L)^\varepsilon}$ (we refer the reader to the paper for the necessary definitions). Consult also the recent works of Lerner, Ombrosi and Pérez [6, 7] for further results on various forms of (1.2). In 2010, both Muckenhoupt-Wheeden conjectures were finally shown to be false, as they do not hold true for $n=1$ and T being the Hilbert transform. See the counterexamples by Reguera [11] and Reguera and Thiele [12], consult also an unpublished manuscript of Nazarov, Reznikov, Vasyunin and Volberg [9]. The latter work contains also related results for Haar multipliers. Let $h=(h_n)_{n\geq 0}$ stand for the usual, unnormalized Haar system on $[0,1)$:

$$\begin{aligned}
&h_0=[0,1),\quad h_1=[0,1/2)-[1/2,1),\\
&h_2=[0,1/4)-[1/4,1/2),\quad h_3=[1/2,3/4)-[3/4,1),\\
&h_4=[0,1/8)-[1/8,1/4),\quad h_5=[1/4,3/8)-[3/8,1/2),\\
&h_6=[1/2,5/8)-[5/8,3/4),\quad h_7=[3/4,7/8)-[7/8,1),
\end{aligned}$$

and so on, where we have identified a set with its indicator function. The collection of all intervals appearing in the above definition will be called the dyadic lattice of $[0,1)$. For a given integrable function $f=\sum_{n=0}^\infty a_nh_n$ on $[0,1)$, let its maximal function M_df be given by $\sup_{N\geq 0}|f_N|$, where $f_N=\sum_{n=0}^N a_nh_n$ is the projection of f onto the space generated by the first $N+1$ Haar functions. For a given sequence $\varepsilon=(\varepsilon_n)_{n\geq 0}$ of real numbers, we define the associated Haar multiplier $T=T_\varepsilon$ by $T\left(\sum_{n=0}^\infty a_nh_n\right)=\sum_{n=0}^\infty\varepsilon_na_nh_n$. The aforementioned result of [9] asserts that for any $c>0$ there is an A_1 weight w, a function f on $[0,1)$ and a sequence ε with values in $\{-1,1\}$ such that the corresponding operator T_ε satisfies

$$w\big(\{x\in[0,1):T_\varepsilon f(x)\geq 1\}\big)>c[w]_{A_1}\int_0^1|f|M_dw\mathrm{d}x.$$

On the other hand, there is a finite universal c such that

$$w\big(\{x\in[0,1):T_\varepsilon f(x)\geq 1\}\big)\leq c[w]_{A_1}\log^{1/7}(1+[w]_{A_1})\int_0^1|f|M_dw\mathrm{d}x.$$

In this paper we will be mostly interested in estimates dual to (1.1) and (1.2). Such inequalities appeared in the works of Lerner, Ombrosi and Pérez [5] in the context of singular integral operators: the strong version is

$$w\left(\{x\in\mathbb{R}^n:|Tf(x)|\geq Mw(x)\}\right)\leq c\int_{\mathbb{R}^n}|f|\mathrm{d}x,\tag{1.3}$$

where w is an arbitrary weight, T is a Calderón-Zygmund operator and c depends only on T and the dimension. The weaker inequality concerns A_1 weights and reads

$$w\left(\{x\in\mathbb{R}^n:|Tf(x)|\geq w(x)\}\right)\leq c[w]_{A_1}\int_{\mathbb{R}^n}|f|\mathrm{d}x,\tag{1.4}$$

where T and c are as above. To see the duality between these bounds and (1.1), suppose that (1.1) holds true for some T and apply the extrapolation theorem of Cruz-Uribe and Pérez [4]. We get that for any $1 < p < \infty$ there is a constant $C_{n,p}$ depending only on the parameters indicated such that

$$\int_{\mathbb{R}^n} |Tf|^p w \mathrm{d}x \leq C_{n,p} \int_{\mathbb{R}^n} |f|^p \left(\frac{Mw}{w}\right)^p w \mathrm{d}x$$

for all f and w. Then by duality we get

$$\int_{\mathbb{R}^n} \left(\frac{|T^* f|}{Mw}\right)^{p'} w \mathrm{d}x \leq C_{n,p} \int_{\mathbb{R}^n} \left(\frac{|f|}{w}\right)^{p'} w \mathrm{d}x,$$

where $p' = p/(p-1) \in (1, \infty)$. Thus (1.3) can be regarded as a limiting weak-type (1,1) version, as $p \to \infty$, of this estimate (applied to the operator T^*).

The question about the validity of the inequality (1.3) and (1.4) seems to be open at the moment. In [5], Lerner, Ombrosi and Pérez proved the following weaker form of (1.3): there is a constant c depending only on n and T such that

$$w\left(\{x \in \mathbb{R}^n : |Tf(x)| \geq M^3 w(x)\}\right) \leq c \int_{\mathbb{R}^n} |f| \mathrm{d}x$$

(here M^3 is the third iteration of the Hardy-Littlewood maximal operator). The purpose of this paper is to study the dual inequalities in the context of Haar multipliers. Our first result is the following.

Theorem 1.1. *For any c, there is a weight w, a function f on $[0, 1)$ and a sequence ε with values in $\{-1, 1\}$ such that the associated Haar multiplier T_ε satisfies*

$$w\left(\{x \in [0, 1) : |T_\varepsilon f(x)| \geq M_d w(x)\}\right) > c \int_0^1 |f| dx. \tag{1.5}$$

Our second result is positive and shows that (1.3) holds true if we increase the operator M_d a little bit. In what follows, for any $r > 1$ we define the r-maximal operator by $\mathcal{M}_r w = (M_d w^r)^{1/r}$.

Theorem 1.2. *For any $r > 1$ there is a constant C_r depending only on r such that for any weight w, any function f on $[0, 1)$ and any sequence ε with values in $[-1, 1]$, the associated Haar multiplier T_ε satisfies*

$$w\left(\{x \in [0, 1) : |T_\varepsilon f(x)| \geq \mathcal{M}_r w(x)\}\right) \leq C_r \int_0^1 |f| dx. \tag{1.6}$$

The above statements do not answer the question about the validity of (1.4) for Haar multipliers. Though we believe that the inequality does not hold, we have been unable to prove this rigorously.

We establish Theorem 1.1 in the next section, by providing a family of appropriate examples. Section 3 is devoted to the proof of Theorem 1.2: we will exploit the so-called Bellman function method and deduce the validity of (1.6) from the existence of a certain special function, involving appropriate majorization and concavity. This is quite different from the argumentation used in [5]-[7], which exploits delicate boundedness properties of the Hardy-Littlewood maximal operator and Rubio de Francia algorithm.

2. A counterexample

For the sake of clarity, we have decided to split the construction into two parts.

2.1. **A building block.** Fix an arbitrary odd number $L = 2\ell + 1 \geq 3$ and consider a function F on $[0,1)$ given by

$$F = \sum_{n=0}^{\infty} a_n h_n.$$

Here the coefficients $a_k \in \{0,1\}$ are defined inductively as follows: $a_0 = 1$ and, if $F_{k-1} = \sum_{n=0}^{k-1} a_n h_n \in \{0, L\}$ on the support of h_k, then $a_k = 0$; otherwise, $a_k = 1$. Next, put

$$G = \sum_{k=0}^{\infty} \varepsilon_k a_k h_k,$$

where $\varepsilon_0 = 0$ and, for $k \geq 1$, the term $\varepsilon_k = 1$ when one of the following conditions is true:

(i) we have $G_{k-1} < 0$ and $F_{k-1} > 1$ on the support of h_k;
(ii) we have $G_{k-1} \geq 0$ and $F_{k-1} = 1$ on the support of h_k.

If neither of the above holds, we set $\varepsilon_k = -1$. Clearly, by the very definition, we see that $G = TF$ for a Haar multiplier T with coefficients ε_0, ε_1, ε_2, ...

The pair (F, G) has a very nice interpretation in terms of martingales, which we will describe now, for the convenience of the reader. If we treat $([0,1), \mathcal{B}(0,1), |\cdot|)$ as a probability space equipped with the dyadic filtration, then $((F_n, G_n))_{n\geq 0}$ is a Markov martingale taking values in the set

$$\big\{(k, m) \in \mathbb{Z} \times \mathbb{Z} : k \geq 0, k + |m| \in \{1, 3, 5, \ldots, L\}\big\},$$

with the distribution uniquely determined by the following requirements:

(i) We have $(F_0, G_0) \equiv (1, 0)$.
(ii) Any point of the form $(1, k)$ with $k = 0, 2, 4, \ldots, L-3$ leads to $(0, k-1)$ or to $(2, k+1)$. Any point of the form $(1, k)$ with $k = -2, -4, \ldots, 3-L$ leads to $(0, k+1)$ or to $(2, k-1)$.
(iii) The point $(1, L-1)$ leads to $(0, L)$ or to $(2, L-2)$. The point $(1, 1-L)$ leads to $(0, -L)$ or to $(2, 2-L)$.
(iv) Any point of the form (k, m) with $k \in \{2, 3, \ldots, L-1\}$ and $m \in \{0, 1, \ldots, L-k\}$ leads to $(k-1, m+1)$ or to $(k+1, m-1)$. Any point of the form (k, m) with $k \in \{2, 3, \ldots, L-1\}$ and $m \in \{-1, -2, \ldots, k-L\}$ leads to $(k-1, m-1)$ or to $(k+1, m+1)$.
(v) All the remaining states are absorbing: once the process gets to such a state, it stays there forever.

In (ii), (iii) and (iv), each possibility has probability $1/2$. See the Figure 1 for the graph of transities for $L = 7$.

In what follows, we will need a small, but crucial modification of the pair (F, G) (after which it will no longer be a Markov process). Namely, let us take a look at the dyadic interval

$$\mathcal{I} = \{s \in [0,1) : F_0(s) = 1, F_1(s) = 2, F_2(s) = 1, F_3(s) = 2, \ldots, F_{L-1}(s) = 1\}.$$

Directly from the above definition it follows that $G_0 = 0$, $G_1 = 1$, $G_2 = 2$, ..., $G_{L-1} = L - 1$ on $\mathcal{I}$. The modification is that we set $F \equiv 1$ and $G \equiv L - 1$ on $\mathcal{I}$. In the language of the Haar expansion, this amounts to cutting off all the coefficients for which the corresponding Haar functions have their support contained in $\mathcal{I}$. This modification also has a clear probabilistic interpretation: we stop the martingale

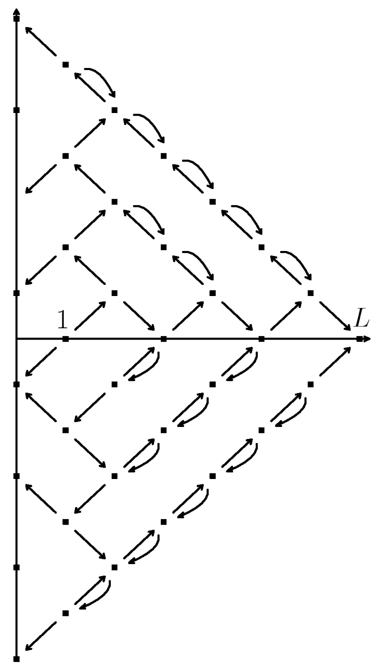

FIGURE 1. The evolution of the process (F,G) in the case $L=7$. The dots stand for the elements of the state space, while the arrows indicate possible movements between the states.

pair (F,G) after $L-1$ steps if it moved according to the trajectory described in the above definition of $\mathcal{I}$.

Let us gather some facts which follow from the above construction. First, observe that the function F takes values in the set $\{0,1,L\}$ and hence

$$1=||F||_{L^1}=1\cdot|\{F=1\}|+L\cdot|\{F=L\}|=|\mathcal{I}|+L|\{F=\ell\}|=2^{1-L}+L|\{F=L\}|,$$

so $|\{F=L\}|=L^{-1}(1-2^{1-L})$. Next, observe that on $\mathcal{I}$ we have $G=L-1=2\ell$ and $M_dF=\max\{F_0,F_1,\ldots,F_{L-1}\}=2$. Consequently, $|\{G\geq \ell M_dF\}|\geq|\mathcal{I}|=2^{1-L}$. Finally, we will use the notation $\mathcal{Z}_{[0,1)}=\{F=0\}$ and $\mathcal{P}_{[0,1)}=\{F=L\}$; note that each of these sets is a countable union of dyadic intervals.

In what follows, we will need a version of the above construction on an arbitrary dyadic interval I. This version is obtained by taking an increasing affine mapping S from I onto $[0,1)$ and setting $F^I=F\circ S$, $G^I=G\circ S$. The pair (F^I,G^I) inherits all the crucial properties. In particular, we see that $G=TF$ for some Haar multiplier T, and

$$|\{G^I\geq \ell M_dF\}|\geq 2^{1-L}|I|.$$

We will also use the notation $\mathcal{Z}_I=\{F^I=0\}$ and $\mathcal{P}^I=\{F^I=L\}$; as previously, each of these sets is a countable union of dyadic intervals.

2.2. **Proof of** (1.5)**.** Now we provide the final counterexample. The appropriate function $f=\sum_{k=0}^\infty a_kh_k:[0,1)\to[0,\infty)$ and the associated function $g=\sum_{k=0}^\infty \alpha_ka_kh_k$, with $\alpha_k\in\{-1,1\}$, will be obtained as the result of several modifications with the use of the building block described previously.

Step 1. Our starting point is to take $a_0 = a_1 = 1/2$, $a_2 = a_3 = \ldots = 0$ and $\alpha_0 = 1$, $\alpha_1 = -1$, $\alpha_3 = \alpha_4 = \ldots = 1$. Then, clearly, $f \in \{0, 1\}$.

Step 2. On the set where $f = 0$, we do not change anything. Let us look at the interval $\{f = 1\} = [0, 1/2)$. Here we let $(f, g)|_{[0,1/2)}$ be equal to $(F^{[0,1/2)}, G^{[0,1/2)})$. Since $(f_1, g_1) = (1, 0) = (F_0^{([0,1/2)}, G_0^{[0,1/2)})$, this is well-defined and hence

$$f = \sum_{k=0}^{\infty} a_k h_k, \qquad g = \sum_{k=0}^{\infty} \alpha_k a_k h_k,$$

for some real numbers a_0, a_1, a_2, $\ldots$ and α_0, α_1, α_2, $\ldots$. Furthermore, since the coefficients ε_1 ,ε_2, $\ldots$ appearing in the definition of G took values in $\{-1, 1\}$, we see that the sequence $(\alpha_k)_{k \geq 0}$ consists of signs only. Finally, note that

$$M_d f \leq L \qquad \text{on } [0, 1) \tag{2.1}$$

and

$$|g| \geq \ell M_d f \qquad \text{on the set } \{f = 1\}.$$

Step 3. We know that the interval $[0, 1)$ splits into the sum of three sets: $\{f = 0\}$, $\{f = 1\}$ and $\{f = L\}$. On the first two of these sets the construction is over; however, on the third we will change f a little bit. Namely, directly from §2.1, the set $\{f = L\}$ is a countable union of pairwise disjoint dyadic intervals J_1, J_2, $\ldots$. We modify the pair (f, g) on each J_m by putting $(f, g)|_{J_m} = (LF^{J_m}, LG^{J_m})$, $m = 1, 2, \ldots$. This makes sense, since before the modification we had $(f, g) = (L, 0) = (LF_0^{J_m}, LG_0^{J_m})$ on each J_m. Now, by the structure of the Haar system, we easily see that f and g still admit the representation

$$f = \sum_{k=0}^{\infty} a_k h_k, \qquad g = \sum_{k=0}^{\infty} \alpha_k a_k h_k, \tag{2.2}$$

for some real numbers a_0, a_1, a_2, $\ldots \in \{0, 1\}$ and some signs α_0, α_1, α_2, $\ldots$. Observe that

$$M_d f \leq L^2 \qquad \text{on } [0, 1) \tag{2.3}$$

and, by (2.1),

$$|g| \geq \ell M_d f \qquad \text{on } \{f = 1\} \cup \{f = L\}.$$

For completeness, let us write formally what happens in the k-th turn.

Step k. The interval $[0, 1)$ splits into the sum of the sets $\{f = 0\}$, $\{f = 1\}$, $\{f = L\}$, $\ldots$, $\{f = L^{k-3}\}$ and $\{f = L^{k-2}\}$. We do not change f on the first $k - 1$ sets. The last set is a union of dyadic intervals, denoted again by J_1, J_2, $\ldots$. On each J_m we set $(f, g)|_{J_m} = (L^{k-2} F^{J_m}, L^{k-2} G^{J_m})$, $m = 1, 2, \ldots$. It is easy to check that after this modification, (2.2) are still valid and we have

$$M_d f \leq L^{k-1} \qquad \text{on } [0, 1)$$

and

$$|g| \geq \ell M_d f \qquad \text{on} \{f = 1\} \cup \{f = L\} \cup \ldots \cup \{f = L^{k-2}\}.$$

Let us carry over K steps. We get a pair (f, g) which satisfies the formulas (2.2) and hence there is a Haar multiplier T with coefficients belonging to $\{-1, 1\}$ such that $g = Tf$. Take $w = \ell f$. Then the left-hand side of (1.5) equals

$$\int_0^1 \ell f \chi_{\{|g| \geq \ell M_d f\}} \mathrm{d}s = \int_0^1 \ell f \mathrm{d}s - \int_0^1 \ell f \chi_{\{f = L^K\}} \mathrm{d}s,$$

since

$$\begin{aligned}[0,1) &= \{f=0\}\cup\{f=1\}\cup\{f=L\}\cup\ldots\cup\{f=L^K\}\\ &= \{f=0\}\cup\{|g|\geq \ell M_d f\}\cup\{f=L^K\}.\end{aligned}$$

Hence

$$\begin{aligned}\int_0^1 w\chi_{\{|g|\geq M_d w\}}\mathrm{d}x &= \int_0^1 \ell f\mathrm{d}s - \ell L^k|f=L^K|\\ &= \int_0^1 \ell f\mathrm{d}s - \ell L^k|F=L|^K\\ &= \ell\int_0^1 f\mathrm{d}s - \ell(1-2^{1-L})^K\\ &\xrightarrow{K\to\infty} \ell\int_0^1 f\mathrm{d}s.\end{aligned}$$

But ℓ was an arbitrary number. Thus, for any positive constant c as in (1.5), we pick $\ell > c$ and take sufficiently large K to obtain a function and a Haar multiplier for which the desired inequality is satisfied.

3. Proof of Theorem 1.2

For fixed $1 < p < 2$ and $\alpha \in (0,1)$, consider a function $U_{p,\alpha} : \mathbb{R}^2 \times (0,\infty)^2 \to \mathbb{R}$, given by

$$U_{p,\alpha}(x,y,u,v) = (x^2+y^2)^{p/2}(u\vee v)^{-\alpha(p-1)} - \frac{2^{1+\alpha(p-1)}(\alpha(p-1)+1)}{(1-\alpha)(p-1)}|x|^p u^{-\alpha(p-1)}.$$

Observe that $U_{p,\alpha}$ satisfies

$$U_{p,\alpha}(x,y,u,u) \leq 0 \qquad \text{if } |y|\leq |x|. \tag{3.1}$$

Indeed, we have $(u\vee v)^{-\alpha(p-1)} \leq u^{-\alpha(p-1)}$, $(x^2+y^2)^{p/2} \leq 2^{p/2}|x|^p \leq 2|x|^p$ and

$$\frac{2^{1+\alpha(p-1)}(\alpha(p-1)+1)}{(1-\alpha)(p-1)} \geq 2.$$

Furthermore, we have $(x^2+y^2)^{p/2} \geq |y|^p$, which implies

$$U_{p,\alpha}(x,y,u,v) \geq |y|^p v^{-\alpha(p-1)} - \frac{2^{1+\alpha(p-1)}(\alpha(p-1)+1)}{(1-\alpha)(p-1)}|x|^p u^{-\alpha(p-1)} \tag{3.2}$$

provided $v \geq u$. The main property of $U_{p,\alpha}$ is the following concavity-type condition.

Theorem 3.1. *Suppose that $(x,y,u,v) \in \mathbb{R}^2\times(0,\infty)^2$ satisfies $u\leq v$. Then for any h, $k\in\mathbb{R}$ such that $|k|\leq|h|$ and any $\ell\in(-u,u)$ we have*

$$2U_{p,\alpha}(x,y,u,v) \geq U_{p,\alpha}(x-h,y-k,u-\ell,v) + U_{p,\alpha}(x+h,y+k,u+\ell,v). \tag{3.3}$$

Proof. Fix x, y, u, v and consider the function $G:[0,1]\to\mathbb{R}$ given by

$$\begin{aligned}G(t) &= ((x+th)^2+(y+tk)^2)^{p/2}v^{-\alpha(p-1)}\\ &\quad - \frac{2^{1+\alpha(p-1)}(\alpha(p-1)+1)}{(1-\alpha)(p-1)}|x+th|^p(u+t\ell)^{-\alpha(p-1)}.\end{aligned}$$

This is almost $U_{p,\alpha}(x+th,y+tk,u+t\ell,v)$; the only difference is that instead of $((u+t\ell)\vee v)^{-\alpha(p-1)}$, we have a somewhat simpler (and larger) term $v^{-\alpha(p-1)}$

appearing in G. Therefore, the claim will follow if we prove that $G(1)+G(-1)\leq 2G(0)$, and this will be accomplished by showing that G is concave. Since G is of class C^1, it is enough to prove that $G''(t)\leq 0$ for all $t\in(0,1)$ such that the second derivative exists. To ease the computations, note that G is a difference of two terms; let us analyze each term separately. First compute that

$$\begin{aligned}&\frac{d}{dt}\big((x+th)^2+(y+tk)^2\big)^{p/2}\\&=p\big((x+th)^2+(y+tk)^2\big)^{p/2-1}\big[(x+th)h+(y+tk)k\big]\end{aligned}$$

Therefore, differentiating once again, we get

$$\begin{aligned}&\frac{d^2}{dt^2}\big((x+th)^2+(y+tk)^2\big)^{p/2}\\&=p(p-2)\big((x+th)^2+(y+tk)^2\big)^{p/2-2}\big[(x+th)h+(y+tk)k\big]^2\\&\quad+p\big((x+th)^2+(y+tk)^2\big)^{p/2-1}(h^2+k^2).\end{aligned}$$

This, after some manipulations, can be expressed as $I_1+I_2+I_3$, where

$$\begin{aligned}I_1&=p((x+th)^2+(y+tk)^2)^{p/2-2}((p-1)(x+th)^2+(y+tk)^2)h^2,\\I_2&=2p(p-2)(x+th)(y+tk)((x+th)^2+(y+tk)^2)^{p/2-2}hk\\&\leq p(2-p)((x+th)^2+(y+tk)^2)^{p/2-1}h^2,\\I_3&=p((x+th)^2+(y+tk)^2)^{p/2-2}((x+th)^2+(p-1)(y+tk)^2)k^2\\&\leq p((x+th)^2+(y+tk)^2)^{p/2-2}((x+th)^2+(p-1)(y+tk)^2)h^2.\end{aligned}$$

This implies

$$\frac{d^2}{dt^2}\big((x+th)^2+(y+tk)^2\big)^{p/2}\leq 2p((x+th)^2+(y+tk)^2)^{p/2-1}h^2.$$

Furthermore, we have $v\geq u\geq(u+t\ell)/2$ and hence

$$\begin{aligned}&\frac{d^2}{dt^2}\big((x+th)^2+(y+tk)^2\big)^{p/2}v^{-\alpha(p-1)}\\&\qquad\leq 2^{1+\alpha(p-1)}p((x+th)^2+(y+tk)^2)^{p/2-1}(u+t\ell)^{-\alpha(p-1)}h^2.\end{aligned}\tag{3.4}$$

Next, let us handle the second "part" of G. If $x+th\neq 0$, we compute that

$$\frac{d^2}{dt^2}\big(|x+th|^p(u+t\ell)^{-\alpha(p-1)}\big)=\big\langle A(h,\ell),(h,\ell)\big\rangle,\tag{3.5}$$

where the 2×2 matrix $A=(A_{ij})_{i,j=1}^2$ has the entries

$$\begin{aligned}A_{11}&=p(p-1)|x+th|^{p-2}(u+t\ell)^{-\alpha(p-1)},\\A_{12}=A_{21}&=-\alpha p(p-1)|x+th|^{p-1}(u+t\ell)^{-\alpha(p-1)-1}\operatorname{sgn}(x+th),\\A_{22}&=\alpha(p-1)(\alpha(p-1)+1)|x+th|^p(u+t\ell)^{-\alpha(p-1)-2}.\end{aligned}$$

We have $A_{11}>0$ and $\det A>0$, so A is nonnegative-definite. Actually, this will still be true if replace A_{11} by γA_{11}, with

$$\gamma=\frac{\alpha p}{\alpha(p-1)+1},$$

since then the determinant of the matrix vanishes. Therefore, we may write

$$\begin{aligned}\frac{d^2}{dt^2}&\left(|x+th|^p(u+t\ell)^{-\alpha(p-1)}\right)\\ &\geq (1-\gamma)A_{11}h^2\\ &= \frac{p(p-1)(1-\alpha)}{\alpha(p-1)+1}|x+th|^{p-2}(u+t\ell)^{-\alpha(p-1)}h^2.\end{aligned} \tag{3.6}$$

Combining this inequality with (3.4), we obtain the desired bound $G''(t) \leq 0$. This proves the claim. □

The function $U_{p,\alpha}$ leads to the following L^p estimate related to (1.6).

Theorem 3.2. *Let $1 < p < 2$ and $r > 1$. For any weight w, any function f on $[0, 1)$ and any sequence ε with values in $[-1, 1]$, the associated multiplier T satisfies*

$$\left|\left|\frac{Tf}{\mathcal{M}_r w}\right|\right|_{L^p(\mathcal{M}_r w)} \leq \left(\frac{2^{1+(p-1)/r}(p+r-1)}{(r-1)(p-1)}\right)^{1/p} \left|\left|\frac{f}{w}\right|\right|_{L^p(w)}. \tag{3.7}$$

Proof. By a straightforward approximation, we may and do assume that $w > 0$. Let $g = Tf$ and put $\alpha = 1/r$. The assertion is equivalent to

$$\int_0^1 |g(s)|^p(\mathcal{M}_r w)^{1-p}(s)\mathrm{d}s \leq \frac{2^{1+\alpha(p-1)}(\alpha(p-1)+1)}{(1-\alpha)(p-1)}\int_0^1 |f(s)|^p w^{1-p}(s)\mathrm{d}s$$

or, if we consider the weight $\tilde{w} = w^r$,

$$\int_0^1 |g(s)|^p(M_d\tilde{w})^{-\alpha(p-1)}(s)\mathrm{d}s \leq \frac{2^{1+\alpha(p-1)}(\alpha(p-1)+1)}{(1-\alpha)(p-1)}\int_0^1 |f(s)|^p \tilde{w}^{-\alpha(p-1)}(s)\mathrm{d}s.$$

We will first show this in the case when f and $\tilde{w}$ (and hence also g) have finite expansion in the Haar system, i.e.,

$$f = f_N = \sum_{k=0}^N a_k h_k, \qquad \tilde{w} = \tilde{w}_N = \sum_{k=0}^N b_k h_k$$

for some integer N and some coefficients $a_0, a_1, \ldots, a_N, b_0, b_1, \ldots, b_N$. By (3.2), this will follow once we have proved that

$$\int_0^1 U_{p,\alpha}\left(f_N, g_N, \tilde{w}_N, \max_{0\leq m\leq N}\tilde{w}_m\right)\mathrm{d}s \leq 0.$$

Note that the integrand makes sense: we have $\tilde{w}_N > 0$ and $\max_{0\leq m\leq N}\tilde{w}_m > 0$ by the positivity of w which we have assumed at the beginning. We will show first that for each $n = 0, 1, \ldots, N-1$ we have

$$\begin{aligned}\int_0^1 &U_{p,\alpha}\left(f_n, g_n, \tilde{w}_n, \max_{0\leq m\leq n}\tilde{w}_m\right)\mathrm{d}s\\ &\geq \int_0^1 U_{p,\alpha}\left(f_{n+1}, g_{n+1}, \tilde{w}_{n+1}, \max_{0\leq m\leq n+1}\tilde{w}_m\right)\mathrm{d}s.\end{aligned} \tag{3.8}$$

Let I be the support of h_{n+1}. The pairs f_n, f_{n+1}; g_n, g_{n+1}; $\tilde{w}_n$, $\tilde{w}_{n+1}$; and $\max_{0\leq m\leq n}\tilde{w}_m$, $\max_{0\leq m\leq n+1}\tilde{w}_m$ coincide outside I, so we must show the appropriate inequality for the integrals over I. However, f_n, g_n, $\tilde{w}_n$ and $\max_{0\leq m\leq n}\tilde{w}_m$

are constant on I; denoting the appropriate values by x, y, u and v, we easily see that (3.8) is equivalent to

$$\begin{aligned}|I|U_{p,\alpha}(x,y,u,v) \geq \frac{|I|}{2}&U_{p,\alpha}(x-a_{n+1},y-\varepsilon_{n+1}a_{n+1},u-b_{n+1},v)\\ &+\frac{|I|}{2}U_{p,\alpha}(x+a_{n+1},y+\varepsilon_{n+1}a_{n+1},u+b_{n+1},v).\end{aligned}$$

This follows directly from (3.3). Thus, (3.8) holds true and hence

$$\begin{aligned}\int_0^1 U_{p,\alpha}\left(f_N,g_N,\tilde{w}_N,\max_{0\leq m\leq N}\tilde{w}_m\right)\mathrm{d}s &\leq \int_0^1 U_{p,\alpha}(f_0,g_0,\tilde{w}_0,\tilde{w}_0)\mathrm{d}s\\ &= U_{p,\alpha}(a_0,\varepsilon_0 a_0,b_0,b_0).\end{aligned}$$

The latter expression is nonpositive, due to (3.1), and hence (3.7) holds true for functions with finite expansions. The passage to the general case exploits a simple limiting argument. Let $f=\sum_{k=0}^\infty a_kh_k$, $\tilde{w}=\sum_{k=0}^\infty b_kh_k$ be arbitrary. By what we have just proved,

$$\begin{aligned}&\int_0^1 |\sum_{k=0}^n \varepsilon_k a_k h_k|^p (M_d\tilde{w})^{-\alpha(p-1)}\mathrm{d}s\\ &\leq \int_0^1 |\sum_{k=0}^n \varepsilon_k a_k h_k|^p \left(\sup_{0\leq m\leq n}\sum_{k=0}^m b_kh_k\right)^{-\alpha(p-1)}\mathrm{d}s\\ &\leq \frac{2^{1+\alpha(p-1)}(\alpha(p-1)+1)}{(1-\alpha)(p-1)}\int_0^1\left|\sum_{k=0}^n a_kh_k\right|^p\left(\sum_{k=0}^n b_kh_k\right)^{-\alpha(p-1)}\mathrm{d}s.\end{aligned}$$

However, the function $(x,u)\mapsto |x|^pu^{-\alpha(p-1)}$ is convex on $\mathbb{R}\times(0,\infty)$: see (3.5) and use the fact that the matrix A appearing there is nonnegative-definite. Furthermore, the pair $(\sum_{k=0}^n a_kh_k,\sum_{k=0}^n b_kh_k)$ is the projection of $(f,\tilde{w})$ on the subspace generated by the first $n+1$ Haar functions. Therefore, by Jensen's inequality,

$$\int_0^1\left|\sum_{k=0}^n a_kh_k\right|^p\left(\sum_{k=0}^n b_kh_k\right)^{-\alpha(p-1)}\mathrm{d}s \leq \int_0^1 |f|^p\tilde{w}^{-\alpha(p-1)}\mathrm{d}s.$$

It remains to plug this into the previous estimate, let $n\to\infty$ and use Fatou's lemma on the left-hand side. □

We will also need the following simple lemma concerning the Haar system. Roughly speaking, it asserts that any Haar function h_i can be replaced, in distribution, by sums of Haar functions with arbitrarily large indices and arbitrarily small coefficients. For any $J\subset\mathbb{N}$ and any nonnegative integer K, we set $J(K)=\{j\in J: j\leq K\}$.

Lemma 3.3. *Let I be a dyadic subinterval of $[0,1)$ and fix positive integers M, N. Then there is a set $J\subset\{N,N+1,N+2,\ldots\}$ such that for each $j\in J$, the support of h_j is contained in I, the sums*

$$\sum_{j\in J(K)} M^{-1}h_j,\qquad K=1,\,2,\,\ldots,$$

take values in $[-1,1]$*, and*

$$\left|\left\{s : \sum_{j\in J} M^{-1}h_j(s) = 1\right\}\right| = \frac{|I|}{2}, \qquad \left|\left\{s : \sum_{j\in J} M^{-1}h_j(s) = -1\right\}\right| = \frac{|I|}{2}.$$

Proof. This is straightforward. By the structural, dilation properties of the Haar system we may assume that $I = [0,1)$. Let $(c_n)_{n\geq 0}$ be a sequence defined inductively by

$$c_0 = 1, \qquad c_n = \begin{cases} 0 & \text{if } \sum_{j=0}^{n-1} c_j M^{-1}h_j \in \{-1,1\} \text{ on the support of } h_n, \\ 1 & \text{otherwise.} \end{cases}$$

It is easy to see that for each n, the sum $\sum_{j=0}^{n} c_j M^{-1}h_j$ takes values in $[-1,1]$ and converges, as $n\to\infty$, to a function taking values ± 1 only (on the sets of measure equal to $1/2$). This is precisely the claim. $\square$

By a simple inductive argument, the above statement implies the following.

Corollary 3.4. *Fix a positive integer* M*. Then there is a sequence* $(d_k)_{k=1}^{\infty}$ *with values in* $[-M^{-1}, M^{-1}]$ *and a sequence* N_0*,* N_1*,* N_2*, …,* N_n *of integer-valued functions on* $[0,1)$ *such that* $N_0 \equiv 0$*,* $N_0 < N_1 < \ldots < N_n$ *on* $[0,1)$ *and*

$$\tilde{H}_n = \left(h_0, \sum_{k=1}^{N_1} d_k h_k, \sum_{k=N_1+1}^{N_2} d_k h_k, \ldots, \sum_{k=N_{n-1}+1}^{N_n} d_k h_k\right)$$

has the same distribution as $H_n = (h_0, h_1, h_2, \ldots, h_n)$*. That is, for any Borel subset* B *of* $\mathbb{R}^{n+1}$*,*

$$|\{s\in[0,1) : \tilde{H}_n(s) \in B\}| = |\{s\in[0,1) : H_n(s)\in B\}|.$$

Furthermore, each sum $\sum_{k=N_{j-1}+1}^{\ell} d_k h_k$ *(where* $N_{j-1}+1 \leq \ell \leq N_j$*) takes values in the interval* $[-1,1]$*.*

Equipped with the above facts, we are ready to establish Theorem 1.2. Actually, we will be able to show a whole family of inequalities (1.6) (indexed by $p\in(1,2)$).

Theorem 3.5. *Let* $r > 1$ *and* $1 < p < 2$*. Then for any weight* w *and any function* $f : [0,1) \to \mathbb{R}$ *we have*

(3.9)

$$w(\{s\in[0,1) : |T_\varepsilon f(s)| \geq \mathcal{M}_r w(s)\}) \leq \left(1 + \frac{2^{1+(2p-1)/r}(p+r-1)}{(r-1)(p-1)}\right)||f||_{L^1}.$$

Proof. We may assume that f, w have finite Haar expansions

$$f = \sum_{k=0}^{n} a_k h_k, \qquad w = \sum_{k=0}^{n} b_k h_k.$$

Furthermore, we may assume that w is bounded away from 0: there is $\eta > 0$ such that $w \geq \eta$ almost surely. Let us also fix $\delta\in(0,\eta)$.

The remainder of the proof splits into two major parts.

Step 1. Reduction to functions having small Haar coefficients. Since f, w have finite Haar expansions, the function $\tilde{w} = w^r$ also has this property: $\tilde{w} = \sum_{k=0}^{n} c_k h_k$: indeed, $\tilde{w}$ is measurable with respect to the σ-algebra generated by h_0, h_1, …, h_n.

Fix a large positive integer M and let $(d_k)_{k=1}^\infty$, $0 = N_0 < N_1 < \ldots < N_n$ be the sequences guaranteed by Corollary 3.4. Put

$$F = a_0h_0 + \sum_{j=1}^n \sum_{k=N_{j-1}+1}^{N_j} a_jd_kh_k, \qquad W = b_0h_0 + \sum_{j=1}^n \sum_{k=N_{j-1}+1}^{N_j} b_jd_kh_k$$

and

$$\tilde{W} = W^r = c_0h_0 + \sum_{j=1}^n \sum_{k=N_{j-1}+1}^{N_j} c_jd_kh_k.$$

Since (a_k), (b_k), (c_k) are finite sequences, they are bounded and hence, increasing M if necessary, we may and do assume that the Haar coefficients of F, W and $\tilde{W}$ (except for the first ones!) are smaller in absolute value than δ. By the above corollary, we see that the distribution of the function $((F_0, W_0, \tilde{W}_0), (F_{N_1}, W_{N_1}, \tilde{W}_{N_1}), \ldots, (F_{N_n}, W_{N_n}, \tilde{W}_n))$ is the same as that of $((f_0, w_0, \tilde{w}_0), (f_1, w_1, \tilde{w}_1), \ldots, (f_n, w_n, \tilde{w}_n))$. In particular, this implies the existence of a sequence $\beta = (\beta_k)_{k=0}^\infty$ with values in $[-1, 1]$ such that $T_\beta F$ has the same distribution as $T_\varepsilon f$ (where ε is an arbitrary sequence with values in $[-1, 1]$).

As we shall see, it suffices to study (3.9) for these new functions F, $T_\beta F$ and W. To this end, we will need to compare the r-maximal functions $\mathcal{M}_r w$ and $\mathcal{M}_r W$. For any integer k we have $\tilde{w}_{k+1} \leq 2\tilde{w}_k$ almost everywhere, since w is a nonnegative function. Indeed, if I is a dyadic subinterval of $[0, 1]$ of measure 2^{-k-1} and J is the parent of I, that is, I is the left- or the right half of J, then

$$\tilde{w}_{k+1}|_I = \frac{1}{|I|}\int_I \tilde{w}\mathrm{d}x \leq \frac{1}{|I|}\int_J \tilde{w}\mathrm{d}x = \frac{2}{|J|}\int_J \tilde{w}\mathrm{d}x = 2\tilde{w}_k|_J = 2\tilde{w}_k|_I.$$

Therefore, $\tilde{W}_{N_{k+1}} \leq 2\tilde{W}_{N_k}$ almost everywhere, by the equality of distributions of the sequences $(\tilde{w}_0, \tilde{w}_1, \ldots, \tilde{w}_n)$ and $(\tilde{W}_0, \tilde{W}_{N_1}, \ldots, \tilde{W}_{N_n})$. In consequence (see the last sentence of the above corollary), for almost all s and any $\ell \in [N_k(s) + 1, N_{k+1}(s)]$, we have $\tilde{W}_\ell(s) \leq 2\tilde{W}_{N_k}(s)$ and hence

$$\mathcal{M}_r W(s) = \sup_\ell ((W^r)_\ell)^{1/r}(s) \leq 2^{1/r} \sup_k ((W^r)_{N_k})^{1/r}(s).$$

The above considerations imply the following upper bound for the left-hand side of (3.9):

$$\begin{aligned}\int_0^1 w1_{\{|T_\varepsilon f| \geq \mathcal{M}_r w\}}\mathrm{d}s &= \int_0^1 W1_{\{|T_\beta F| \geq \sup_k ((W^r)_{N_k})^{1/r}\}}\mathrm{d}s \\ &\leq \int_0^1 W1_{\{|T_\beta F| \geq 2^{-1/r}\mathcal{M}_r W\}}\mathrm{d}s.\end{aligned}$$

Hence we need to provide an appropriate bound for the latter expression, which concerns functions/weights with small Haar coefficients.

Step 2. An extrapolation and stopping-time arguments. Let $\tau : [0, 1) \to \mathbb{N} \cup \{\infty\}$ be given by

$$\tau(s) = \inf\{n : |F_n(s)| > W_n(s)\},$$

with the usual convention $\inf\emptyset=\infty$. We have

$$\begin{aligned}
&\int_0^1 W(s)1_{\{|T_\beta F(s)|\geq 2^{-1/r}\mathcal{M}_rW(s)\}}\mathrm{d}s\\
&\qquad=\int_0^1 W(s)1_{\{|T_\beta F(s)|\geq 2^{-1/r}\mathcal{M}_rW(s),\,\tau(s)<\infty\}}\mathrm{d}s\\
&\qquad\quad+\int_0^1 W(s)1_{\{|T_\beta F(s)|\geq 2^{-1/r}\mathcal{M}_rW(s),\tau(s)=\infty\}}\mathrm{d}s.
\end{aligned}\tag{3.10}$$

Let us handle each term on the right separately. We have

$$\begin{aligned}
\int_0^1 W(s)1_{\{|T_\beta F(s)|\geq 2^{-1/r}\mathcal{M}_rW(s),\,\tau(s)<\infty\}}\mathrm{d}s &\leq \int_0^1 W(s)1_{\{\tau(s)<\infty\}}\mathrm{d}s\\
&=\sum_{n=0}^\infty\int_0^1 W(s)1_{\{\tau(s)=n\}}\mathrm{d}s\\
&=\sum_{n=0}^\infty\int_0^1 W_n(s)1_{\{\tau(s)=n\}}\mathrm{d}s,
\end{aligned}$$

where in the last line we have used the fact that the set $\{\tau(s)=n\}$ is measurable with respect to the σ-algebra generated by h_0, h_1, $\ldots$, h_n. But on this set we have $|F_n(s)|/W_n(s)>1$, so

$$\begin{aligned}
\sum_{n=0}^\infty\int_0^1 W_n(s)1_{\{\tau(s)=n\}}\mathrm{d}s &\leq \sum_{n=0}^\infty\int_0^1 |F_n(s)|1_{\{\tau(s)=n\}}\mathrm{d}s\\
&\leq \sum_{n=0}^\infty\int_0^1 |F(s)|1_{\{\tau(s)=n\}}\mathrm{d}s\leq\int_0^1|F(s)|\mathrm{d}s.
\end{aligned}$$

To deal with the second term in (3.10), consider the "stopped" function F^τ given by $F^\tau(s)=F_{\tau(s)}(s)$. Then $T_\beta(F^\tau)=(T_\beta F)^\tau$: both sides define a "truncated" version of $T_\beta F$ in which the summation runs over all indices smaller than τ. One easily checks that we have the pointwise bound $\mathcal{M}_rW^\tau\leq\mathcal{M}_rW$, so

$$\begin{aligned}
&\int_0^1 W(s)1_{\{|T_\beta F(s)|\geq 2^{-1/r}\mathcal{M}_rW(s),\tau(s)=\infty\}}\mathrm{d}s\\
&\qquad\leq\int_0^1 W(s)1_{\{|T_\beta F^\tau(s)|\geq 2^{-1/r}\mathcal{M}_rW^\tau(s),\tau(s)=\infty\}}\mathrm{d}s\\
&\qquad\leq\int_0^1 W(s)1_{\{|T_\beta F^\tau(s)|\geq 2^{-1/r}\mathcal{M}_rW^\tau(s)\}}\mathrm{d}s\\
&\qquad=\int_0^1 W^\tau(s)1_{\{|T_\beta F^\tau(s)|\geq 2^{-1/r}\mathcal{M}_rW^\tau(s)\}}\mathrm{d}s\\
&\qquad\leq\int_0^1 \mathcal{M}_rW^\tau(s)1_{\{|T_\beta F^\tau(s)|\geq 2^{-1/r}\mathcal{M}_rW^\tau(s)\}}\mathrm{d}s\\
&\qquad\leq 2^{p/r}\int_0^1\frac{|T_\beta F^\tau(s)|^p}{(\mathcal{M}_rW^\tau(s))^{p-1}}\mathrm{d}s,
\end{aligned}$$

for any $p \in (1,2)$. Thus, by (3.7),

$$\int_0^1 W(s) 1_{\{|T_\beta F(s)| \geq 2^{-1/r} \mathcal{M}_r w(s), \tau(s)=\infty\}} \mathrm{d}s$$
$$\leq \frac{2^{1+(2p-1)/r}(p+r-1)}{(r-1)(p-1)} \int_0^1 |F^\tau(s)| \left(\frac{|F^\tau(s)|}{W^\tau(s)}\right)^{p-1} \mathrm{d}s.$$

However, combining the definition of τ with the inequality $W \geq \eta$ and the fact that the Haar coefficients of F and W are smaller than δ, we get

$$\frac{|F^\tau(s)|}{W^\tau(s)} \leq \frac{|F_{\tau(s)-1}(s)| + \delta}{|W_{\tau(s)-1}| - \delta} \leq \frac{W_{\tau(s)-1}(s) + \delta}{W_{\tau(s)-1}(s) - \delta} \leq 1 + \frac{2\delta}{\eta - \delta}.$$

Furthermore, we have $\int_0^1 |F^\tau(s)| \mathrm{d}s \leq \int_0^1 |F(s)| \mathrm{d}s$, by Jensen's inequality. Putting all the above facts together, we get

$$\int_0^1 W(s) 1_{\{|T_\beta F(s)| \geq 2^{-1/r} \mathcal{M}_r W(s)\}} \mathrm{d}s \leq C_{p,r} \int_0^1 |F(s)| \mathrm{d}s, \tag{3.11}$$

where

$$C_{p,r} = 1 + \frac{2^{1+(2p-1)/r}(p+r-1)}{(r-1)(p-1)} \left(1 + \frac{2\delta}{\eta - \delta}\right)^{p-1}.$$

It remains to let $\delta \to 0$ to get the claim. □

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