ON THE BELLMAN FUNCTION OF NAZAROV, TREIL AND VOLBERG

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ABSTRACT. We find the explicit formula for the Bellman function associated with the dual bound related to the unconditional constant of the Haar system.

1. INTRODUCTION

Let $\mathfrak{h} = (\mathfrak{h}_n)_{n\geq 0}$ denote the standard Haar system on $[0, 1)$. Recall that this fmaily of functions is given by

$$\begin{aligned} &\mathfrak{h}_0 = [0, 1), && \mathfrak{h}_1 = [0, 1/2) - [1/2, 1), \\ &\mathfrak{h}_2 = [0, 1/4) - [1/4, 1/2), && \mathfrak{h}_3 = [1/2, 3/4) - [3/4, 1), \\ &\mathfrak{h}_4 = [0, 1/8) - [1/8, 1/4), && \mathfrak{h}_5 = [1/4, 3/8) - [3/8, 1/2), \\ &\mathfrak{h}_6 = [1/2, 5/8) - [5/8, 3/4), && \mathfrak{h}_7 = [3/4, 7/8) - [7/8, 1),\ \ldots \end{aligned}$$

where we have identified a set with its indicator function. A classical result of Schauder [11] states that the Haar system forms a basis of $L^p = L^p(0, 1)$, $1 \leq p < \infty$ (with the underlying measure being the Lebesgue measure). That is, for every $f \in L^p$ there is a unique sequence $a = (a_n)_{n\geq 0}$ of real numbers satisfying $||f - \sum_{k=0}^n a_k\mathfrak{h}_k||_p \to 0$. Let $\beta_p(\mathfrak{h})$ be the unconditional constant of $\mathfrak{h}$, i.e. the least extended real number β with the following property: if n is a nonnegative integer and $a_0, a_1, \ldots, a_n$ are real numbers such that $||\sum_{k=0}^n a_k\mathfrak{h}_k||_p \leq 1$, then

$$\left|\left|\sum_{k=0}^n \varepsilon_k a_k \mathfrak{h}_k\right|\right|_p \leq \beta \tag{1.1}$$

for all choices of signs $\varepsilon_0, \varepsilon_1, \ldots, \varepsilon_n$. Using Paley's inequality [10], Marcinkiewicz [5] proved that $\beta_p(\mathfrak{h}) < \infty$ if and only if $1 < p < \infty$. The precise value of $\beta_p(\mathfrak{h})$ was determined by Burkholder in [1]: we have

$$\beta_p(\mathfrak{h}) = p^* - 1, \qquad 1 < p < \infty, \tag{1.2}$$

where $p^* = \max\{p, p/(p-1)\}$. Actually, the constant remains the same if we allow the coefficients $a_0, a_1, a_2, \ldots$ to take values in a Hilbert space $\mathcal{H}$ (cf. [2]). This result can be further generalized: if $(a_n)_{n\geq 0}$, $(b_n)_{n\geq 0}$ are sequences with $\mathcal{H}$-valued terms satisfying

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$|a_n| \leq |b_n|$ for each n, then

$$\left\| \sum_{k=0}^{n} a_k \mathfrak{h}_k \right\|_p \leq (p^* - 1) \left\| \sum_{k=0}^{n} b_k \mathfrak{h}_k \right\|_p, \qquad n = 0, 1, 2, \ldots, \quad 1 < p < \infty, \tag{1.3}$$

and the constant $p^* - 1$ cannot be replaced by a smaller number. The original proof of this fact exploits the properties of a certain special object, the associated Bellman function (for details, see Burkholder [1, 2, 3]).

In the nineties, Nazarov, Treil and Volberg (cf. [7] and a preprint version of [8]) proposed a different, dual approach to the above $p^* - 1$ problems. Namely, they proved that (1.2), (1.3) can be deduced from the existence of a function B_p defined on the set

$$\mathcal{D} = \big\{(\zeta, \eta, Z, H) \in \mathcal{H} \times \mathcal{H} \times [0, \infty) \times [0, \infty) : Z \geq |\zeta|^p,\ H \geq |\eta|^q\big\},$$

satisfying the following two conditions:

(I) We have $0 \leq B_p(\zeta, \eta, Z, H) \leq (p^* - 1) Z^{1/p} H^{1/q}$ on $\mathcal{D}$.

(II) For any $a_\pm = (\zeta_\pm, \eta_\pm, Z_\pm, H_\pm) \in \mathcal{D}$, we have the concavity-type condition

$$B_p\left(\frac{a_- + a_+}{2}\right) - \frac{B_p(a_-) + B_p(a_+)}{2} \geq \left|\frac{\zeta_+ - \zeta_-}{2}\right| \left|\frac{\eta_+ - \eta_-}{2}\right|.$$

The existence of such a function can be extracted from Burkholder's works [1] and [2]. As shown later by Nazarov and Volberg [9] and Dragičević and Volberg [4], this special object can be further exploited to yield interesting tight L^p bounds for various classes of Fourier multipliers. There is an intriguing question about the explicit formulas for the above functions B_p. What may be quite surprising, this problem has been solved so far only in the particular case $p = 2$. For this value of the parameter p, Nazarov, Treil and Volberg [7, 9] showed that

$$\mathbb{B}_2(\zeta, \eta, Z, H) = \sqrt{(Z - |\zeta|^2)(H - |\eta|^2)} \tag{1.4}$$

works fine. The paper [7] contains also some attempts to find B_p explicitly for other values of p, but with no success. Nevertheless, the authors managed to construct, for each $1 < p < \infty$, a function which satisfies (II) and a version of (I), in which $p^* - 1$ is replaced by a slightly larger constant. The purpose of this note is to fill this gap and give an explicit formula for B_p satisfying (I) and (II), for all $1 < p < \infty$.

Suppose that $1 < p \leq 2$ and introduce the function $\mathcal{B}_p : \mathcal{D} \to \mathbb{R}$ as follows: if $|\eta|^q Z \geq |\zeta|^p H$, then

$$\mathcal{B}_p(\zeta, \eta, Z, H) = \frac{(H - |\eta|^q)^{1/q}(Z - |\zeta|^p)^{1/p}}{p - 1}.$$

On the other hand, if $|\eta|^q Z < |\zeta|^p H$, then

$$\mathcal{B}_p(\zeta, \eta, Z, H) = \gamma Z^{1/p} H^{1/q} - |\zeta||\eta| Y,$$

where (γ, Y), $0 \leq Y < \gamma < (p-1)^{-1}$ is the unique solution to the system of equations

$$\frac{(1 - (p-1)Y)(1 + Y)^{p-1}}{(1 - (p-1)\gamma)(1 + \gamma)^{p-1}} = \frac{Z}{|\zeta|^p}, \qquad \frac{Y(1 + Y)^{p-2}}{\gamma(1 + \gamma)^{p-2}} = \left(\frac{|\eta|^q Z}{|\zeta|^p H}\right)^{1/q} \tag{1.5}$$

(we will show the existence and the uniqueness of the pair (γ, Y) later on).

Here is the precise statement of our main result. In what follows, $q = p/(p-1)$ denotes the harmonic conjugate to p.

Theorem 1.1. *For any $1 < p \leq 2$, the function $\mathcal{B}_p$ satisfies (I) and (II). If $p > 2$, then the function $(\zeta, \eta, Z, H) \mapsto \mathcal{B}_q(\eta, \zeta, H, Z)$ satisfies (I) and (II).*

It is not difficult to check that when $p = 2$, we get the function (1.4): the system (1.5) can be solved explicitly, and in both cases $|\eta|^2 Z \geq |\zeta|^2 H$, $|\eta|^2 Z < |\zeta|^2 H$ we get the expression $\sqrt{(Z-|\zeta|^2)(H-|\eta|^2)}$. For other values of the parameter p, no compact formula for $\mathcal{B}_p$ seems to exist.

A few words about the proof of the above statement are in order. One can establish the theorem by the direct verification of the conditions (I) and (II), but this approach is extremely technical, and it does not give the indication on how the special function was constructed. Thus, to simplify and clarify the reasoning, we have decided to propose a different proof. There is an abstract formula for a function satisfying the conditions (I) and (II), due to Nazarov and Treil [7] (see also Nazarov and Volberg [9] and Dragičević and Volberg [4]). We will derive the formula explicitly, actually with the use of a slightly more general, probabilistic setting. This approach has also the advantage that it shows how to handle complicated Bellman functions (i.e., depending on many variables) by solving associated less dimensional problems.

We have organized the remainder of this paper as follows. In the next section we present the abstract formula of Nazarov and Treil, for the function satisfying (I) and (II), and express it in the probabilistic language of martingales. Section 3 contains some auxiliary material: we establish there a family of auxiliary L^p estimates for martingales. The final two sections are devoted to the proof of our main result, Theorem 1.1.

2. An abstract formula

Let us start with introducing the necessary notation. Let $\mathfrak{D}$ denote the lattice of dyadic subintervals of $[0,1)$. Given $I \in \mathfrak{D}$, its left and right halves will be denoted by I_- and I_+, respectively. Furthermore, for $I \in \mathfrak{D}$ and a locally integrable function φ on $[0,1)$, we denote by φ_I the average of φ over I: $\varphi_I = \frac{1}{|I|}\int_I \varphi$. For a fixed $(\zeta,\eta,Z,H) \in \mathcal{D}$, consider all integrable φ, ψ on $[0,1)$ which satisfy $\varphi_{[0,1)} = \zeta$, $\psi_{[0,1)} = \eta$, $(|\varphi|^p)_{[0,1)} \leq Z$ and $(|\psi|^q)_{[0,1)} \leq H$ (it is not difficult to see that such functions exist). Then, as shown by Nazarov and Treil [7], the function

$$\mathbb{B}_p(\zeta,\eta,Z,H) = \frac{1}{4}\sup \sum_{I\in\mathfrak{D}} |\varphi_{I_+} - \varphi_{I_-}||\psi_{I_-} - \psi_{I_+}||I| \tag{2.1}$$

satisfies (I) and (II). Here the supremum is taken over all φ, ψ as above. We will show that the function of Theorem 1.1 coincides with $\mathbb{B}_p$. Observe that the roles of φ and ψ are symmetric, and therefore we immediately see that $\mathbb{B}_p(\zeta,\eta,Z,H) = \mathbb{B}_q(\eta,\zeta,H,Z)$ for all $(\zeta,\eta,Z,H) \in \mathcal{D}$. Consequently, we will be done with Theorem 1.1 if we manage to establish the equality $\mathbb{B}_p = \mathcal{B}_p$ for $1 < p < 2$.

Actually, it will be convenient for us to work with an appropriate probabilistic version of (2.1). Assume that $(\Omega,\mathcal{F},\mathbb{P})$ is a probability space, equipped with the filtration $(\mathcal{F}_n)_{n\geq 0}$, a nondecreasing sequence of sub-σ-algebras of $\mathcal{F}$. Let f, g be $\mathcal{H}$-valued martingales adapted to $(\mathcal{F}_n)_{n\geq 0}$, and denote by $(df_n)_{n\geq 0}$, $(dg_n)_{n\geq 0}$ the associated difference sequences:

$$df_0 = f_0, \qquad df_n = f_n - f_{n-1}, \quad n = 1,\, 2,\, \ldots,$$

and similarly for dg. Following Burkholder [1], we say that g is *differentially subordinate* to f, if for any $n \geq 0$ we have $|dg_n| \leq |df_n|$ almost surely.

The triple $([0,1),\mathcal{B}([0,1)),|\cdot|)$ forms a probability space and $\mathfrak{D}$ gives rise to the corresponding dyadic filtration (for each n, the σ-algebra $\mathcal{F}_n$ is generated by the Haar functions $\mathfrak{h}_0$, $\mathfrak{h}_1$, ..., $\mathfrak{h}_n$). The adapted martingales in this special setting are called *dyadic* (or Haar) martingales. We easily check that the formula (2.1) can be rewritten as

$$\mathbb{B}_p(\zeta, \eta, Z, H) = \sup \mathbb{E} \sum_{n=1}^{\infty} |df_n||dh_n|,$$

where the supremum is taken over the class of all dyadic martingales $f = (f_n)_{n\geq 0}$, $h = (h_n)_{n\geq 0}$ such that $f_0 \equiv \zeta$, $\sup_n \mathbb{E}|f_n|^p \leq Z$, $h_0 \equiv \eta$ and $\sup_n \mathbb{E}|h_n|^q \leq H$. Let us transform this formula to a more convenient form. First, note that we can write

$$\mathbb{B}_p(\zeta, \eta, Z, H) = \sup \mathbb{E} \sum_{n=1}^{\infty} \langle dg_n, dh_n\rangle$$

($\langle\cdot,\cdot\rangle$ is the scalar product in $\mathcal{H}$), where the supremum is taken over all f, h as above and all dyadic martingales g which are differentially subordinate to f. This can be further simplified. Pick the martingales f, g, h as above, and note that the first two of them are bounded in L^p, while the last one is bounded in L^q. Thus, using classical results from the martingale theory, there are random variables f_∞, g_∞ and h_∞ such that $f_n \to f_\infty$, $g_n \to g_\infty$ in L^p and $h_n \to h_\infty$ in L^q. Thus, by the orthogonality of the martingale differences, we get that

$$\begin{aligned} \mathbb{B}_p(\zeta, \eta, Z, H) &= \sup \mathbb{E}\left\langle \sum_{n=1}^{\infty} dg_n, \sum_{n=1}^{\infty} dh_n \right\rangle \\ &= \sup \mathbb{E}\big\langle g_\infty - g_0, h_\infty - h_0\big\rangle \\ &= \sup\big\{\mathbb{E}\big\langle g_\infty, h_\infty\big\rangle - \langle \mathbb{E}g_\infty, \mathbb{E}h_\infty\rangle\big\}, \end{aligned} \tag{2.2}$$

where the supremum is taken over all dyadic martingale triples (f, g, h) such that $f_0 \equiv \zeta$, $\mathbb{E}|f_\infty|^p \leq Z$, $h_0 \equiv \eta$, $\mathbb{E}|h_\infty|^q \leq H$ and g is differentially subordinate to f. This formula immediately shows that $\mathbb{B}_p(\zeta, \eta, Z, H) = \mathcal{B}_p(\zeta, \eta, Z, H)$ if $|\zeta|^p = Z$ or $|\eta|^q = H$; indeed, then the corresponding martingale (f or h) must be constant and hence $\mathbb{B}_p(\zeta, \eta, Z, H) = 0$. Thus, in our considerations below, we will assume that the strict estimates $|\zeta|^p < Z$ and $|\eta|^q < H$ hold true. Another crucial observation, particularly helpful during the study of lower bounds for $\mathbb{B}_p$, is that in the above formula one can consider all (i.e., not necessarily dyadic) martingales. This follows from the results of Maurey [6], see also Section 10 in Burkholder's paper [1].

The proof of Theorem 1.1 will rest on the careful analysis of the above formula for $\mathbb{B}_p$. It will consist of several ingredients, which are presented in the three sections below.

3. L^p BOUNDS FOR DIFFERENTIALLY SUBORDINATE MARTINGALES

We start with a family of certain auxiliary martingale inequalities. For fixed $1 < p < 2$ and $0 < \gamma \leq (p-1)^{-1}$, introduce the function $b_{p,\gamma} : \mathcal{H} \times \mathcal{H} \to \mathbb{R}$ by

$$b_{p,\gamma}(x, y) = \begin{cases} \left(\dfrac{\gamma}{\gamma+1}\right)^{p-2} (|x| + |y|)^{p-1} \left(|y| - \dfrac{|x|}{p-1}\right) & \text{if } |y| < \gamma|x|, \\ |y|^p - \dfrac{(2-p)\gamma^{p-1} + \gamma^{p-2}}{p-1}|x|^p & \text{if } |y| \geq \gamma|x|. \end{cases}$$

It can be easily verified that $b_{p,\gamma}$ is of class C^1 on $\mathcal{H} \times \mathcal{H}$. We will establish the following statement.

Theorem 3.1. *Suppose that f, g are $\mathcal{H}$-valued martingales such that $(f_0, g_0) \equiv (x, y)$ and $\mathbb{P}(|dg_n| \leq |df_n|) = 1$ for all $n \geq 1$. Then for any p and γ as above we have*

$$\mathbb{E}|g_n|^p \leq \frac{(2-p)\gamma^{p-1} + \gamma^{p-2}}{p-1}\mathbb{E}|f_n|^p + b_{p,\gamma}(x, y), \qquad n = 0,\, 1,\, 2,\, \ldots. \tag{3.1}$$

To show this theorem, we will require the following properties of $b_{p,\gamma}$.

Lemma 3.1. *(i) There is an absolute constant $c_{p,\gamma}$ depending only on the parameters indicated, such that*

$$|b_{p,\gamma}(x, y)| \leq c_{p,\gamma}(|x|^p + |y|^p)$$

and

$$\left|\frac{\partial b_{p,\gamma}(x, y)}{\partial x}\right| + \left|\frac{\partial b_{p,\gamma}(x, y)}{\partial y}\right| \leq c_{p,\gamma}(|x|^{p-1} + |y|^{p-1}).$$

(ii) For any x, $y \in \mathcal{H}$ we have the majorization

$$b_{p,\gamma}(x, y) \geq |y|^p - \frac{(2-p)\gamma^{p-1} + \gamma^{p-2}}{p-1}|x|^p. \tag{3.2}$$

(iii) For any $x,\ y,\ h,\ k \in \mathcal{H}$ such that $|k| \leq |h|$, the function

$$F_{x,y,h,k}(t) = b_{p,\gamma}(x + th, y + tk), \qquad t \in \mathbb{R},$$

is concave.

Proof. (i) This is straightforward: we leave the details to the reader.

(ii) Clearly, we may assume that $\mathcal{H} = \mathbb{R}$ and $x,\ y \geq 0$. Furthermore, it suffices to show the majorization for $y < \gamma x$. Finally, by homogeneity, we may assume that $x + y = 1$. Then the bound can be rewritten as

$$\left(\frac{\gamma}{\gamma+1}\right)^{p-2}\left(1 - \frac{px}{p-1}\right) - (1-x)^p + \frac{(2-p)\gamma^{p-1} + \gamma^{p-2}}{p-1}x^p \geq 0$$

for $x \geq (\gamma + 1)^{-1}$. Denoting the left-hand side by $G(x)$, we easily verify that $G((\gamma + 1)^{-1}) = G'((\gamma + 1)^{-1}) = 0$ and

$$\begin{aligned}
G''(x) &= p(p-1)x^{p-2}\left[\frac{(2-p)\gamma^{p-1} + \gamma^{p-2}}{p-1} - \left(\frac{1-x}{x}\right)^{p-2}\right] \\
&\geq p(p-1)x^{p-2}\left[\frac{(2-p)\gamma^{p-1} + \gamma^{p-2}}{p-1} - \gamma^{p-2}\right] \\
&= p(2-p)(\gamma x)^{p-2}(1+\gamma) \geq 0.
\end{aligned}$$

Thus, (3.2) follows.

(iii) The function $F_{x,y,h,k}$ is of class C^1, so we will be done if we check that $F''_{x,y,h,k}(t) \leq 0$ for t such that $0 < |y + tk| < \gamma|x + th|$ or $0 < |x + th| < |y + tk|/\gamma$. In the first case, we go back to Burkholder's calculation (cf. page 17 in [3]): actually, the function

$$t \mapsto (|x + th| + |y + tk|)^{p-1}\left(|y + tk| - \frac{|x + th|}{p-1}\right)$$

is concave on $\mathbb{R}$ for *any* x, y, h, k with $|k| \leq |h|$. To handle $F''_{x,y,h,k}(t)$ for $0 < |x + th| < |y + tk|/\gamma$, note that we have the translation property $F_{x,y,h,k}(t + s) = F_{x+th,y+tk,h,k}(s)$

for all $t,\ s \in \mathbb{R}$, and hence it is enough to study the sign of the second derivative at $t = 0$. We compute that

$$\begin{aligned} &\frac{\mathrm{d}^2}{\mathrm{d}t^2}\left[|y+tk|^p - \frac{(2-p)\gamma^{p-1}+\gamma^{p-2}}{p-1}|x+th|^p\right]\Bigg|_{t=0} \\ &= p|y|^{p-2}|k|^2 + p(p-2)|y|^{p-4}\langle y,k\rangle^2 \\ &\qquad - \frac{(2-p)\gamma^{p-1}+\gamma^{p-2}}{p-1}\left(p(p-2)|x|^{p-4}\langle x,h\rangle^2 + p|x|^{p-2}|k|^2\right). \end{aligned} \tag{3.3}$$

Now, since p is smaller than 2, we immediately see that $p|y|^{p-2}|k|^2 \le p(\gamma|x|)^{p-2}|k|^2$, $p(p-2)|y|^{p-4}\langle y,k\rangle^2 \le 0$ and

$$p(p-2)|x|^{p-4}\langle x,h\rangle^2 + p|x|^{p-2}|k|^2 \ge p(p-1)|x|^{p-2}|h|^2.$$

Hence the second derivative (3.3) is not larger than $p(p-2)\gamma^{p-1}|x|^{p-2}|k|^2 \le 0$, and the claim follows. □

We turn our attention to the main result of this section.

Proof of Theorem 3.1. There is a well-known procedure which enables the extraction of (3.1) from the special function $b_{p,\gamma}$. Fix $f,\ g,\ n$ as in the statement. Of course we may and do assume that $\mathbb{E}|f_n|^p < \infty$, since otherwise the bound is trivial. Then $\mathbb{E}|f_k|^p < \infty$ for all $0 \le k \le n$, and hence also $df_k,\ dg_k$ are p-integrable for these values of k. The key observation is that by Lemma 3.1 (iii) and the smoothness of $b_{p,\gamma}$, we have

$$\begin{aligned} b_{p,\gamma}(f_{k+1},g_{k+1}) &= b_{p,\gamma}(f_k+df_{k+1},g_k+dg_{k+1}) \\ &\le b_{p,\gamma}(f_k,g_k) + \left\langle \frac{\partial b_{p,\gamma}(f_k,g_k)}{\partial x}, df_{k+1}\right\rangle + \left\langle \frac{\partial b_{p,\gamma}(f_k,g_k)}{\partial y}, dg_{k+1}\right\rangle, \end{aligned}$$

for $k = 0,\ 1,\ 2,\ \ldots,\ n-1$. Now by Lemma 3.1 (i) and the aforementioned p-integrability of the differences of f and g, we see that both sides above are integrable. Taking expectation yields $\mathbb{E}b_{p,\gamma}(f_{k+1},g_{k+1}) \le \mathbb{E}b_{p,\gamma}(f_k,g_k)$ and hence, by (3.2),

$$\begin{aligned} \mathbb{E}\left[|g_n|^p - \frac{(2-p)\gamma^{p-1}+\gamma^{p-2}}{p-1}|f_n|^p\right] &\le \mathbb{E}b_{p,\gamma}(f_n,g_n) \\ &\le \mathbb{E}b_{p,\gamma}(f_0,g_0) = b_{p,\gamma}(x,y). \end{aligned}$$

This is precisely the assertion of the theorem. □

Let us conclude this section by making a simple observation which will be needed later. Namely, if the martingale f in Theorem 3.1 is assumed to be L^p bounded, then so is g and we may let $n \to \infty$ in (3.1), obtaining

$$\mathbb{E}|g_\infty|^p - \frac{(2-p)\gamma^{p-1}+\gamma^{p-2}}{p-1}\mathbb{E}|f_\infty|^p \le b_{p,\gamma}(x,y). \tag{3.4}$$

4. Proof of $\mathbb{B}_p \le \mathcal{B}_p$

The purpose of this section is to deduce the above upper bound for $\mathbb{B}_p$ from Theorem 3.1. We start with three technical facts.

Lemma 4.1. *Let $1 < p < 2$ and fix $(\zeta,\eta,Z,H) \in \mathcal{D}$ such that $Z > |\eta|^p$, $H > |\eta|^q$ and $|\eta|^q Z < |\zeta|^p H$. Then there is a unique pair (γ, Y) satisfying the system (1.5).*

Proof. It is convenient to split the reasoning into a few parts.

Step 1. Auxiliary functions. Consider κ, $\delta : [0,\infty) \to [0,\infty)$ given by $\kappa(t) = (1-(p-1)t)(1+t)^{p-1}$ and $\delta(t) = t(1+t)^{p-2}$. A direct differentiation shows that

$$\kappa'(t) = -p(p-1)t(1+t)^{p-2} < 0, \quad \delta'(t) = (1+t)^{p-3}(1+(p-1)t) > 0$$

and

$$\delta''(t) = (p-2)(1+t)^{p-4}(2+(p-1)t) < 0.$$

Step 2. An easy case. If $|\eta| = 0$, the assertion of the lemma is clear: the second equality in (1.5) implies $Y = 0$, and plugging this into the first equation gives $\kappa(\gamma) = |\zeta|^p/Z \in (0,1)$. But, as we have observed above, κ is strictly decreasing and satisfies $\kappa(0) = 1$, $\kappa((p-1)^{-1}) = 0$; thus the claim follows at once from Darboux property. Hence, from now on, we may assume that $\eta \neq 0$.

Step 3. An extra function. As we have shown above, δ is strictly increasing; so, for a given $Y > 0$ there is a unique $G(Y) > Y$ satisfying

$$\delta(Y) = \left(\frac{|\eta|^q Z}{|\zeta|^p H}\right)^{1/q} \delta\big(G(Y)\big).$$

Of course, G is a smooth function on $(0,\infty)$. Differentiating both sides above gives

$$G'(Y) = \frac{\delta'(Y)}{\delta'(G(Y))}\left(\frac{|\zeta|^p H}{|\eta|^q Z}\right)^{1/q},$$

and hence $G'(Y) > 1$. Indeed, $|\zeta|^p H/(|\eta|^q Z) > 1$ by the assumption of the lemma, and $\delta'(Y)/\delta'(G(Y)) > 1$, because $G(Y) > Y$ and $\delta'' < 0$.

Step 4. Completion of the proof. The assertion of the lemma will follow if we show that there is a unique $Y > 0$ for which $G(Y) < (p-1)^{-1}$ and

$$F(Y) := \kappa(Y) - \frac{Z}{|\zeta|^p}\kappa(G(Y)) = 0.$$

However, we have

$$F'(Y) = \kappa'(Y) - \frac{Z}{|\zeta|^p}\kappa'(G(Y))G'(Y) > \kappa'(Y) - \frac{Z}{|\zeta|^p}\kappa'(G(Y)),$$

since $G'(Y) > 1$ and $\kappa'(G(Y)) < 0$. Thus,

$$F'(Y) = -p(p-1)Y(1+Y)^{p-2}\left[1 - \frac{Z}{|\zeta|^p}\left(\frac{|\zeta|^p H}{|\eta|^q Z}\right)^{1/q}\right] > 0$$

and it remains to note that $\lim_{Y\to 0} F(Y) = 0$ (since $\gamma(Y) \to 0$ as $Y \to 0$), and $F(Y)$ is positive when $G(Y)$ approaches $(p-1)^{-1}$. □

Lemma 4.2. *Fix nonzero ζ, $\eta \in \mathcal{H}$ and two numbers Z, H satisfying $Z > |\zeta|^p$ and $H > |\eta|^q$. Consider the function*

$$L(\gamma, Y) = -Y|\eta| + H^{1/q}\Bigg(\left(\frac{\gamma}{\gamma+1}\right)^{p-2}(1+Y)^{p-1}\left(Y - \frac{1}{p-1}\right) + \frac{(2-p)\gamma^{p-1} + \gamma^{p-2}}{p-1}\frac{Z}{|\zeta|^p}\Bigg)^{1/p},$$

defined for $0 \le Y \le \gamma \le (p-1)^{-1}$, and assume that L attains its minimum at the point (γ_0, Y_0).

(i) If $|\eta|^q Z \ge |\zeta|^p H$, then $\gamma_0 = Y_0 = (p-1)^{-1}$.

(ii) If $|\eta|^q Z < |\zeta|^p H$, then (γ_0, Y_0) is the unique solution to the system (1.5).

Proof. Observe first that L is continuous, so its minimum is attained and hence (γ_0, Y_0) exists. A little computation shows that if Y lies in the interval $[0,(p-1)^{-1})$ and $\gamma \in (Y,(p-1)^{-1})$, then

$$\begin{aligned}\frac{\partial L(\gamma,Y)}{\partial\gamma} &= (2-p)\gamma^{p-3}(1-(p-1)\gamma)\left[\frac{(1+Y)^{p-1}(1-(p-1)Y)}{(1+\gamma)^{p-1}(1-(p-1)\gamma)} - \frac{Z}{|\zeta|^p}\right]\\ &= (2-p)\gamma^{p-3}(1-(p-1)\gamma)\left[\frac{\kappa(Y)}{\kappa(\gamma)} - \frac{Z}{|\zeta|^p}\right],\end{aligned}$$

where κ is the function introduced in the proof of Lemma 4.1. This function is decreasing and vanishes at $(p-1)^{-1}$, so for each Y as above there is a unique $\gamma(Y) \in (Y,(p-1)^{-1})$ at which the partial derivative vanishes. Here the one-dimensional restriction $\gamma \mapsto L(\gamma, Y)$ attains its minimum. Therefore, we have three possibilities for the location of (γ_0, Y_0). Namely,

a) $(\gamma_0, Y_0) = (\gamma(0), 0)$,
b) $(\gamma_0, Y_0) = ((p-1)^{-1}, (p-1)^{-1})$

or

c) (γ_0, Y_0) lies in the triangle $\big\{(\gamma, Y) : 0 < Y < \gamma < (p-1)^{-1}\big\}$.

However, the first possibility cannot occur. To see this, we compute that

$$\begin{aligned}\frac{\partial L(\gamma,Y)}{\partial Y} &= -|\eta| + H^{1/q}\left(\frac{\gamma}{\gamma+1}\right)^{p-2}(1+Y)^{p-2}Y\times\\ &\times\left(\left(\frac{\gamma}{\gamma+1}\right)^{p-2}(1+Y)^{p-1}\left(Y-\frac{1}{p-1}\right) + \frac{(2-p)\gamma^{p-1}+\gamma^{p-2}}{p-1}\frac{Z}{|\zeta|^p}\right)^{1/p-1},\end{aligned}$$

which becomes negative when $Y \to 0$. Thus, b) or c) holds true.

If $|\eta|^q Z < |\zeta|^p H$, we easily check that $\partial L/\partial Y$ is positive when γ, Y are sufficiently close to $(p-1)^{-1}$ and hence b) is impossible. Therefore, c) must hold and (γ_0, Y_0) satisfies

$$\frac{\partial L(\gamma_0,Y_0)}{\partial\gamma} = \frac{\partial L(\gamma_0,Y_0)}{\partial Y} = 0.$$

One easily verifies that this condition is precisely (1.5).

It remains to consider the case $|\eta|^q Z \geq |\zeta|^p H$. Suppose that c) holds; then (γ_0, Y_0) would have to satisfy (1.5). But the first equality in this system would imply $\gamma_0 > Y_0$ (by $Z/|\zeta|^p > 1$ and the aforementioned monotonicity of κ), while the second equality would give $\gamma_0 \leq Y_0$ (we have $|\eta|^p Z/(|\zeta|^q H) \geq 1$ and the function δ of Lemma 4.1 is increasing). The contradiction shows that b) must be true, and this completes the proof of the lemma. □

Finally, let us state a simple fact, the proof of which is left to the reader.

Lemma 4.3. *The function*

$$\gamma \mapsto \frac{(2-p)\gamma^{p-1}+\gamma^{p-2}}{p-1}$$

is strictly decreasing on the interval $(0,(p-1)^{-1}]$ and its value at $(p-1)^{-1}$ equals $(p-1)^{-p}$.

We proceed to the bound $\mathbb{B}_p \leq \mathcal{B}_p$. Fix ζ, $\eta \in \mathcal{H}$ and $Z > |\zeta|^p$, $H > |\eta|^q$, and pick martingales f, g, h as in the definition of $\mathbb{B}_p(\zeta, \eta, Z, H)$. Clearly, we may assume that $g_0 \equiv 0$ (the formula does not depend on the starting variable of g). By Hölder inequality, we see that for any $\gamma \in (0, (p-1)^{-1}]$ and any $y \in \mathcal{H}$ such that $\langle y, \eta\rangle = |y||\eta|$, we have

$$\begin{aligned}\mathbb{E}\langle g_\infty, h_\infty\rangle &= -\langle y, \eta\rangle + \mathbb{E}\langle g_\infty + y, h_\infty\rangle \\ &\leq -|y||\eta| + (\mathbb{E}|g_\infty + y|^p)^{1/p} H^{1/q} \\ &\leq -|y||\eta| + \left(\mathbb{E}|g_\infty + y|^p - \frac{(2-p)\gamma^{p-1} + \gamma^{p-2}}{p-1}(\mathbb{E}|f_\infty|^p - Z)\right)^{1/p} H^{1/q} \\ &\leq -|y||\eta| + H^{1/q}\left(b_{p,\gamma}(\zeta, y) + \frac{(2-p)\gamma^{p-1} + \gamma^{p-2}}{p-1} Z\right)^{1/p},\end{aligned}$$

where in the last line we have used (3.4). It will be convenient to write $b^{\mathbb{R}}_{p,\gamma}$ to indicate that we consider the function $b_{p,\gamma}$ defined on $\mathbb{R}\times\mathbb{R}$. The above chain of inequalities, combined with (2.2), implies that

$$\begin{aligned}&\mathbb{B}_p(\zeta, \eta, Z, H) \\ &\leq \inf\left\{-s|\eta| + H^{1/q}\left(b^{\mathbb{R}}_{p,\gamma}(|\zeta|, s) + \frac{(2-p)\gamma^{p-1} + \gamma^{p-2}}{p-1} Z\right)^{1/p}\right\},\end{aligned} \tag{4.1}$$

where the infimum is taken over all $\gamma \in (0, (p-1)^{-1}]$ and all $s \geq 0$. The remainder of this section is devoted to showing that this infimum is equal to $\mathcal{B}_p(\zeta, \eta, Z, H)$. For the sake of convenience and clarity, we have decided to split the reasoning into a few separate parts.

1° The case $\zeta = 0$. Then we have $b^{\mathbb{R}}_{p,\gamma}(|\zeta|, s) = s^p$. Furthermore,

$$\frac{(2-p)\gamma^{p-1} + \gamma^{p-2}}{p-1} \geq (p-1)^{-p} \qquad \text{for } \gamma \in (0, (p-1)^{-1}],$$

by virtue of Lemma 4.3. Consequently, we see that the infimum in (4.1) equals

$$\inf_{s\geq 0}\left(-s|\eta| + H^{1/p}(s^p + (p-1)^{-p}Z)^{1/p}\right). \tag{4.2}$$

However, a straightforward analysis of a derivative shows that the expression in the parentheses attains its minimal value for s satisfying $s^p = (p-1)^{-p}|\eta|^q Z/(H - |\eta|^q)$. Plugging this s into the expression in (4.2), we get that the infimum equals

$$\frac{Z^{1/p}(H - |\eta|^q)^{1/q}}{p-1} = \mathcal{B}_p(0, \eta, Z, H).$$

2° The case $\zeta \neq 0$, $|\zeta|^p H \leq |\eta|^q Z$. The function $b^{\mathbb{R}}_{p,\gamma}$ is homogeneous of order p. Take $|\zeta|$ out from the expression on the right in (4.1). We get

$$\begin{aligned}&\mathbb{B}_p(\zeta, \eta, Z, H) \\ &\leq |\zeta| \inf\left\{-Y|\eta| + H^{1/q}\left(b^{\mathbb{R}}_{p,\gamma}(1, Y) + \frac{(2-p)\gamma^{p-1} + \gamma^{p-2}}{p-1}\frac{Z}{|\zeta|^p}\right)^{1/p}\right\} \\ &= |\zeta| \inf w(\gamma, Y),\end{aligned} \tag{4.3}$$

where the infimum is taken over the set $\{(\gamma, Y) : \gamma \in (0, (p-1)^{-1}],\, Y = s/|\zeta| \geq 0\}$. Let us analyze the function w separately on the following subsets of this domain:

$$S_1 = \{(\gamma, Y) : 0 \leq Y \leq \gamma \leq (p-1)^{-1},\, Y < \gamma^{-1}\},$$
$$S_2 = \big\{(\gamma, Y) : \gamma < \min\{Y, (p-1)^{-1}\}\big\},$$
$$S_3 = \{(\gamma, Y) : \gamma = (p-1)^{-1},\, Y \geq (p-1)^{-1}\}.$$

First, note that the infimum in (4.3) cannot be attained at S_1. This follows from Lemma 4.2 (i), since for $Y \leq \gamma$ we have $w = L$. On the other hand, the infimum cannot be attained on S_2 either. Indeed, for $(\gamma, Y) \in S_2$ we have

$$w(\gamma, Y) = -Y|\eta| + H^{1/q}\left(Y^p + \frac{(2-p)\gamma^{p-1} + \gamma^{p-2}}{p-1}\left(\frac{Z}{|\zeta|^p} - 1\right)\right)^{1/p},$$

which is strictly decreasing with respect to γ: see Lemma 4.3. Therefore, we see that during the computation of the right-hand side of (4.3), we may assume that $\gamma = (p-1)^{-1}$ and $Y \geq (p-1)^{-1}$. This leads us to the problem of finding the minimal value of the function

$$F(Y) = -Y|\eta| + H^{1/q}\left(Y^p + (p-1)^{-p}\left(Z/|\zeta|^p - 1\right)\right)^{1/p} \tag{4.4}$$

on $[(p-1)^{-1}, \infty)$. A straightforward analysis shows that this function attains its minimum for

$$Y = (p-1)^{-1}\left(\frac{Z - |\zeta|^p}{H - |\eta|^q}\frac{|\eta|^q}{|\zeta|^p}\right)^{1/p} \tag{4.5}$$

(note that this value of Y is at least $(p-1)^{-1}$, by the assumption $|\zeta|^p H \leq |\eta|^q Z$). It suffices to note that the minimum is precisely

$$\frac{(Z - |\zeta|^p)^{1/p}(H - |\eta|^q)^{1/q}}{p-1} = \mathcal{B}(\zeta, \eta, Z, H).$$

3° *The case $\zeta \neq 0$, $|\eta|^q Z < |\zeta|^p H$.* We proceed as previously: observe that (4.3) holds true, and let us try to analyze w on the sets

$$S_1 = \{(\gamma, Y) : 0 \leq Y \leq \gamma \leq (p-1)^{-1}\},$$
$$S_2 = \big\{(\gamma, Y) : \gamma < \min\{Y, (p-1)^{-1}\}\big\},$$
$$S_3 = \{(\gamma, Y) : \gamma = (p-1)^{-1},\, Y > (p-1)^{-1}\}$$

separately. On the first subset, we make use of Lemma 4.2: we have $w = L$, so by part (ii) of that statement, the infimum (at least over S_1) is attained at the point satisfying (1.5). The same analysis as above shows that the set S_2 does not contribute to the infimum. Thus, all that remains is to check the behavior of w on S_3, and this leads us to the function F given by (4.4). However, this function is strictly increasing on $[(p-1)^{-1}, \infty)$ (since $|\eta|^q Z < |\zeta|^p H$, the point Y given by (4.5) lies below $(p-1)^{-1}$) and hence the claim follows.

5. Proof of $\mathbb{B}_p \geq \mathcal{B}_p$

We turn our attention to the proof of the lower bound for $\mathbb{B}_p$, which will show that the functions $\mathbb{B}_p$ and $\mathcal{B}_p$ actually coincide. This will be accomplished by constructing appropriate examples. Fix a small $\delta > 0$, numbers $\gamma \in (0, (p-1)^{-1})$, $Y \in [0, \gamma)$ and let $(\mathfrak{f}, \mathfrak{g})$ be a Markov martingale with values in $[0, \infty) \times \mathbb{R}$, satisfying the following conditions:

(i) We have $(\mathfrak{f}_0, \mathfrak{g}_0) \equiv (1, Y)$.

(ii) A point of the form (x, y) with $0 < y < \gamma x$, leads to $\left(\frac{x+y}{\gamma+1}, \frac{\gamma(x+y)}{\gamma+1}\right)$ or to $(x + y, 0)$.

(iii) A point of the form (x, y) with $-\gamma x < y < 0$, leads to $\left(\frac{x-y}{\gamma+1}, \frac{\gamma(-x+y)}{\gamma+1}\right)$ or to $(x - y, 0)$.

(iv) A point of the form $(x, 0)$ leads to $(x(1 + \delta), \delta x)$, $(x(1 + \delta), -\delta x)$, $\left(\frac{x}{\gamma+1}, \frac{\gamma x}{\gamma+1}\right)$ or to $\left(\frac{x}{\gamma+1}, -\frac{\gamma x}{\gamma+1}\right)$, with probabilities $\gamma/(2\gamma + 2\delta(\gamma + 1))$, $\gamma/(2\gamma + 2\delta(\gamma + 1))$, $\delta(\gamma + 1)/(2\gamma + 2\delta(\gamma + 1))$ and $\delta(\gamma + 1)/(2\gamma + 2\delta(\gamma + 1))$, respectively.

(v) All the points not mentioned in (ii) and (iii) are absorbing.

We need not specify the probabilities in (ii) and (iii), they are uniquely determined by the martingale property. To gain some intuition about this martingale pair, let us briefly describe its behavior for $Y > 0$. The pair starts from $(1, Y)$ and then it moves along the line of slope -1, either to the point on the line $y = \gamma x$, or to the x-axis. If the first possibility occurs, the pair stops; if it went to the x-axis (so it is at the point $(1 + Y, 0)$ at the moment), it continues its evolution as follows. We pick independently the random slope $s \in \{-1, 1\}$ (each choice has probability $1/2$), and then move the pair $(\mathfrak{f}, \mathfrak{g})$ along the line of slope s, either to the point on the line $y = -s\gamma x$, or to the point $(1+Y+\delta, \delta s)$. If the pair visits the line $y = -s\gamma x$, the evolution stops. Otherwise, the pair moves along the line of slope $-s$, either to the line $y = s\gamma x$ or to $(1 + Y + 2\delta, 0)$. In the first case the evolution stops, while in the second, we pick a new random slope s, and the pattern is repeated.

Let us list several properties of $(\mathfrak{f}, \mathfrak{g})$, which follow directly from the above definition. First, it is easy to see that $|d\mathfrak{g}_n| \equiv |d\mathfrak{f}_n|$ for each $n \geq 1$. Second, the above analysis clearly shows that $(\mathfrak{f}, \mathfrak{g})$ converges almost surely to a random variable $(\mathfrak{f}_\infty, \mathfrak{g}_\infty)$ satisfying $|\mathfrak{g}_\infty| = \gamma \mathfrak{f}_\infty$ almost surely. The final observation is that conditionally on the set $\{\mathfrak{g}_1 = 0\}$, the random variable $\mathfrak{g}_\infty$ is symmetric, while on $\{\mathfrak{g}_1 > 0\}$, the variable is equal to $\gamma(1 + Y)/(\gamma + 1)$. Consequently, we get

$$\begin{aligned}\mathbb{E}\mathfrak{g}_\infty|\mathfrak{g}_\infty|^{p-2} = \mathbb{E}\big\{\mathbb{E}\big[\mathfrak{g}_\infty|\mathfrak{g}_\infty|^{p-2}|\mathfrak{g}_1\big]\big\} &= \left(\frac{\gamma(1+Y)}{\gamma+1}\right)^{p-1} \mathbb{P}\left(\mathfrak{g}_1 = \frac{\gamma(1+Y)}{\gamma+1}\right) \\ &= Y\left(\frac{\gamma(1+Y)}{\gamma+1}\right)^{p-2}.\end{aligned}$$

In what follows, we will require the asymptotic behavior of the p-th moment of $\mathfrak{f}_\infty$ as $\delta \to 0$. It will be convenient to use the notation $A \simeq B$ when $\lim_{\delta \to 0} A/B = 1$. Directly from (i)-(v), we derive that $\mathbb{P}(\mathfrak{f}_\infty \geq (1 + Y)/(\gamma + 1)) = 1$ and, for $k \geq 1$,

$$\mathbb{P}\left(\mathfrak{f}_\infty \geq \frac{1+Y}{\gamma+1}(1+2\delta)^k\right) = \frac{\gamma - Y}{(1+Y)(\gamma + \delta(\gamma+1))}\mathcal{P}^{k-1},$$

where

$$\mathcal{P} = \frac{\gamma + \delta(\gamma - 1)}{(1+2\delta)(\gamma + \delta(\gamma+1))}.$$

Therefore, we have

$$\mathbb{P}\left(\mathfrak{f}_\infty = \frac{1+Y}{\gamma+1}\right) \simeq \frac{(\gamma+1)Y}{(1+Y)\gamma}$$

and, for $k \geq 1$,

$$\mathbb{P}\left(\mathfrak{f}_\infty = \frac{1+Y}{\gamma+1}(1+2\delta)^k\right) = \frac{\gamma - Y}{(\gamma+\delta(\gamma+1))(1+Y)}\mathcal{P}^{k-1}(1-\mathcal{P}).$$

Consequently,

$$\begin{aligned}
\mathbb{E}|\mathfrak{f}_\infty|^p &\simeq \frac{(\gamma+1)Y}{(1+Y)\gamma}\left(\frac{1+Y}{\gamma+1}\right)^p + \frac{\gamma-Y}{\gamma(1+Y)}\sum_{k=1}^\infty \left(\frac{1+Y}{\gamma+1}(1+2\delta)^k\right)^p \mathcal{P}^{k-1}(1-\mathcal{P}) \\
&\simeq \frac{Y}{\gamma}\left(\frac{1+Y}{\gamma+1}\right)^{p-1} \\
&\quad + \left(\frac{1+Y}{\gamma+1}\right)^{p-1}\frac{\gamma-Y}{\gamma^2}\cdot 2\delta \sum_{k=1}^\infty \left[\frac{(1+2\delta)^{p-1}(\gamma+\delta(\gamma-1))}{\gamma+\delta(\gamma+1)}\right]^{k-1} \\
&\simeq \frac{Y}{\gamma}\left(\frac{1+Y}{\gamma+1}\right)^{p-1} + \left(\frac{1+Y}{\gamma+1}\right)^{p-1}\frac{\gamma-Y}{\gamma}\times \\
&\qquad\qquad \times \frac{2\delta}{\gamma(1-(1+2\delta)^{p-1}) + \delta(\gamma+1) - \delta(\gamma-1)(1+2\delta)^{p-1}} \\
&\simeq \frac{Y}{\gamma}\left(\frac{1+Y}{\gamma+1}\right)^{p-1} + \frac{\gamma-Y}{\gamma(1-\gamma(p-1))}\left(\frac{1+Y}{\gamma+1}\right)^{p-1} \\
&= \left(\frac{1+Y}{\gamma+1}\right)^{p-1}\frac{1-(p-1)Y}{1-(p-1)\gamma}.
\end{aligned}$$

Here in the third passage we have used the fact that $\gamma < (p-1)^{-1}$: this guarantees that the geometric series converges and the martingale $\mathfrak{f}$ is bounded in L^p.

Equipped with the above facts concerning $(\mathfrak{f}, \mathfrak{g})$, we are ready to prove the estimate $\mathbb{B}_p \geq \mathcal{B}_p$. Pick $(\zeta, \eta, Z, H) \in \mathcal{D}$ with $Z > |\zeta|^p$, $H > |\eta|^q$ and assume first that $|\eta|^q Z < |\zeta|^p H$. Let us decrease Z and H a little: that is, choose $\bar{Z} \in (|\zeta|^p, Z)$ and $\bar{H} \in (|\eta|^q, H)$ for which the condition $|\eta|^q \bar{Z} < |\zeta|^p \bar{H}$ is still satisfied. Let γ, Y be the numbers coming from the system (1.5) (with the parameters ζ, η, $\bar{Z}$ and $\bar{H}$). Put $f = \zeta\mathfrak{f}$, $g = |\zeta|\eta'\mathfrak{g}$ and let h be the martingale adapted to the filtration of f and g, with the terminal value h_∞ given by

$$h_\infty = \mathfrak{g}_\infty|\mathfrak{g}_\infty|^{p-2}\cdot\left(\frac{\bar{H}|\zeta|^p}{\bar{Z}\gamma^p}\right)^{1/q}\eta'.$$

Here $\eta' = \eta/|\eta|$ if $\eta \neq 0$, and $0'$ is an arbitrary vector of length 1. Since $\mathfrak{g}_\infty$ belongs to L^p, the martingale h is bounded in L^q. We have $\mathbb{E}f_\infty = \zeta\mathbb{E}\mathfrak{f}_\infty = \zeta$; furthermore, as δ approaches 0, the p-th moment $\mathbb{E}|f_\infty|^p$ converges to $\bar{Z}$ (by the above calculation). Therefore, we have $\mathbb{E}|f_\infty|^p \leq Z$ for sufficiently small δ. Next,

$$\mathbb{E}h = \left(\frac{\bar{H}|\zeta|^p}{\bar{Z}\gamma^p}\right)^{1/q} Y\left(\frac{\gamma(1+Y)}{1+\gamma}\right)^{p-2}\eta' = \frac{\bar{H}^{1/q}|\zeta|^{p/q}}{\bar{Z}^{1/q}}\frac{Y(1+Y)^{p-2}}{\gamma(1+\gamma)^{p-2}}\eta' = |\eta|\eta' = \eta,$$

where in the third passage we have exploited (1.5). Furthermore, we have

$$\mathbb{E}|h|^q = \frac{\bar{H}|\zeta|^p}{\bar{Z}\gamma^p}\mathbb{E}|\mathfrak{g}_\infty|^p \xrightarrow{\delta\to 0} \frac{\bar{H}|\zeta|^p}{\bar{Z}}\cdot\left(\frac{1+Y}{\gamma+1}\right)^{p-1}\frac{1-(p-1)Y}{1-(p-1)\gamma} = \bar{H},$$

and hence $\mathbb{E}|h|^q \leq H$ if δ is small enough. Therefore, by the very definition of $\mathbb{B}_p$,

$$\begin{aligned}\mathbb{B}(\zeta,\eta,Z,H) &\geq \mathbb{E}\langle g_\infty, h_\infty\rangle - \langle \mathbb{E}g_\infty, \mathbb{E}h_\infty\rangle\\ &= \mathbb{E}|\mathfrak{g}_\infty|^p\cdot|\zeta|\left(\frac{\bar{H}|\zeta|^p}{\bar{Z}\gamma^p}\right)^{1/q} - |\eta\zeta|Y.\end{aligned}$$

However, as we have already observed above, $\mathbb{E}|\mathfrak{g}_\infty|^p$ converges to $\bar{Z}\gamma^p/|\zeta|^p$ as $\delta \to 0$. This implies

$$\mathbb{B}_p(\zeta,\eta,Z,H) \geq \frac{\bar{Z}\gamma^p}{|\zeta|^p}|\zeta|\left(\frac{\bar{H}|\zeta|^p}{\bar{Z}\gamma^p}\right)^{1/q} - |\eta\zeta|Y = \mathcal{B}_p(\zeta,\eta,\bar{Z},\bar{H}),$$

and letting $\bar{Z} \to Z$, $\bar{H} \to H$ gives the desired bound $\mathbb{B}_p(\zeta,\eta,Z,H) \geq \mathcal{B}_p(\zeta,\eta,Z,H)$.

Finally, we turn our attention to the case $|\eta|^q Z \geq |\zeta|^p H$. As previously, we slightly decrease Z and H: pick $\bar{Z} \in (|\zeta|^p, Z)$, $\bar{H} \in (|\eta|^q, H)$ such that $|\eta|^q\bar{Z} \geq |\zeta|^p\bar{H}$. We will need the following modification of the above exemplary martingale pair $(\mathfrak{f},\mathfrak{g})$. Let Y be the number given by (4.5) (with Z, H replaced by $\bar{Z}$, $\bar{H}$), fix $\gamma < (p-1)^{-1}$ and take

$$\varepsilon = (1-(p-1)\gamma)\cdot\frac{Y(1+\gamma)^{p-1}}{(1+Y)^p}\left(\frac{\bar{Z}}{|\zeta|^p}-1\right).$$

Let $(\mathfrak{f},\mathfrak{g})$ be a martingale satisfying

(i) $(\mathfrak{f}_0,\mathfrak{g}_0) = (1,Y)$,

(ii) At the first step, the pair moves to $(1-\varepsilon, Y+\varepsilon)$ or to $(1+Y,0)$.

(iii) Starting with the second step, the pair moves according to the rules (ii)-(v) listed in the previous case.

We easily see that the condition $|d\mathfrak{g}_n| = |d\mathfrak{f}_n|$, $n \geq 1$, is satisfied. Now, put $f = \zeta\mathfrak{f}$, $g = |\zeta|\eta'\mathfrak{g}$ and let h be the martingale with the terminal random variable

$$h_\infty = Y^{-1}(Y+\varepsilon)^{2-p}\mathfrak{g}_\infty|\mathfrak{g}_\infty|^{p-2}\eta.$$

We have $\mathbb{E}f_\infty = \zeta\mathbb{E}\mathfrak{f}_\infty = \zeta$ and, by the above definition of ε,

$$\mathbb{E}|f_\infty|^p = \frac{Y}{Y+\varepsilon}|\zeta|^p(1-\varepsilon)^p + \frac{\varepsilon}{Y+\varepsilon}|\zeta|^p\frac{(1+Y)^p}{(1-(p-1)\gamma)(1+\gamma)^{p-1}} \to \bar{Z},$$

as $\delta \to 0$ and $\gamma \to (p-1)^{-1}$. Next, we check that

$$\mathbb{E}h_\infty = \frac{Y}{Y+\varepsilon}\cdot\frac{Y+\varepsilon}{Y}\eta = \eta$$

and, by (4.5),

$$\begin{aligned}\mathbb{E}|h_\infty|^q = |\eta|^q Y^{-q}(Y+\varepsilon)^{(2-p)q}\mathbb{E}|\mathfrak{g}_\infty|^p &\to \frac{|\eta|^p}{Y^p}\left[Y^p + \gamma^p\left(\frac{\bar{Z}}{|\zeta|^p}-1\right)\right]\\ &= |\eta|^p + \frac{(p-1)^{-p}\left(\bar{Z}/|\zeta|^p-1\right)|\eta|^p}{Y^p} = \bar{H},\end{aligned}$$

as $\delta \to 0$ and $\gamma \to (p-1)^{-1}$. Consequently, since $\bar{Z} < Z$ and $\bar{H} < H$, we can write, for δ sufficiently small and γ sufficiently close to $(p-1)^{-1}$,

$$\begin{aligned}\mathbb{B}(\zeta,\eta,Z,H) &\geq \mathbb{E}\langle g_\infty, h_\infty\rangle - \langle \mathbb{E}g_\infty, \mathbb{E}h_\infty\rangle\\ &= |\zeta||\eta|\frac{(Y+\varepsilon)^{2-p}}{Y}\mathbb{E}|\mathfrak{g}_\infty|^p - |\zeta||\eta|Y.\end{aligned}$$

Letting $\delta \to 0$ and $\gamma \to (p-1)^{-1}$ (then $\varepsilon \to 0$), we see that the latter expression converges to

$$\frac{|\zeta||\eta|}{Y^{p-1}}\left[Y^p + (p-1)^{-p}\left(\frac{\bar{Z}}{|\zeta|^p} - 1\right)\right] - |\zeta||\eta|Y = \frac{|\zeta||\eta|(p-1)^{-p}(\bar{Z}/|\zeta|^p - 1)}{Y^{p-1}}.$$

Plugging the formula (4.5) for Y we obtain $\mathbb{B}_p(\zeta,\eta,Z,H) \geq \mathcal{B}_p(\zeta,\eta,\bar{Z},\bar{H})$. It remains to let $\bar{Z} \to Z$ and $\bar{H} \to H$ to complete the proof.

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