SHARP MAXIMAL INEQUALITIES FOR THE MOMENTS OF MARTINGALES AND NONNEGATIVE SUBMARTINGALES

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Abstract. In the paper we study sharp maximal inequalities for martingales and nonnegative submartingales: if f, g are martingales satisfying

$$|dg_n| \leq |df_n|, \qquad n = 0, 1, 2, \ldots$$

almost surely, then

$$\|\sup_{n\geq 0} |g_n|\|_p \leq p\|f\|_p, \qquad p \geq 2,$$

and the inequality is sharp. Furthermore, if $\alpha \in [0,1]$, f is a nonnegative submartingale and g satisfies

$$|dg_n| \leq |df_n| \text{ and } |\mathbb{E}(dg_{n+1}|\mathcal{F}_n)| \leq \alpha \mathbb{E}(df_{n+1}|\mathcal{F}_n), \qquad n = 0, 1, 2, \ldots$$

almost surely, then

$$\|\sup_{n\geq 0} |g_n|\|_p \leq (\alpha+1)p\|f\|_p, \qquad p \geq 2,$$

and the inequality is sharp. As an application, we establish related estimates for stochastic integrals and Itô processes. The inequalities strengthen the earlier classical results of Burkholder and Choi.

1. Introduction

The purpose of the paper is to provide the best constants in some maximal inequalities for martingales and nonnegative submartingales. Let us start with introducing the necessary notation. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a nonatomic probability space, equipped with a filtration $(\mathcal{F}_n)_{n\geq 0}$, that is, a nondecreasing family of sub-σ-fields of $\mathcal{F}$. Let $f = (f_n)$, $g = (g_n)$ be adapted and real-valued integrable processes. The difference sequences $df = (df_n)$, $dg = (dg_n)$ of f and g are defined by the equations

$$f_n = \sum_{k=0}^{n} df_k, \quad g_n = \sum_{k=0}^{n} dg_k, \quad n = 0, 1, 2, \ldots.$$

We will be particularly interested in those pairs (f, g), for which a certain domination relation is satisfied. Following Burkholder [6], we say that g is *differentially subordinate* to f, if for any $n \geq 0$ we have

$$\mathbb{P}(|dg_n| \leq |df_n|) = 1.$$

As an example, let g be a transform of f by a predictable sequence $v = (v_n)$ bounded in absolute value by 1; that is, we have $\mathbb{P}(|v_n| \leq 1) = 1$ and $df_n = v_n dg_n$,

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$n \geq 0$. Here by predictability we mean that v_0 is $\mathcal{F}_0$-measurable and v_n is $\mathcal{F}_{n-1}$-measurable for $n \geq 1$. In the particular case when each v_n is deterministic and takes values in $\{-1, 1\}$, we will say that g is ± 1 transform of f.

Another domination we will consider is so called α-strong subordination, where α is a fixed nonnegative number. This notion was introduced by Burkholder in [10] in the special case $\alpha = 1$ and extended to a general case by Choi [12]: the process g is α-strongly subordinate to f, if it is differentially subordinate to f and, for any $n \geq 0$,

$$|\mathbb{E}(dg_{n+1}|\mathcal{F}_n)| \leq \alpha |\mathbb{E}(df_{n+1}|\mathcal{F}_n)|$$

almost surely.

There is a vast literature concerning the comparison of the sizes of f and g under the assumption of one of the dominations above and the further condition that f is a martingale or nonnegative submartingale; we refer the interested reader to the papers [6]–[21] and the references therein. In addition, these inequalities have found their applications in many areas of mathematics: Banach space theory [4], [5]; harmonic analysis [8], [13], [14]; functional analysis [6], [7], [20]; analysis [1], [2]; stochastic integration [6], [11], [17], [20], [21]; and more. To present our motivation, we state here only two theorems. Let us start with a fundamental result of Burkholder [6]. We use the notation $||f||_p = \sup_n ||f_n||_p$, $p \in [1, \infty]$.

Theorem 1.1 (Burkholder). *Assume that f, g are martingales and g is differentially subordinate to f. Then for any $1 < p < \infty$,*

$$||g||_p \leq (p^* - 1)||f||_p, \tag{1.1}$$

where $p^* = \max\{p, p/(p-1)\}$. The constant $p^* - 1$ is the best possible, it is already the best possible if g is assumed to be ± 1 transform of f.

Here by the optimality of the constant we mean that for any $r < p^* - 1$ there exists a martingale f and its ± 1 transform g, for which $||g||_p > r||f||_p$.

The submartingale version of the estimate above is the following result of Choi [12].

Theorem 1.2 (Choi). *Assume that f is a nonnegative submartingale and g is α-differentially subordinate to f, $\alpha \in [0, 1]$. Then for any $1 < p < \infty$,*

$$||g||_p \leq (p^*_\alpha - 1)||f||_p, \tag{1.2}$$

where $p^*_\alpha = \max\{(\alpha + 1)p, p/(p-1)\}$. The constant is the best possible.

In the paper we deal with a considerably harder problem and determine the optimal constants in the related moment estimates involving the *maximal functions* of f and g. For $n \geq 0$, let $f^*_n = \sup_{0 \leq k \leq n} |f_k|$ and $f^* = \sup_{k \geq 0} |f_k|$. Here is our first main result.

Theorem 1.3. *Let f, g be martingales with g being differentially subordinate to f. Then for any $p \geq 2$,*

$$||g^*||_p \leq p||f||_p \tag{1.3}$$

and the constant p is the best possible. It is already the best possible in the following weaker inequality: if f is a martingale and g is its ± 1 transform, then

$$||g^*||_p \leq p||f^*||_p. \tag{1.4}$$

Note that the validity of the estimates (1.3) and (1.4) is an immediate consequence of (1.1) and Doob's bound $||f^*||_p \leq \frac{p}{p-1}||f||_p$, $p > 1$. The nontrivial (and quite surprising) part is the optimality of the constant p.

Now let us state the submartingale version of the theorem above.

Theorem 1.4. *Fix $\alpha \in [0,1]$. Let f be a nonnegative submartingale and g be real valued and α-strongly subordinate to f. Then for any $p \geq 2$,*

$$||g^*||_p \leq (\alpha+1)p||f||_p \tag{1.5}$$

and the constant $(\alpha+1)p$ is the best possible. It is already the best possible in the weaker estimate

$$||g^*||_p \leq (\alpha+1)p||f^*||_p. \tag{1.6}$$

There is a natural question what is the best constant in the inequalities above in the case $1 < p < 2$. Unfortunately, we have been unable to answer it; our argumentation works only for the case $p \geq 2$.

The proof of (1.5) is based on a technique invented by Burkholder in [11]. It enables to translate the problem of proving a maximal inequality for martingales to that of finding a certain special function, an upper solution to a corresponding nonlinear problem. The method can be easily extended to the submartingale setting (see e.g. [17]) and we construct the function in Section 3. For the sake of construction, we need a solution to a certain differential equation, which is analyzed in Section 2. The next two sections are devoted to the proofs of the announced results: Section 4 contains the proof of the estimate (1.5) and the final part concerns the optimality of the constants appearing in (1.4) and (1.6). In the final section we present some applications: sharp estimates for stochastic integrals and Itô processes.

2. A differential equation

For a fixed $\alpha \in (0,1]$ and $p \geq 2$, let $C = C_{p,\alpha} = [(\alpha+1)p]^p(p-1)$. A central role in the paper is played by a certain solution to the differential equation

$$\gamma'(x) = \frac{-1 + C(1-\gamma(x))\gamma(x)x^{p-2}}{1 + C(1-\gamma(x))x^{p-1}}. \tag{2.1}$$

Lemma 2.1. *There is a solution $\gamma : [((\alpha+1)p)^{-1}, \infty) \to \mathbb{R}$ of (2.1), satisfying the initial condition*

$$\gamma\left(\frac{1}{(\alpha+1)p}\right) = 1 - [(\alpha+1)p]^{-1}. \tag{2.2}$$

The solution is nondecreasing, concave and bounded from above by 1.

Proof. Let γ be a solution to (2.1), satisfying (2.2) and extended to a maximal subinterval I of $[((\alpha+1)p)^{-1}, \infty)$. It is convenient to split the proof into a few steps.

Step 1: $I = [((\alpha+1)p)^{-1}, \infty)$ In view of Picard-Lindelöf's theorem, this will be established if we show that $\gamma < 1$ on I. To this end, suppose that the set $\{x \in I : \gamma(x) = 1\}$ is nonempty and let y denote its smallest element. Then, by (2.1), we have $\gamma'(y) = -1$, which, by minimality of y, implies $\gamma'(((\alpha+1)p)^{-1}) > 1$ and contradicts (2.2).

Step 2: Concavity of γ. Suppose that the set $\{x \in I : \gamma''(x) > 0\}$ is nonempty and let z denote its infimum. Consider the functions $F, G : \big(((\alpha+1)p)^{-1}, \infty\big) \to \mathbb{R}$ given by

$$F(x) = \gamma(x) - x\gamma'(x),$$
$$G(x) = (1-\gamma(x))x^{p-2}.$$

Observe that

$$G > 0 \text{ on } I \qquad \text{and} \qquad F > 0 \text{ on } (((\alpha+1)p)^{-1}, z+\varepsilon) \tag{2.3}$$

for some $\varepsilon > 0$. The statement about G is clear, while the positivity of F follows from

$$F'(x) = -x\gamma''(x) \geq 0, \; x \in \big(((\alpha+1)p)^{-1}, z\big]$$

and

$$F\big(((\alpha+1)p)^{-1}+\big) = \frac{1}{p} > 0.$$

Now multiply (2.1) throughout by $1+C(1-\gamma(x))x^{p-1}$ and differentiate both sides. We obtain an equality which is equivalent to

$$\gamma''(x)(1+CxG(x)) = CF(x)G'(x), \qquad x > \frac{1}{(\alpha+1)p}. \tag{2.4}$$

As a first consequence, we have $z > ((\alpha+1)p)^{-1}$. To see this, tend with x down to $((\alpha+1)p)^{-1}$ and observe that F and G have strictly positive limits; furthermore,

$$G'(x) = x^{p-3}[(p-2)(1-\gamma(x)) - x\gamma'(x)] =: x^{p-3}J(x) \tag{2.5}$$

with $J(((\alpha+1)p)^{-1}) = -\frac{\alpha(p-1)}{(\alpha+1)p} < 0$. Combining (2.3) and (2.4) we see that, for some $\varepsilon > 0$, $G' \leq 0$ on $(z-\varepsilon, z)$ and $G' > 0$ on $(z, z+\varepsilon)$. Consequently, by (2.5), $J \leq 0$ on $(z-\varepsilon, z)$ and $J > 0$ on $(z, z+\varepsilon)$. This implies $J'(z) > 0$ and since $J'(z) = -(p-1)\gamma'(z)$, we get $\gamma'(z) < 0$. However, this contradicts $G'(z) = 0$, in view of (2.5) and $\gamma(z) < 1$. Let us stress that here, in the last passage, we use the inequality $p \geq 2$.

Step 3: γ is nondecreasing. It follows from (2.4), the concavity on γ and positivity of F and G, that $G' \leq 0$, or, by (2.5),

$$(p-2)(1-\gamma(x)) - x\gamma'(x) \leq 0. \tag{2.6}$$

The claim follows. □

Let us extend γ to the whole halfline $[0, \infty)$ by

$$\gamma(x) = [(p-1)(\alpha+1)-1]x + \frac{1}{p}, \qquad x \in \left[0, \frac{1}{(\alpha+1)p}\right).$$

It can be verified readily that γ is of class C^1 on $(0, \infty)$.

Let $H : [((\alpha+1)p)^{-1}, \infty) \to [1, \infty)$ be given by $H(x) = x + \gamma(x)$ and let h be the inverse to H. Clearly, we have

$$x - 1 \leq h(x) \leq x, \qquad x \geq 1. \tag{2.7}$$

We conclude this section by providing a formula for h' to be used later. As

$$h'(x) = \frac{1}{H'(h(x))} = \frac{1}{1+\gamma'(h(x))}, \qquad x > 1, \tag{2.8}$$

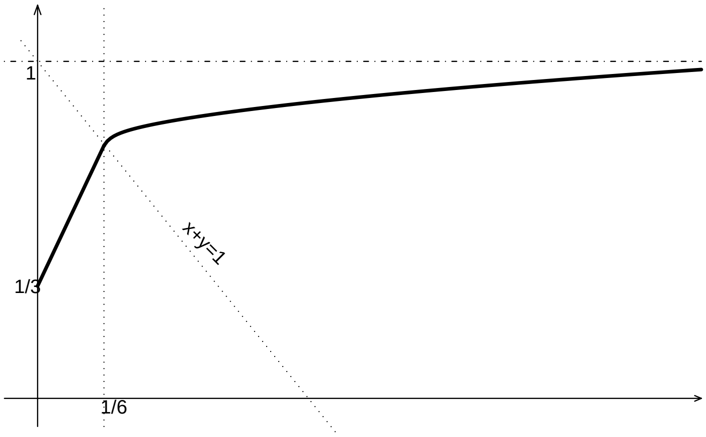

FIGURE 1. The graph of γ (the bold line) in the case $p = 3$, $\alpha = 1$. Note that γ is linear on $[0, 1/6]$ and solves (2.1) on $(1/6, \infty)$.

it can be derived that, in view of (2.1),

$$h'(x) = \frac{1 + ((\alpha+1)p)^p (p-1)(h(x) - x + 1)h(x)^{p-1}}{((\alpha+1)p)^p (p-1)(h(x) - x + 1)h(x)^{p-2}x} \tag{2.9}$$

3. The special function

Throughout this section, $\alpha \in (0, 1]$ and $p \geq 2$ are fixed. Let S denote the strip $[0, \infty) \times [-1, 1]$. Consider the following subsets of S.

$$\begin{aligned} D_0 &= \{(x, y) \in S : |y| \leq \gamma(x)\}, \\ D_1 &= \{(x, y) \in S : |y| > \gamma(x),\ x + |y| \leq 1\}, \\ D_2 &= \{(x, y) \in S : |y| > \gamma(x),\ x + |y| > 1\}. \end{aligned}$$

Introduce the function $u : S \to \mathbb{R}$ by

$$u(x, y) = \begin{cases} 1 - [(\alpha+1)p]^p x^p & \text{on } D_0, \\ 1 - \left(\frac{px + p|y| - 1}{p-1}\right)^{p-1} [p(p(\alpha+1) - 1)x - p|y| + 1] & \text{on } D_1, \\ 1 - [(\alpha+1)p]^p h(x + |y|)^{p-1} [px - (p-1)h(x + |y|)] & \text{on } D_2. \end{cases}$$

Let $U : [0, \infty) \times \mathbb{R} \times (0, \infty) \to \mathbb{R}$ be given by

$$U(x, y, z) = (|y| \vee z)^p u\left(\frac{x}{|y| \vee z}, \frac{y}{|y| \vee z}\right).$$

As we will see below, the function U is the key to the inequality (1.5). Let us study the properties of this function.

Lemma 3.1. *The function U is of class C^1. Furthermore, there exists an absolute constant K such that for all $x > 0$, $y \in \mathbb{R}$, $z > 0$ we have*

$$U(x, y, z) \le K(x + |y| + z)^p \tag{3.1}$$

and

$$U_x(x, y, z) \le K(x + |y| + z)^{p-1}, \qquad U_x(x, y, z) \le K(x + |y| + z)^{p-1}. \tag{3.2}$$

Proof. The continuity of the partial derivatives can be verified readily. The inequality (3.1) is evident for those (x, y, z), for which $(\frac{x}{|y|\vee z}, \frac{y}{|y|\vee z}) \in D_0 \cup D_1$; for remaining (x, y, z) it suffices to use (2.7). Finally, the inequality (3.2) is clear if $(\frac{x}{|y|\vee z}, \frac{y}{|y|\vee z}) \in D_0 \cup D_1$. For the remaining points one applies (2.7) and (2.8), the latter inequality implying $h' < 1$. □

Now let us deal with the following majorization property.

Lemma 3.2. *For any $(x, y, z) \in [0, \infty) \times \mathbb{R} \times (0, \infty)$, we have*

$$U(x, y, z) \ge (|y| \vee z)^p - [(\alpha + 1)p]^p x^p. \tag{3.3}$$

Proof. The inequality is equivalent to $u(x, y) \ge 1 - [(\alpha + 1)p]^p x^p$ and we need to establish it only on D_1 and D_2. On D_1, substitution $X = px$ and $Y = p|y| - 1$ (note that $Y \ge 0$) transforms it into

$$(\alpha + 1)^p X^p \ge \left(\frac{X + Y}{p - 1}\right)^{p-1} [(p(\alpha + 1) - 1)X - Y].$$

This inequality is valid for all nonnegative X, Y. To see this, observe that by homogeneity we may assume $X + Y = 1$, and then the estimate reads

$$F(X) := (\alpha + 1)^p X^p - (p - 1)^{-p+1}[p(\alpha + 1)X - 1] \ge 0, \qquad X \in [0, 1].$$

Now it suffices to note that F is convex on $[0, 1]$ and satisfies

$$F\left(\frac{1}{(p - 1)(\alpha + 1)}\right) = F'\left(\frac{1}{(p - 1)(\alpha + 1)}\right) = 0.$$

It remains to show the majorization on D_2. It is dealt with in the similar manner: setting $s = x + |y| > 1$, we see that (3.3) is equivalent to

$$G(x) := x^p - h(s)^{p-1}[px - (p - 1)h(s)] \ge 0, \qquad s - 1 < x < h(s).$$

It is easily verified that G is convex and satisfies $G(h(s)) = G'(h(s)) = 0$. This completes the proof of (3.3). □

The main property of the function U is the concavity along the lines of slope belonging to $[-1, 1]$.

Lemma 3.3. *For fixed y, z satisfying $z > 0$, $|y| \le z$, and any $a \in [-1, 1]$, the function $\Phi = \Phi_{y,z,a} : [0, \infty) \to \mathbb{R}$ given by*

$$\Phi(t) = U(t, y + at, z)$$

is concave.

Before we turn to the proof, let us first establish some useful consequences.

Corollary 3.4. *(i) The function U has the following property: for any x, y, z, k_x, k_y such that x, $x+k_x \geq 0$, $z>0$, $|y| \leq z$ and $|k_y| \leq |k_x|$ we have*

$$U(x+k_x, y+k_y, z) \leq U(x,y,z) + U_x(x,y,z)k_x + U_y(x,y,z)k_y \tag{3.4}$$

(for $x=0$ we replace $U_x(0,y,z)$ by right-sided derivative $U_x(0+,y,z)$).

(ii) For any $x \geq 1$ we have

$$U(x,1,1) \leq 0. \tag{3.5}$$

Proof. (i) This follows immediately.

(ii) We have $\Phi_{0,1,x^{-1}}(0) = U(0,0,1) = 1$ and $\Phi_{0,1,x^{-1}}(((\alpha+1)p)^{-1}) = U(((\alpha+1)p)^{-1}, x^{-1}((\alpha+1)p)^{-1}, 1) = 0$, since $(((\alpha+1)p)^{-1}, x^{-1}((\alpha+1)p)^{-1}, 1) \in D_0$. Since $x \geq 1 > ((\alpha+1)p)^{-1}$, the lemma above gives $U(x,1,1) = \Phi_{0,1,x^{-1}}(x) \leq 0$. □

Proof of Lemma 3.3. By homogeneity, we may assume $z=1$. As Φ is of class C^1, it suffices to verify that $\Phi''(t) \leq 0$ for those t, for which $(t, y+at)$ lies in the interior of D_0, D_1, D_2 or outside the strip S. Since $U(x,y,z) = U(x,-y,z)$, we may restrict ourselves to the case $y+at \geq 0$. If $(t, y+at)$ belongs to D_0^o, the interior of D_0, then $\Phi''(t) = -[(\alpha+1)p]^p \cdot p(p-1)t^{p-2} < 0$, while for $(t, y+at) \in D_1^o$ we have

$$\Phi''(t) = -\frac{p^3(pt+p(y+at)-1)^{p-3}(1+a)}{(p-1)^{p-2}}(I_1+I_2),$$

where

$$\begin{aligned}
I_1 &= pt[(p-2)(1+a)(p(\alpha+1)-1) + 2(p(\alpha+1)-1-a)] \geq 0,\\
I_2 &= (p(y+at)-1)(2\alpha+1-a) \geq 0.
\end{aligned}$$

The remaining two cases are a bit more complicated. If $(t, y+at) \in D_2^o$, then

$$\frac{\Phi''(t)}{Cp(1+a)^2} = J_1 + J_2 + J_3,$$

where

$$\begin{aligned}
J_1 =& h(t+y+at)^{p-2}h''(t+y+at)[h(t+y+at)-t],\\
J_2 =& h(t+y+at)^{p-3}[h'(t+y+at)]^2[(p-1)h(t+y+at)-(p-2)t],\\
J_3 =& -\frac{2}{a+1}h(t+y+at)^{p-2}h'(t+y+at).
\end{aligned}$$

Now if we change y and t keeping $s = t+y+at$ fixed, then $J_1+J_2+J_3$ is a linear function of $t \in [s-1, h(s)]$. Therefore, to prove it is nonpositive, it suffices to verify this for $t = h(s)$ and $t = s-1$. For $t = h(s)$, we have

$$J_1+J_2+J_3 = h(s)^{p-2}h'(s)\left[h'(s) - \frac{2}{a+1}\right] \leq 0,$$

since $0 \leq h'(s) \leq 1$ (see (2.8)). If $t = s-1$, rewrite (2.9) in the form

$$Cs(h(s)+1-s)h(s)^{p-2}h'(s) = 1 + C(h(s)+1-s)h(s)^{p-1}$$

and differentiate both sides; as a result, we obtain

$$\begin{aligned}
Cs\left[J_1+J_2+J_3 + h(s)^{p-2}h'(s)\left(\frac{2}{a+1}-1\right)\right]&\\
=& Ch(s)^{p-2}[(h'(s)-1)h(s) + (p-2)(h(s)+1-s)h'(s)].
\end{aligned}$$

As $h' \geq 0$ and $2/(a+1) \geq 1$, we will be done if we show the right-hand side is nonpositive. This is equivalent to

$$h'(s)[h(s)+(p-2)(h(s)+1-s)] \leq h(s).$$

Now use (2.8) and substitute $h(s) = r$, noting that $h(s)+1-s = 1-\gamma(r)$, to obtain

$$r + (p-2)(1-\gamma(r)) \leq r(1+\gamma'(r)),$$

or $r\gamma'(r) \geq (p-2)(1-\gamma(r))$, which is (2.6).

Finally, suppose that $y + at > 1$. For such t we have $\Phi(t) = (y+at)^p u(t/(y+at), 1)$, hence, setting $X = t/(y+t)$, $Y = y + at$, we easily check that $\Phi''(t)$ equals

$$Y^{p-2}\left[p(p-1)a^2u(X,1) + 2a(p-1)(1-aX)u_x(X,1) + (1-aX)^2u_{xx}(X,1)\right].$$

First let us derive the expressions for the partial derivatives. Using (2.9), we have

$$\begin{aligned} u_x(X,1) =& \frac{p}{X+1}[1 + C(h(X+1)-X)h(X+1)^{p-1}] - \frac{Cph(X+1)^{p-1}}{p-1}, \\ u_{xx}(X,1) =& \frac{p(p-1)}{(X+1)^2}[1 + C(h(X+1)-X)h(X+1)^{p-1}] \\ &- \frac{Cph(X+1)^{p-1}}{X+1} - \frac{Cph(X+1)^{p-2}h'(X+1)}{X+1}. \end{aligned}$$

Now it can be checked that

$$\Phi''(t)Y^{2-p}/p = K_1 + K_2 + K_3,$$

where

$$\begin{aligned} K_1 =& (p-1)\left(\frac{a+1}{X+1}\right)^2 [1 + C(h(X+1)-X)h(X+1)^{p-1}], \\ K_2 =& -\frac{Ch(X+1)^{p-1}}{X+1}(1+2a-a^2X), \\ K_3 =& -\left(\frac{1-aX}{X+1}\right)^2 \cdot \frac{1 + C(h(X+1)-X)h(X+1)^{p-1}}{h(X+1)-X} \\ \leq& -\left(\frac{1-aX}{X+1}\right)^2 \cdot Ch(X+1)^{p-1}. \end{aligned}$$

We may write

$$\begin{aligned} K_2 + K_3 \leq& -\frac{Ch(X+1)^{p-1}}{(X+1)^2}[(1+2a-a^2X)(X+1) + (1-aX)^2] \\ =& -\frac{Ch(X+1)^{p-1}(a+1)}{(X+1)^2}[2 + X(1-a)] \leq -\left(\frac{a+1}{X+1}\right)^2 Ch(X+1)^{p-1}, \end{aligned}$$

where in the last passage we have used $a \leq 1$. On the other hand, as h is nondecreasing, we have

$$1 = \frac{Ch(1)^p}{p-1} \leq \frac{Ch(X+1)^{p-1}h(1)}{p-1}.$$

Moreover, since $x \mapsto h(x+1)-x$ is nonincreasing (see (2.8)), we have $h(X+1)-X \leq h(1)$. Combining these two facts, we obtain

$$\begin{aligned} K_1 \leq &(p-1)\left(\frac{a+1}{X+1}\right)^2 \left[1+Ch(1)h(X+1)^{p-1}\right] \\ \leq &\left(\frac{a+1}{X+1}\right)^2 Ch(X+1)^{p-1}\left[h(1)+(p-1)h(1)\right] \\ \leq &\left(\frac{a+1}{X+1}\right)^2 Ch(X+1)^{p-1}, \end{aligned}$$

as $ph(1) = (\alpha+1)^{-1} \leq 1$. This implies $K_1+K_2+K_3 \leq 0$ and completes the proof. □

The final property we will need is the following.

Lemma 3.5. *For any x, y, z such that $x \geq 0$, $z > 0$ and $|y| \leq z$ we have*

$$U_x(x,y,z) \leq -\alpha|U_y(x,y,z)| \tag{3.6}$$

(if $x = 0$, then U_x is replaced by right-sided derivative).

Proof. It suffices to show that for fixed y, z, $|y| \leq z$, and $a \in [-\alpha,\alpha]$, the function $\Phi = \Phi_{y,z,a} : [0,\infty) \to \mathbb{R}$ given by $\Phi(t) = U(t, y+at, z)$ is nonincreasing. Since $\alpha \leq 1$, we know from the previous lemma that Φ is concave. Hence all we need is $\Phi'(0+) \leq 0$. By symmetry, we may assume $y \geq 0$. If $y \leq 1/p$, then the derivative equals 0; in the remaining case, we have

$$\Phi'(0+) = -\frac{p^2(py-1)^{p-1}}{(p-1)^{p-1}}(\alpha - a) \leq 0. \quad \square$$

4. The proof of (1.5)

First let us observe that it suffices to show (1.5) for strictly positive α. This is an immediate consequence of the fact that α-strong subordination implies α'-strong subordination for $\alpha < \alpha'$.

Suppose f, g are as in Theorem 1.4. We may restrict ourselves to the case $||f||_p < \infty$. Hence, by Choi's inequality (1.2), we have $||g||_p < \infty$. It suffices to show that for any $n = 0, 1, 2, \ldots$ we have

$$\mathbb{E}\left[(g_n^*)^p - (\alpha+1)^p p^p f_n^p\right] \leq 0.$$

Clearly, we may assume that $\mathbb{P}(g_0 > 0) = 1$, simply replacing f, g by $f+\varepsilon$, $g+\varepsilon$ if necessary (here ε is a small positive number). In particular, this implies $f_0 > 0$ almost surely. In view of the majorization (3.3), we will be done if we show that the expectation $\mathbb{E}U(f_n, g_n, g_n^*)$ is nonpositive for any n. As a matter of fact, we will show more, namely, that the process $(U(f_n, g_n, g_n^*)_{n\geq 0})$ is a supermartingale and $\mathbb{E}U(f_0, g_0, g_0^*) \leq 0$.

To this end, fix $n \geq 1$ and observe that $g_n^* \leq |g_0|+|g_1|+\ldots+|g_n|$, so g_n^* belongs to L^p. Thus, by Lemma 3.1 and Hölder's inequality, the variables $U(f_n, g_n, g_n^*)$, $U_x(f_{n-1}, g_{n-1}, g_{n-1}^*)df_n$ and $U_y(f_{n-1}, g_{n-1}, g_{n-1}^*)dg_n$ are integrable. Moreover, by definition of U and the inequality (3.4),

$$\begin{aligned}\mathbb{E}(U(f_n,g_n,g_n^*)|\mathcal{F}_{n-1}) =&\mathbb{E}(U_n(f_n,g_n,g_{n-1}^*)|\mathcal{F}_{n-1})\\ =&\mathbb{E}(U(f_{n-1}+df_n,g_{n-1}+dg_n,g_{n-1}^*)|\mathcal{F}_{n-1})\\ \leq&\mathbb{E}\big[U(f_{n-1},g_{n-1},g_{n-1}^*)+U_x(f_{n-1},g_{n-1},g_{n-1}^*)df_n\\ &\quad+U_y(f_{n-1},g_{n-1},g_{n-1}^*)dg_n|\mathcal{F}_{n-1}\big]\\ \leq&U(f_{n-1},g_{n-1},g_{n-1}^*).\end{aligned}$$

The latter inequality is the consequence of the following. By (3.6) and the submartingale property of f,

$$\begin{aligned}\mathbb{E}(U_x(f_{n-1},g_{n-1},g_{n-1}^*)df_n|\mathcal{F}_{n-1}) &= U_x(f_{n-1},g_{n-1},g_{n-1}^*)\mathbb{E}(df_n|\mathcal{F}_{n-1})\\ &\leq -\alpha|U_y(f_{n-1},g_{n-1},g_{n-1}^*)|\mathbb{E}(df_n|\mathcal{F}_{n-1})\\ &\leq -U_y(f_{n-1},g_{n-1},g_{n-1}^*)\mathbb{E}(dg_n|\mathcal{F}_{n-1})\\ &= -\mathbb{E}(U_y(f_{n-1},g_{n-1},g_{n-1}^*)dg_n|\mathcal{F}_{n-1}),\end{aligned}$$

where the second inequality is due to α-domination.

To complete the proof, it suffices to show that $\mathbb{E}U(f_0,g_0,g_0^*)\leq 0$. However, $U(f_0,g_0,g_0^*)=U(f_0,g_0,g_0)=g_0^pU(f_0/g_0,1,1)$ almost surely and the estimate follows from Corollary 3.4 (ii).

5. Sharpness

We start with inequality (1.4) and restrict ourselves to the case when g is a ±1 transform of f. Suppose the best constant in this estimate equals $\beta>0$. This implies the existence of a function $W:\mathbb{R}\times\mathbb{R}\times[0,\infty)\times[0,\infty)\to\mathbb{R}$ which satisfies the following properties:

$$W(1,1,1,1)\leq 0, \tag{5.1}$$

$$W(x,y,z,w)=W(x,y,|x|\vee z,|y|\vee w),\quad \text{if } x,y\in\mathbb{R},\ w,z\geq 0, \tag{5.2}$$

$$(|y|\vee w)^p-\beta^p(|x|\vee z)^p\leq W(x,y,z,w),\quad \text{if } x,y\in\mathbb{R},\ w,z\geq 0 \tag{5.3}$$

and, furthermore,

(5.4)
$$\begin{gathered}aW(x+t_1,y+\varepsilon t_1,z,w)+(1-a)W(x+t_2,y+\varepsilon t_2,z,w)\leq W(x,y,z,w),\\ \text{for any } |x|\leq z,\ |y|\leq w,\ \varepsilon\in\{-1,1\},\ a\in(0,1) \text{ and } t_1,\ t_2\\ \text{with } at_1+(1-a)t_2=0.\end{gathered}$$

Indeed, one puts

$$W(x,y,z,w)=\sup\{\mathbb{E}(|g_n|\vee w)^p-\beta^p\mathbb{E}(|f_n|\vee z)^p\}, \tag{5.5}$$

where the supremum is taken over all integers n and all martingales f, g satisfying $\mathbb{P}((f_0,g_0)=(x,y))=1$ and $df_k=\pm dg_k$, $k=1,2,\ldots$ (see [11] for details). This formula allows us to assume that W is homogeneous: $W(tx,ty,tz,tw)=tW(x,y,z,w)$ for all $x,y\in\mathbb{R}$, $z,w\geq 0$ and $t>0$.

Now the idea is to exploit the above properties of W to get $\beta\geq p$. To this end, let δ be a small number belonging to $(0,1/p)$. By (5.4) applied to $x=0$, $y=w=1$,

$z = \delta/(1+2\delta)$, $\varepsilon = 1$ and $t_1 = \delta$, $t_2 = -1/p$, we obtain

$$\begin{aligned} W(0,1,\frac{\delta}{1+2\delta},1) \geq & \frac{p\delta}{1+p\delta} W\left(-\frac{1}{p}, 1-\frac{1}{p}, \frac{\delta}{1+2\delta}, 1\right) \\ & + \frac{1}{1+p\delta} W\left(\delta, 1+\delta, \frac{\delta}{1+2\delta}, 1+\delta\right). \end{aligned} \tag{5.6}$$

Now, by (5.2) and (5.3),

$$W\left(-\frac{1}{p}, 1-\frac{1}{p}, \frac{\delta}{1+2\delta}, 1\right) = W\left(-\frac{1}{p}, 1-\frac{1}{p}, \frac{1}{p}, 1\right) \geq 1 - \left(\frac{\beta}{p}\right)^p. \tag{5.7}$$

Furthermore, by (5.2),

$$W\left(\delta, 1+\delta, \frac{\delta}{1+2\delta}, 1+\delta\right) = W\left(\delta, 1+\delta, \delta, 1+\delta\right),$$

which, by (5.4) (with $x = z = \delta$, $y = w = 1+\delta$, $\varepsilon = -1$ and $t_1 = -\delta$, $t_2 = \frac{1}{p} + \delta(\frac{1}{p} - 1)$) can be bounded from below by

$$\frac{p\delta}{1+\delta} W\left(\frac{1+\delta}{p}, 1-\frac{1}{p}+\delta\Big(2-\frac{1}{p}\Big), \delta, 1+\delta\right) + \frac{1+\delta-p\delta}{1+\delta} W(0, 1+2\delta, \delta, 1+\delta).$$

Using (5.3), we get

$$W\left(\frac{1+\delta}{p}, 1-\frac{1}{p}+\delta\Big(2-\frac{1}{p}\Big), \delta, 1+\delta\right) \geq (1+\delta)^p \left[1 - \left(\frac{\beta}{p}\right)^p\right],$$

furthermore, by (5.2) and the homogeneity of W,

$$W(0, 1+2\delta, \delta, 1+\delta) = W(0, 1+2\delta, \delta, 1+2\delta) = (1+2\delta)^p W\left(0, 1, \frac{\delta}{1+2\delta}, 1\right).$$

Now plug all the above estimates into (5.6) to get

$$\begin{aligned} W\left(0, 1, \frac{\delta}{1+2\delta}, 1\right) \left[1 - \frac{(1+\delta-p\delta)(1+2\delta)^p}{(1+\delta)(1+p\delta)}\right] \geq & \\ \frac{p\delta}{1+p\delta}\left[1 - \left(\frac{\beta}{p}\right)^p\right](1 + (1+\delta)^{p-1}). & \end{aligned} \tag{5.8}$$

Now it follows from the definition (5.5) of W that

$$W\left(0, 1, \frac{\delta}{1+2\delta}, 1\right) \leq W(0,1,0,1).$$

Furthermore, one easily checks that the function

$$F(s) = 1 - \frac{(1+s-ps)(1+2s)^p}{(1+s)(1+ps)}, \qquad s > -\frac{1}{p},$$

satisfies $F(0) = F'(0) = 0$. Hence

$$1 - \left(\frac{\beta}{p}\right)^p \leq \frac{W(0,1,0,1) \cdot F(\delta) \cdot (1+p\delta)}{p\delta(1 + (1+\delta)^{p-1})}$$

and letting $\delta \to 0$ yields $1 - \left(\frac{\beta}{p}\right)^p \leq 0$, or $\beta \geq p$.

The argumentation for the inequality (1.6) is essentially the same: suppose the best constant in the estimate equals $\gamma > 0$. Introduce the function $V : [0,\infty) \times \mathbb{R} \times [0,\infty) \times [0,\infty) \to \mathbb{R}$ by

$$V(x,y,z,w) = \sup\{\mathbb{E}(|g_n| \vee w)^p - \gamma^p \mathbb{E}(|f_n| \vee z)^p\},$$

where the supremum is taken over all integers n, all nonnegative submartingales f and all integrable sequences g satisfying $\mathbb{P}((f_0, g_0) = (x, y)) = 1$ and, for $k = 1, 2, \ldots$,

$$|df_k| \geq |dg_k|, \qquad \alpha\mathbb{E}(df_k|\mathcal{F}_{k-1}) \geq |\mathbb{E}(dg_k|\mathcal{F}_{k-1})|$$

with probability 1. We see that V is homogeneous and satisfies the properties analogous to (5.1) - (5.4) (with obvious changes: in (5.2), (5.3) one must assume $x \geq 0$, in (5.3) the number β is replaced by γ and, in (5.4), we impose x, $x+t_1$, $x+t_2 \geq 0$). In addition, there is an extra property of V, which corresponds to the fact that we deal with the inequality for submartingales:

$$V(x+d, y+\alpha d, z, w) \leq V(x,y,z,w), \quad \text{if } x \geq 0\,, y \in \mathbb{R},\ w, z \geq 0,\ d \geq 0. \tag{5.9}$$

Now fix $\delta \in (0, 1/p)$ and apply this property with $x = 0$, $y = w = 1$, $z = \delta/(1 + (\alpha+1)p)$, $d = \delta$ and then use (5.2) to obtain

$$\begin{aligned} V\left(0, 1, \frac{\delta}{1+(\alpha+1)\delta}, 1\right) &\geq V\left(\delta, 1+\alpha\delta, \frac{\delta}{1+(\alpha+1)\delta}, 1\right) \\ &= V(\delta, 1+\alpha\delta, \delta, 1+\alpha\delta). \end{aligned} \tag{5.10}$$

Using (5.2), (5.3) and (5.4) as above, we have

$$\begin{aligned} V(\delta, 1+\alpha\delta, \delta, 1+\alpha\delta) &\geq \frac{\delta(\alpha+1)p}{1+\alpha\delta}(1+\alpha\delta)^p\left[1 - \left(\frac{\gamma}{(\alpha+1)p}\right)^p\right] \\ &+ \frac{1+\alpha\delta - \delta(\alpha+1)p}{1+\alpha\delta}(1+(\alpha+1)\delta)^p V\left(0, 1, \frac{\delta}{1+(\alpha+1)\delta}, 1\right), \end{aligned}$$

which, combined with (5.10), gives

$$\begin{aligned} V\left(0, 1, \frac{\delta}{1+(\alpha+1)\delta}, 1\right) &\left[1 - \frac{1+\alpha\delta - \delta(\alpha+1)p}{1+\alpha\delta}(1+(\alpha+1)\delta)^p\right] \\ &\geq \delta(\alpha+1)p(1+\alpha\delta)^{p-1}\left[1 - \left(\frac{\gamma}{(\alpha+1)p}\right)^p\right]. \end{aligned}$$

Now it suffices to use

$$V\left(0, 1, \frac{\delta}{1+(\alpha+1)\delta}, 1\right) \leq V(0,1,0,1)$$

and the fact that the function

$$G(s) = 1 - \frac{1+\alpha s - s(\alpha+1)p}{1+\alpha s}(1+(\alpha+1)s)^p, \quad s > -1/\alpha,$$

satisfies $G(0) = G'(0) = 0$, to obtain

$$1 - \left(\frac{\gamma}{(\alpha+1)p}\right)^p \leq \frac{V(0,1,0,1)G(\delta)}{\delta(\alpha+1)p(1+\alpha\delta)^{p-1}}.$$

Letting $\delta \to 0$ gives $1 - \left(\frac{\gamma}{(\alpha+1)p}\right)^p \leq 0$, or $\gamma \geq (\alpha+1)p$. This completes the proof.

6. Inequalities for stochastic integrals and Itô processes

In this section we present applications of the results above. Theorem 1.4 in the special case $\alpha = 1$ yields an interesting inequality for the stochastic integrals. Suppose $(\Omega, \mathcal{F}, \mathbb{P})$ is a complete probability space, filtered by a nondecreasing right-continuous family $(\mathcal{F}_t)_{t\geq 0}$ of sub-σ-fields of $\mathcal{F}$. In addition, let $\mathcal{F}_0$ contain all the events of probability 0. Suppose $X = (X_t)_{t\geq 0}$ is an adapted nonnegative right-continuous submartingale with left limits and let Y be the Itô integral of H with respect to X,

$$Y_t = H_0 X_0 + \int_{(0,t]} H_s dX_s, \quad t \geq 0.$$

Here H is a predictable process with values in $[-1, 1]$. Denote $||X||_p = \sup_{t\geq 0} ||X_t||_p$ and $X^* = \sup_{t\geq 0} |X_t|$. We will establish the following extension of Theorem 1.4.

Theorem 6.1. *Under the above conditions, we have, for any $p \geq 2$,*

$$||Y^*||_p \leq 2p||X||_p, \tag{6.1}$$

and the constant $2p$ is the best possible. It is already the best possible in the weaker estimate

$$||Y^*||_p \leq 2p||X^*||_p.$$

Proof. The constant $2p$ is optimal even in the dicrete-time setting, so all we need is to show (6.1). This is a consequence of approximation results of Bichteler [3]. We proceed as follows: consider the family $\mathbf{Y}$ of all processes Y of the form

$$Y_t = H_0 X_0 + \sum_{k=1}^{n} h_k [X_{\tau_k \wedge t} - X_{\tau_{k-1} \wedge t}], \tag{6.2}$$

where n is a positive integer, h_k belongs to $[-1, 1]$ and the stopping times τ_k take only a finite number of finite values, with $0 = \tau_0 \leq \tau_1 \leq \ldots \leq \tau_n$. Let

$$f = (X_{\tau_0}, X_{\tau_1}, \ldots, X_{\tau_n}, X_{\tau_n}, \ldots)$$

and let g be the transform of f by $(H_0, h_1, h_2, \ldots, h_n, 0, 0, \ldots)$. In virtue of Doob's optional sampling theorem, f is a submartingale. Therefore, by Theorem 1.4, if $\tau_n \leq t$ almost surely, then for Y as in (6.2),

$$||Y_t^*||_p = ||g_n^*||_p \leq 2p||f_n||_p \leq 2p||X_t||_p.$$

Now we have that X and H satisfy the conditions of Proposition 4.1 of Bichteler [3]. Thus by (2) of that proposition, if Y is as in the statement of the theorem above, then there is a sequence (Y^j) of elements of $\mathbf{Y}$ such that $\lim_{j\to\infty}(Y^j - Y)^* = 0$ almost surely. Hence, by Fatou's lemma,

$$||Y_t^*||_p \leq 2p||X_t||_p.$$

Now take $t \to \infty$ to complete the proof. □

The result above can be further strengthened. Assume that X is a nonnegative submartingale and $X = X_0 + M + A$ stands for its Doob-Meyer decomposition, uniquely determined by the condition that A is predictable. Let $\alpha \in [0, 1]$ be fixed and suppose ϕ, ψ are predictable processes satisfying $|\phi_s| \leq 1$ and $|\psi_s| \leq \alpha$ for all s. Consider the Itô process Y such that $|Y_0| \leq X_0$ and

$$Y_t = Y_0 + \int_{0+}^{t} \phi_s dM_s + \int_{0+}^{t} \psi_s dA_s$$

for all $t \geq 0$. We have the following sharp bound.

Theorem 6.2. *For X, Y as above, we have*

$$||Y^*||_p \leq (\alpha+1)p||X||_p$$

and the inequality is sharp. So is the weaker estimate

$$||Y^*||_p \leq (\alpha+1)p||X^*||_p.$$

This result can be established using essentially the same approximation arguments as above; we omit the details. We would only like to mention here that there is an alternative way of proving Theorems 6.1 and 6.2, based on Itô's formula applied to the function u (as the function is not of class C^2, one needs some additional „smoothing" arguments to overcome this difficulty). See e.g. [19] or [20] for similar reasoning.

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