SHARP ESTIMATES FOR THE DIAMETER OF A MARTINGALE

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Abstract. We establish sharp weak type and logarithmic estimates for the diameter of the stopped Brownian motion. By standard embedding theorems, the results extend to the case of general real-valued martingales.

1. Introduction

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space filtered by $(\mathcal{F}_t)_{t\geq 0}$, a nondecreasing sequence of sub-σ-algebras of $\mathcal{F}$. Let $B = (B_t)_{t\geq 0}$ be a standard one-dimensional Brownian motion with respect to $(\mathcal{F}_t)_{t\geq 0}$ and let

$$B_t^* = \sup_{0\leq s\leq t} B_s, \qquad B_{t*} = \inf_{0\leq s\leq t} B_s$$

be the maximum and the minimum of B up to time t. We define the *diameter* of the Brownian motion as the process $(B_t^* - B_{t*})_{t\geq 0}$. In the paper we compare the sizes of a Brownian motion and its diameter, when both these processes are stopped at a certain (arbitrary) stopping time τ. A problem of this type was considered in the recent work of [4] by Dubins, Gilat and Meilijson. That paper contains the proof of the estimate

$$||B_\tau^* - B_{\tau *}||_1 \leq \sqrt{3}||B_\tau||_2 \tag{1.1}$$

for any stopping time τ of B satisfying $\mathbb{E}\tau < \infty$. Furthermore, it is shown that the number $\sqrt{3}$ cannot be replaced by a smaller constant. There are several extensions of (1.1) which might be of interest. The martingale maximal weak-type inequality of Doob [3] implies that for any $p \in [1, \infty)$ and any $\tau \in L^{p/2}$,

$$|| \sup_{0\leq s\leq \tau} |B_s| ||_{p,\infty} := \sup_{\lambda>0} \left[\lambda^p \mathbb{P}(\sup_{0\leq s\leq\tau} |B_s| \geq \lambda)\right]^{1/p} \leq ||B_\tau||_p.$$

This gives the following estimate for the diameter:

$$||B_\tau^* - B_{\tau *}||_{p,\infty} \leq 2||B_\tau||_p \tag{1.2}$$

for p and τ as above. Similarly, Doob's moment bound yields

$$|| \sup_{0\leq s\leq\tau} |B_s| \,||_p \leq \frac{p}{p-1}||B_\tau||_p, \tag{1.3}$$

for $1 < p \leq \infty$ and $\tau \in L^{p/2}$, and hence implies the analogous inequality for the diameter:

$$||B_\tau^* - B_{\tau *}||_p \leq \frac{2p}{p-1}||B_\tau||_p.$$

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The following Hardy-Littlewood logarithmic estimate can be regarded as the limit case $p \to 1^+$ of (1.3). As proved by Gilat [6] (consult also Peskir [8]), for $K > 1$ and any stopping time τ of B satisfying $\mathbb{E}\tau^{r/2} < \infty$ for some $r > 1$, we have

$$|| \sup_{0 \le s \le \tau} |B_s| \, ||_1 \le K\mathbb{E}|B_\tau| \log^+ |B_\tau| + M(K).$$

Here $M(K) = 1+(e^K(K-1))^{-1}$ is the best possible. This yields a related inequality for a stopped diameter:

$$||B^*_\tau - B_{\tau *}||_1 \le 2K\mathbb{E}|B_\tau| \log^+ |B_\tau| + 2M(K). \tag{1.4}$$

The principal purpose of the present paper is to establish sharp versions of the inequalities (1.2) and (1.4). Let us start with the logarithmic estimate. Denote by $L(K)$ the smallest extended number L for which the inequality

$$||B^*_\tau - B_{\tau *}||_1 \le K\mathbb{E}|B_\tau| \log^+ |B_\tau| + L \tag{1.5}$$

is valid for all stopping times τ of B satisfying $\mathbb{E}\tau^{r/2} < \infty$ for some $r > 1$. Furthermore, let

$$c = (e^K(K-1))^{-1}. \tag{1.6}$$

Theorem 1.1. *If $c + 2 \le e^K$, then*

$$L(K) = \log \frac{2c(c+2)}{c+1} - \sqrt{c^2+2c} \cdot \arctan \sqrt{c^2+2c}. \tag{1.7}$$

By standard embedding theorems of Dambis, Dubins and Schwarz (see [**?**], [5]; the monograph [10] is also a convenient reference), the result above extends to the class of all real martingales starting from 0.

Theorem 1.2. *Let $c + 2 \le e^K$ and let M be a cadlág $(\mathcal{F}_n)$-martingale starting from 0. Then*

$$|| \sup_t M_t - \inf_t M_t||_1 \le K \sup_t \mathbb{E}|M_t| \log^+ |M_t| + L(K),$$

and the constant $L(K)$ is the best possible.

Let us stress here that we prove the above Hardy-Littlewood bounds for K bounded away from 1: the condition $c+2 \le e^K$ can be rewritten as $K \ge K_0$, where $K_0 = 1.215\ldots$ is the solution to $2 + (e^{K_0}(K_0-1))^{-1} = e^{K_0}$. We believe that for $K \in (1, K_0)$ the estimate (1.5) remains valid for some $L < \infty$ not depending on τ, however, we do not know the value of $L(K)$ in this case.

Concerning the weak-type estimate, we will establish the following statement.

Theorem 1.3. *Let $p \in [1,2]$. For any stopping time τ of B satisfying $\mathbb{E}\tau^{p/2} < \infty$ we have*

$$||B^*_\tau - B_{\tau *}||_{p,\infty} \le \left(\frac{p+2}{2}\right)^{1/p} ||B_\tau||_p \tag{1.8}$$

and inequality is sharp.

As previously, the inequality above extends to the case of general martingales.

Theorem 1.4. *Let $p \in [1,2]$. For any cadlág $(\mathcal{F}_n)$-martingale M starting from 0,*

$$|| \sup_t M_t - \inf_t M_t||_{p,\infty} \le \left(\frac{p+2}{2}\right)^{1/p} ||M||_p$$

and the constant $\left(\frac{p+2}{2}\right)^{1/p}$ *is the best possible.*

A few words about the proof and the organization of the paper. As we have already mentioned, it suffices to focus on the estimates (1.5) and (1.8) only. We start with the Hardy-Littlewood estimate. By standard means, it is reduced in Section 2 to an optimal stopping problem. The solution to this problem is guessed in the Section 3 and then rigorously verified in Section 4. The final part of the paper is devoted to the weak-type estimate.

2. An optimal stopping problem

While studying the inequality (1.5), it is natural to consider the following optimal stopping problem:

$$U_0 = \sup_\tau \mathbb{E}G(B_\tau, B_\tau^*, B_{\tau*}), \tag{2.1}$$

where the gain function G is given by

$$G(x,y,z) = y - z - K|x|\log^+|x|$$

for $(x,y,z) \in \mathcal{A} = \{(x,y,z) : z \le x \le y\}$. Here the supremum is taken over all stopping times τ of B such that τ belongs to some $L^{p/2}$, $p > 1$. In order to treat the problem succesfully, we extend it so that the process $((B_t, B_t^*, B_{t*}))_{t\ge 0}$ can start at the arbitrary points of $\mathcal{A}$. This is straightforward: for $(x,y,z) \in \mathcal{A}$,

$$(B_t, B_t^*, B_{t*})^{(x,y,z)} = (x + B_t, y \vee \sup_{0\le s\le t}(x+B_s), z \wedge \inf_{0\le s\le t}(x+B_s))$$

starts from (x,y,z), is Markov under $\mathbb{P}$ and hence $\mathbb{P}_{x,y,z}$ =Law$((B_t, B_t^*, B_{t*})^{(x,y,z)}|\mathbb{P})$, $(x,y,z) \in \mathcal{A}$, is a Markovian family of probability measures on the canonical space. Therefore we can now extend the optimal stopping problem (2.1) to

$$U_0(x,y,z) = \sup_\tau \mathbb{E}_{x,y,z} G(B_\tau, B_\tau^*, B_{\tau*}). \tag{2.2}$$

As usual, we start the analysis by introducing the continuation set

$$C_0 = \{(x,y,z) \in \mathcal{A} : U_0(x,y,z) > G(x,y,z)\}$$

and the stopping set

$$D_0 = \{(x,y,z) \in \mathcal{A} : U_0(x,y,z) = G(x,y,z)\}.$$

From the general theory of optimal stopping for the Markov processes (see Chapter I in [9]) we infer that the stopping time which gives equality in (2.2), should be defined by

$$\tau_{D_0} = \inf\{t : (B_t, B_t^*, B_{t*}) \in D_0\}.$$

Therefore we have reduced the problems (2.1) and (2.2) to determining the stopping set D_0 and the value function U_0 outside D_0. We see that the underlying process, and hence also the optimal stopping problem, is three dimensional. As we shall see, the solution to it is quite complicated.

3. On the search for the function U_0

The purpose of this section is to *guess* the formula of U_0. Let us stress here that all the calculations carried out below are based on a certain *conjecture* about the form of the stopping set D_0. The rigorous analysis of the optimal stopping problem (2.2) is postponed to the next section.

To get some intuition and ideas, we look first at the inequality (1.1). Dubins, Gilat and Meilijson study in [4] a related optimal stopping problem

$$\overline{U}(x,y,z)=\sup_\tau \mathbb{E}_{x,y,z}\left[B_\tau^*-B_{\tau*}-cB_\tau^2\right]=\sup_\tau \mathbb{E}_{x,y,z}\left[B_\tau^*-B_{\tau*}-c\tau\right],\qquad c>0,$$

where the supremum is taken over all the stopping times of B satisfying $\mathbb{E}\tau<\infty$, and show that $\overline{U}(0,0,0)\leq\frac{3}{4c}$ (this can be easily shown to be equivalent to (1.1)). The optimal stopping rule can be carried out in two steps: first we wait until the diameter of size $1/c$ is obtained; at this moment B must be either at its hitherto maximum, or minimum. Then, if it is at its maximum, wait until the drawdown of size $1/(2c)$ is obtained; if it is at its minimum, wait for the rise of size $1/(2c)$. Here the drawdown and rise at time t are given by $B_t^*-B_t$ and B_t-B_{t*}, respectively. For details, we refer the interested reader to the paper [4].

It is natural to conjecture that in (2.2) we have a similar two-stage stopping rule, which can roughly be implemented as follows. First step is to wait until the diameter grows large, and the second one is to - depending on whether we are at the maximum or at the minimum at the end of the first stage - wait until a drawdown or rise of certain size is observed. In other words, we guess that the optimal rule τ_{D_0} is of the following form. First let

$$\tau_1=\inf\{t:(B_t^*,B_{t*})\in D_1\}\tag{3.1}$$

for some $D_1\subset\{(y,z):y\geq 0\geq z\}$ to be determined, and

$$\tau_{D_0}=\inf\{t>\tau_1:B_t\leq\alpha(B_t^*,B_{t*})\}\tag{3.2}$$

or

$$\tau_{D_0}=\inf\{t>\tau_1:B_t\geq\beta(B_t^*,B_{t*})\},\tag{3.3}$$

for some functions α, β on D_1, satisfying $\alpha\geq\beta$, to be found. Here we take (3.2) or (3.3) depending on whether $B_{\tau_1}=B_{\tau_1}^*$ or $B_{\tau_1}=B_{\tau_1*}$. In the remaining part of this section we present some steps which lead to the explicit formula for the value function U_0, the continuation set C_0 and the stopping set D_0. In order to stress that we deal with the *candidates* for these objects, we will omit the subscripts and write U, C, D instead of U_0, C_0, D_0. For the sake of clarity and convenience, we have split the reasoning into a few numbered parts.

1°. Here is some initial insight into D_1 and the functions α, β, coming directly from their definitions. First, it is clear that D_1 should satisfy the condition that if $(y,z)\in D_1$, then $(y',z')\in D_1$ for any $y'\geq y$, $z'\leq z$. The next observation is that $(-B_t)_{t\geq 0}$ is a Brownian motion and $(-B_t)^*=-B_{t*}$, $(-B_t)_*=-B_t^*$. Therefore we must have that $(y,z)\in D_1$ if and only if $(-z,-y)\in D_1$ and, furthermore, $\alpha(y,z)=-\beta(-z,-y)$ for all $(y,z)\in D_1$. The symmetry of the Brownian motion affects also the function U: since $G(x,y,z)=G(-x,-z,-y)$, we must have, in view of (2.2),

$$U(x,y,z)=U(-x,-z,-y)\qquad\text{for any }(x,y,z)\in\mathcal{A}.\tag{3.4}$$

Note that in terms of D_1, α, β, the stopping set and the continuation set become

$$D_0 = \{(x,y,z) : (y,z) \in D_1 \text{ and } \alpha(y,z) \geq x \geq \beta(y,z)\}, \quad C_0 = \mathcal{A} \setminus D_0.$$

2°. The optimal stopping problem (2.2) leads to the following free-boundary problem (3.5) – (3.12) for the function U. First, by the definition of C and D, we have

$$U > G \qquad \text{on } C, \tag{3.5}$$

$$U = G \qquad \text{on } D. \tag{3.6}$$

The next observation is that the Markov process (B_t, B_t^*, B_{t*}) can change in the second or third coordinate only when it hits one of the planes $x = y$ or $x = z$. It is not optimal to stop there, unless $y = 1$ or $z = -1$ (in fact, as we will see below, the points $(1,1,z)$ and $(-1,y,-1)$ also belong to the continuation set). Indeed, suppose that for some y, z not both equal to 0 and $y \neq 1$, the point (y,y,z) lies in D and let ε be a small positive number. We have

$$\lim_{\varepsilon \to 0^+} \varepsilon^{-1} \mathbb{E}_{y,y,z}(B^*_{\varepsilon^2} - y) = \sqrt{\frac{2}{\pi}}$$

and

$$\lim_{\varepsilon \to 0^+} \varepsilon^{-1} \mathbb{E}_{y,y,z}(B_{\varepsilon^2 *} - z) = 0, \quad \lim_{\varepsilon \to 0^+} \varepsilon^{-1} \mathbb{E}_{y,y,z}(B_{\varepsilon^2} \log^+ B_{\varepsilon^2} - y \log^+ y) = 0,$$

which gives $U(y,y,z) \geq \mathbb{E}_{y,y,z} G(B_{\varepsilon^2}, B^*_{\varepsilon^2}, B_{\varepsilon^2 *}) > G(y,y,z)$ for small ε and hence (y,y,z) lies in the continuation set. The points from the plane $x = z$ are dealt with in the same manner (or one can use the symmetry (3.4)). Thus we have proved that $\alpha(y,z) < y$ and $\beta(y,z) > -z$ for $(y,z) \in D_1$, $y \neq 1$, $z \neq -1$. Hence, using Markovian arguments (see [9]), we infer that U should satisfy

$$U_y(y,y,z) = 0 \qquad \text{for } y \geq 0 \geq z, \tag{3.7}$$

and

$$U_z(z,y,z) = 0 \qquad \text{for } y \geq 0 \geq z. \tag{3.8}$$

Off the planes $x = y$, $x = z$, the process (B_t, B_t^*, B_{t*}) changes only in the first coordinate and hence its behavior is that of one-dimensional Brownian motion. Therefore, again using Markovian arguments (see [9]), we see that the value function U must satisfy the following conditions:

$$U(\cdot, y, z) : x \mapsto U(x,y,z) \text{ is concave for any } (y,z), \tag{3.9}$$

$$U_{xx}(x,y,z) = 0 \quad \text{for } (x,y,z) \in C, \quad x \notin \{y,z\}. \tag{3.10}$$

Now, before we proceed, it is convenient to fix (y,z) and look at the function $U(\cdot,y,z)$. Observe that if $(y,z) \notin D_1$, then for any $x \in [z,y]$ the point (x,y,z) lies in the continuation set. As a consequence of (3.5) and (3.10), $U(\cdot,y,z)$ is linear and strictly majorizes $G(\cdot,y,z)$ (see Figure 1). If $(y,z) \in D_1$, then $(x,y,z) \in C$ if and only if $x \in (\alpha(y,z), y]$ or $x \in [z, \beta(y,z))$. Hence, by (3.5) and (3.10), $U(\cdot,y,z)$ is linear on those intervals, and equals $G(\cdot,y,z)$ on $[\beta(y,z), \alpha(y,z)]$ (see Figures 2 and 3 below). It is clear that neither α nor β takes value in the interval $(-1,1)$: this would contradict the concavity of U.

The next condition we impose comes from the following observation. It is clear that if $\alpha(y,z) > 1$, then, by (3.9), $U(\cdot,y,z)$ has continuous first derivative at

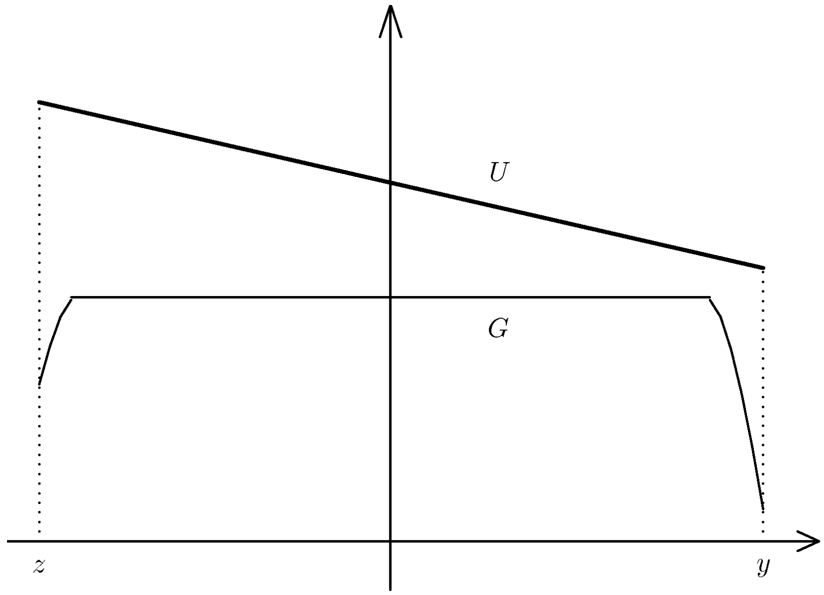

FIGURE 1. The case $(y,z) \notin D_1$. All the points (x,y,z) lie in the continuation set, so $U > G$.

$\alpha(y,z)$: indeed, otherwise (3.5) or (3.9) would not hold (see Figure 2). Therefore, the following principle of smooth-fit is satisfied:

$$U_x(x,y,z) - G_x(x,y,z)|_{x=\alpha(y,z)} = 0 \qquad \text{for } (y,z) \in D_1,\ \alpha(y,z) > 1, \tag{3.11}$$

and, by symmetry,

$$U_x(x,y,z) - G_x(x,y,z)|_{x=\beta(y,z)} = 0 \qquad \text{for } (y,z) \in D_1,\ \beta(y,z) < -1. \tag{3.12}$$

Now, we will try to solve the free-boundary problem formulated above. In the remaining four steps we will specify the functions α and $U(\cdot,y,z)$ on $[\alpha(y,z),y]$.

3° First let us focus on the set $\{(y,z) : \alpha(y,z) > 1\}$ (the situation is illustrated on Figure 2). In view of (3.6), (3.10) and (3.11) we have, for $x \geq \alpha(y,z)$,

$$\begin{aligned} U(x,y,z) &= G(\alpha(y,z),y,z) + G_x(\alpha(y,z),y,z)(x-\alpha(y,z)) \\ &= y - z - Kx[1+\log\alpha(y,z)] + K\alpha(y,z). \end{aligned} \tag{3.13}$$

Applying (3.7) gives

$$1 - \frac{Ky\alpha_y(y,z)}{\alpha(y,z)} + K\alpha_y(y,z) = 0$$

and we easily verify that $\alpha(y,z) = \frac{K-1}{K}y$ solves the equation. We *assume* that this is the right formula for α on the set $\{(y,z) \in D_1 : y > K/(K-1)\}$ (so that $\alpha(y,z) > 1$ here). This implies the following expression for U on this set:

$$U(x,y,z) = Ky - z - Kx\left[1 + \log\left(\frac{K-1}{K}y\right)\right] \qquad \text{for } x \geq \frac{K-1}{K}y > 1. \tag{3.14}$$

4° Now suppose $(y,z) \in D_1$ is such that $\alpha(y,z) = 1$ (see Figure 3). Obviously,

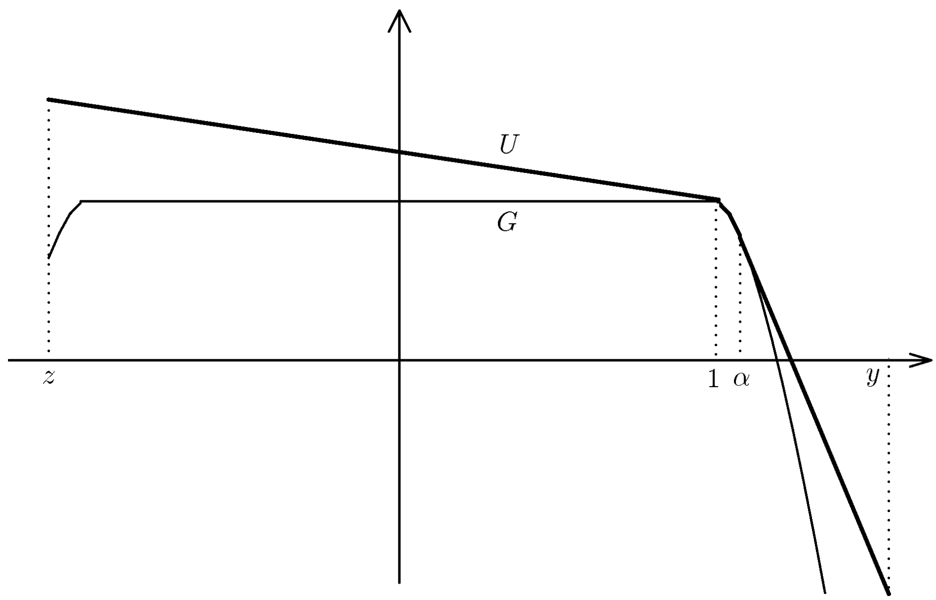

FIGURE 2. The case $(y, z) \in D_1$, $\alpha(y, z) > \beta(y, z) = 1$. Observe $U(\cdot, y, z)$ is not of class C^1 at 1, but has continuous derivative at $\alpha(y, z)$.

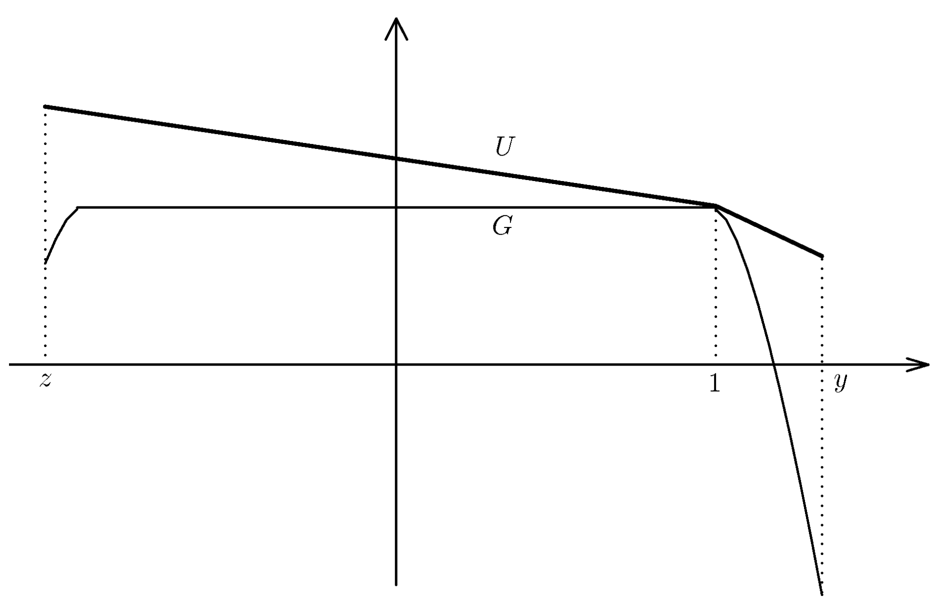

FIGURE 3. The case $(y, z) \in D_1$ and $\alpha(y, z) = \beta(y, z) = 1$. U is not of class C^1 at 1

we have $y \geq 1$ and it follows from (3.6) and (3.10) that if $x \in [1, y]$, then

$$\begin{aligned} U(x,y,z) &= \frac{y-x}{y-1} \cdot G(1,y,z) + \frac{x-1}{y-1} \cdot U(y,y,z) \\ &= \frac{y-x}{y-1} \cdot (y-z) + \frac{x-1}{y-1} \cdot U(y,y,z). \end{aligned} \tag{3.15}$$

Using (3.7), we get

$$\frac{d}{dy} U(y,y,z) = U_x(y,y,z) = -\frac{y-z}{y-1} + \frac{U(y,y,z)}{y-1}.$$

Solving this differential equation gives

$$U(y,y,z) = -(y-1)\log(y-1) + 1 - z - \kappa_1(y-1), \tag{3.16}$$

where κ_1 is a real number. We expect $y \mapsto U(y,y,z)$ to be continuous, hence taking $y = K/(K-1)$ and using the formula for U from the previous step, we obtain

$$\kappa_1 = K - 1 + \log(K-1) = -1 - \log c.$$

To complete the description of α, we must consider the set where it takes values not larger than -1. Here we make an assumption that the minimum of α equals -1; as we will see in the next section, this leads to the condition $c+2 \leq e^K$. Hence, suppose that $(y,z) \in D_1$ and $\alpha(y,z) = -1$. Then for $x \in [-1, y]$ we have

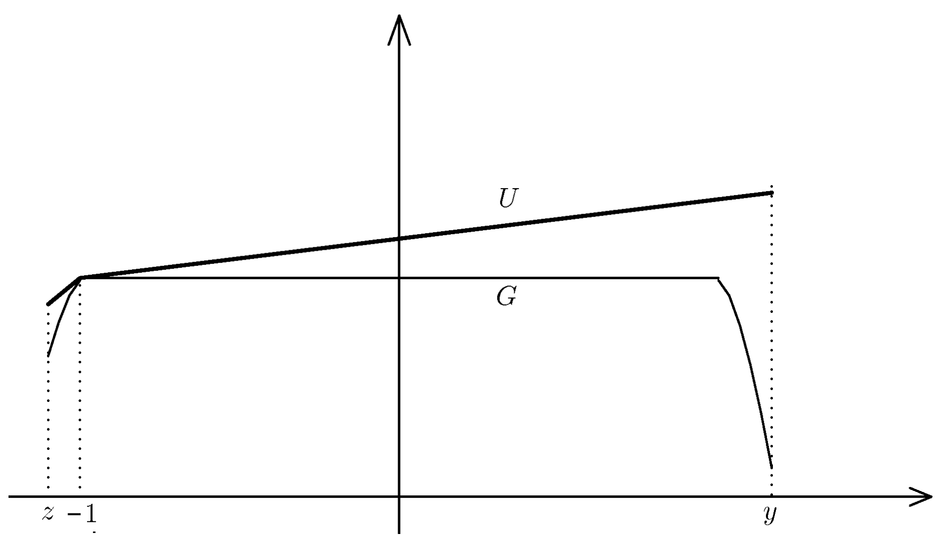

FIGURE 4. The case $(y,z) \in D_1$ and $\alpha(y,z) = \beta(y,z) = -1$. $U(\cdot,y,z)$ is not of class C^1 at -1

$$U(x,y,z) = \frac{y-x}{y+1} \cdot (y-z) + \frac{x+1}{y+1} U(y,y,z) \tag{3.17}$$

and hence (3.7) implies

$$\frac{d}{dy} U(y,y,z) = U_x(y,y,z) = -\frac{y-z}{y+1} + \frac{U(y,y,z)}{y+1}.$$

This gives

$$U(y,y,z)=-(y+1)\log(y+1)-1-z+\kappa_2(y+1). \tag{3.18}$$

Now it is clear that the passage from $\alpha(y,z)=-1$ to $\alpha(y,z)=1$ should take place at such y_0, for which $U(\cdot,y_0,z)$ is constant and equal to $G(1,y_0,z)=y_0-z$ (see Figures 3 and 4). Combining this with the right hand sides of (3.16) and (3.18) yields $y_0=c+1$ and $\kappa_2=1+\log(c+2)$. Therefore (3.15) and (3.17) give us the following expressions for U: if $\alpha(y,z)=1$ (that is, $(y,z)\in D_1$ and $y\in[y_0,K/(K-1)]$), then

$$U(x,y,z)=y-z-(x-1)\log\frac{y-1}{c},\qquad x\in[1,y]. \tag{3.19}$$

If $\alpha(y,z)=-1$ (which is equivalent to $(y,z)\in D_1$ and $y<y_0$), then

$$U(x,y,z)=y-z-(x+1)\log\frac{y+1}{c+2},\qquad x\in[-1,y]. \tag{3.20}$$

Using the condition (3.4), we obtain the formulae for U on the sets

$$\{(x,y,z):(y,z)\in D_1,\ z\in[-K/(K-1),-y_0],\ x\in[z,-1]\}$$

and

$$\{(x,y,z):(y,z)\in D_1,\ z>-y_0,\ x\in[z,1]\}.$$

5° The next step is to guess the set D_1. Intuitively, the boundary of this set should describe the passage from the situation illustrated on Figure 1 (no intersection with the stopping set, smooth $U(\cdot,y,z)$) to the one from the Figures 3 and 4 (the one-point intersection with D, which is the only point where $U(\cdot,y,z)$ may be not differentiable). Therefore it is natural to conjecture that $(y,z)\in\partial D_1$ if $\alpha(y,z)=\pm1$ and $U(\cdot,y,z)$ is of class C^1, or

$$\partial D_1=\{(y,z):\alpha(y,z)=\beta(y,z)=\pm1 \text{ and } U_x(\alpha(y,z)+,y,z)=U_x(\beta(y,z)-,y,z)\}.$$

If $\alpha(y,z)=\beta(y,z)=-1$, then we have, by (3.4), $U_x(-1-,y,z)=-U_x(1+,-z,-y)$, and hence, in view of (3.19) and (3.20), $U_x(-1+,y,z)=U_x(-1-,y,z)$ reads $-\log\frac{y-1}{c}=\log\frac{-z+1}{c+2}$, or

$$(y-1)(-z+1)=c(2+c).$$

A symmetric condition (corresponding to the case $\alpha(y,z)=\beta(y,z)=1$) reads $(y+1)(-z-1)=c(2+c)$. Thus we have obtained

$$D_1=\{(y,z):\min\{(y-1)(-z+1),(y+1)(-z-1)\}\geq c(2+c)\}.$$

It is clear why we have taken the inequality "$\geq$" in the definition above: see the property of D_1 formulated just at the beginning of 1°. Before we proceed, let us introduce the function $s:[0,(c+1)^2]\to\mathbb{R}$ such that

$$s(y)=\sup\{z:(y,z)\in D_1\}. \tag{3.21}$$

To gain some intuition about this function, pick $y\in[0,(c+1)^2]$. Directly from the definition of D_1, we see that if z is close to 0, then $(y,z)\notin D_1$; however, there is a level (which is precisely $s(y)$) below which the inclusion $(y,z)\in D_1$ holds. To put it in other words, $s(y)$ is the smallest number such that if a trajectory of B up to time t runs strictly between $s(y)$ and y, then we have $\tau_1>t$.

It can be easily verified that

$$s(y)=\begin{cases}-c(2+c)(y+1)^{-1}-1 & \text{if } y\in[0,c+1],\\ -c(2+c)(y-1)^{-1}+1 & \text{if } y\in[c+1,(c+1)^2].\end{cases} \tag{3.22}$$

6° Finally, we will determine the expression for U on the set $\{(x,y,z)\in\mathcal{A}:(y,z)\notin D_1\}$. It turns out to be the most elaborate step. We start from the observation that from (3.7), (3.8), (3.10) and Itô's formula it is apparent that

$$U(x,y,z)=\mathbb{E}_{x,y,z}U(B_{\tau_1},B^*_{\tau_1},B_{\tau_1*}). \tag{3.23}$$

Now we will provide an expression for U_y. Fix $(x,y,z)\in\mathcal{A}$ and $a>0$ such that $(y+a,z)\notin D_1$. In order to compare $U(x,y,z)$, $U(x,y+a,z)$ and work under the same probability $\mathbb{P}_{0,0,0}$, introduce the stopping times

$$\sigma_1=\inf\{t:(y\vee(x+B^*_t),z\wedge(x+B_{t*}))\in D_1\},$$
$$\sigma_2=\inf\{t:((y+a)\vee(x+B^*_t),z\wedge(x+B_{t*}))\in D_1\}.$$

By (3.23), we have that $U(x,y,z)=\mathbb{E}_{0,0,0}X$ and $U(x,y+a,z)=\mathbb{E}_{0,0,0}Y$, where

$$X=U(x+B_{\sigma_1},y\vee(x+B^*_{\sigma_1}),z\wedge(x+B_{\sigma_1*})),$$
$$Y=U(x+B_{\sigma_2},(y+a)\vee(x+B^*_{\sigma_2}),z\wedge(x+B_{\sigma_2*})).$$

Now consider the events $A_1=\{x+B^*_{\sigma_1}<y\}$, $A_2=\{x+B^*_{\sigma_2}>y+a\}$ and $A_3=\Omega\setminus(A_1\cup A_2)=\{x+B^*_{\sigma_1}\geq y,\, x+B^*_{\sigma_2}\leq y+a\}$. On A_1, the Brownian motion $x+B$ does not rise to y before $(y\vee(x+B^*),z\wedge(x+B_*))$ reaches D_1. Therefore $y\vee(x+B^*_{\sigma_1})=y$ and hence $z\wedge(x+B_{\sigma_1*})=x+B_{\sigma_1*}=s(y)$. In other words, $x+B$ reaches $s(y)$ before it visits y, and hence it drops to $s(y+a)$ before it rises to $y+a$ (see Figure 5); consequently, $x+B_{\sigma_2}=s(y+a)$, so we have

$$(Y-X)I_{A_1}=[U(s(y+a),y+a,s(y+a))-U(s(y),y,s(y))]I_{A_1}.$$

Arguing as previously, we see that on A_2, $x+B$ hits $y+a$ before $s(y+a)$, and therefore visits y before $s(y)$. This implies that the random variables in the definitions of σ_1 and σ_2 are the same, hence these stopping times are equal.

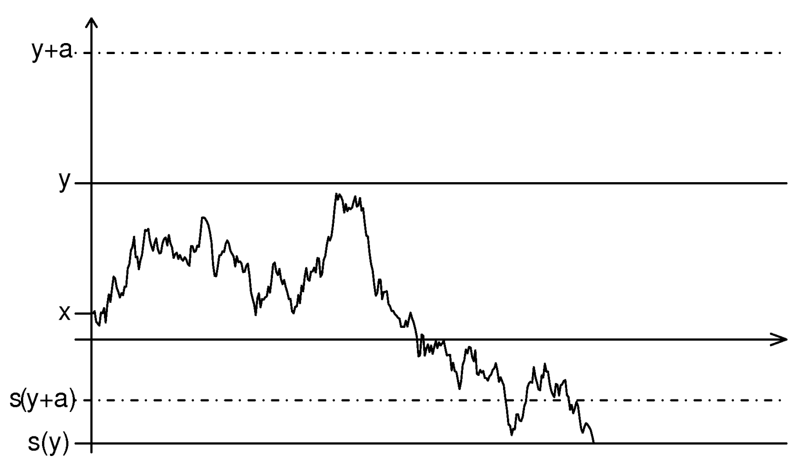

FIGURE 5. An example of a trajectory from A_1. The Brownian motion hits $s(y)$ before it visits y.

In consequence, we have $X = Y$ on A_2. To deal with A_3, observe first that $\mathbb{P}_{0,0,0}(Y = U(s(y+a), y+a, s(y+a))|A_3) = 1$; indeed, $x + B^*_{\sigma_2} \le y + a$ implies $x + B^*_{\sigma_2} < y + a$ almost surely and hence, almost all trajectories from A_3, visit the point $s(y+a)$ before $y+a$. To study X, we split A_3 into two events. Introduce an auxiliary stopping time $\eta = \inf\{t : x + B_t \in \{s(y+a), y\}\}$, and note that the set A_3 is the sum of $A_4 = A_3 \cap \{x + B_\eta = s(y+a)\}$ and $A_5 = A_3 \cap \{x + B_\eta = y\}$. It is evident that we have

$$A_4 = \{x + B_\eta = s(y+a), \text{ then } x + B \text{ reaches } y \text{ before } s(y)\} \tag{3.24}$$

$$A_5 = \{x + B_\eta = y, \text{ then } x + B \text{ reaches } s(y+a) \text{ before } y+a\}$$

By the strong Markov property, this gives

$$\mathbb{P}_{0,0,0}(A_4) = \frac{y-x}{y-s(y+a)} \cdot \frac{s(y+a)-s(y)}{y-s(y)}$$

and

$$\mathbb{P}_{0,0,0}(A_5) = \frac{x-s(y+a)}{y-s(y+a)} \cdot \frac{a}{y+a-s(y+a)}.$$

Now we will let $a \to 0^+$. From the above estimates it follows that

$$\lim_{a\to 0+} \frac{1}{a}\mathbb{P}_{0,0,0}(A_3) = \frac{(y-x)s'(y+) + x - s(y)}{(y-s(y))^2}.$$

Moreover, it is evident that

$$\lim_{a\to 0+} \frac{1}{a}\mathbb{E}_{0,0,0} X I_{A_4} = \frac{(y-x)s'(y+)}{(y-s(y))^2} U(y,y,s(y)), \tag{3.25}$$

$$\lim_{a\to 0+} \frac{1}{a}\mathbb{E}_{0,0,0} X I_{A_5} = \frac{x-s(y)}{(y-s(y))^2} U(s(y),y,s(y)). \tag{3.26}$$

Indeed, to see (3.25), one must look carefully at (3.24) and (3). Having visited y, the trajectory has an overwhelming probability of terminating at time σ_1 somewhere between y and $y+a$ (the remaining trajectories must drop below $s(y+a)$ before they rise to $y+a$). The limit (3.26) holds for similar reasons.

Combining all the above facts together, we obtain

$$\begin{aligned}
U_y(x, y+, z) &= \lim_{a\to 0^+} \frac{1}{a}\mathbb{E}(Y-X) = \lim_{a\to 0^+} \frac{1}{a}\mathbb{E}\left[(Y-X)I_{A_1} + YI_{A_3} - XI_{A_4} - XI_{A_5}\right] \\
&= \mathbb{P}(A_1)\cdot \frac{d^+}{dy}U(s(y),y,s(y)) + \frac{(y-x)s'(y+) + x - s(y)}{(y-s(y))^2}U(s(y),y,s(y)) \\
&\quad - \frac{(y-x)s'(y+)}{(y-s(y))^2}U(y,y,s(y)) - \frac{x-s(y)}{(y-s(y))^2}U(s(y),y,s(y)) \\
&= \frac{y-x}{y-s(y)}\frac{d^+}{dy}U(s(y),y,s(y)) \\
&\quad + \frac{(y-x)s'(y+)}{(y-s(y))^2}\left[U(s(y),y,s(y)) - U(y,y,s(y))\right] \\
&= \frac{y-x}{y-s(y)}\frac{d^+}{dy}U(s(y),y,s(y)) - \frac{(y-x)s'(y+)}{y-s(y)}U_x(s(y),y,s(y)) \\
&= \frac{y-x}{y-s(y)}U_y(s(y),y,s(y)),
\end{aligned}$$

where in the fifth passage we have used the fact that $x \mapsto U(x,y,s(y))$ is linear on $[s(y),y]$ and in the latter one we have exploited (3.8). Now it follows from (3.4), (3.19), (3.20) and (3.22) that $U_y(s(y),y,s(y))=1$ and hence

$$U_y(x,y+,z)=\frac{(y-x)(y+1)}{(y+1)^2+c(2+c)} \tag{3.27}$$

if $y\leq c+1$, and

$$U_y(x,y+,z)=\frac{(y-x)(y-1)}{(y-1)^2+c(2+c)} \tag{3.28}$$

if $y\geq c+1$. As

$$U(x,y,z)=U(x,s^{-1}(z),z)-\int_y^{s^{-1}(z)}U_y(x,t+,z)dt,$$

we are able to derive the formulae for U. By (3.19) and (3.20),

$$U(x,s^{-1}(z),z)=\begin{cases} s^{-1}(z)-z-(x+1)\log(\frac{s^{-1}(z)+1}{c+2}) & \text{if } s^{-1}(z)\leq c+1,\\ s^{-1}(z)-z-(x-1)\log(\frac{s^{-1}(z)-1}{c}) & \text{if } s^{-1}(z)\geq c+1.\end{cases}$$

Therefore, if $(y,z)\notin D_1$ and $y\geq c+1$, then

$$\begin{aligned}U(x,y,z)=y-z+\frac{x-1}{2}&\left[\log\frac{(s^{-1}(z)-1)^2+c^2+2c}{(y-1)^2+c(2+c)}-2\log\frac{s^{-1}(z)-1}{c}\right]\\ &+\sqrt{c^2+2c}\left[\arctan\frac{s^{-1}(z)-1}{\sqrt{c^2+2c}}-\arctan\frac{y-1}{\sqrt{c^2+2c}}\right].\end{aligned} \tag{3.29}$$

if $(y,z)\notin D_1$ and $s^{-1}(z)\leq c+1$, then

$$\begin{aligned}U(x,y,z)=y-z+\frac{x+1}{2}&\left[\log\frac{(s^{-1}(z)+1)^2+c^2+2c}{(y+1)^2+c(2+c)}-2\log\frac{s^{-1}(z)+1}{c+2}\right]\\ &+\sqrt{c^2+2c}\left[\arctan\frac{s^{-1}(z)+1}{\sqrt{c^2+2c}}-\arctan\frac{y+1}{\sqrt{c^2+2c}}\right].\end{aligned} \tag{3.30}$$

Finally, if $(y,z)\notin D_1$ and $y<c+1<s^{-1}(z)$, then

$$\begin{aligned}U(x,y,z)=y-z-(x-1)\log\frac{s^{-1}(z)-1}{c}&\\ +\frac{x+1}{2}\log\frac{2(c+1)(c+2)}{(y+1)^2+c^2+2c}&+\frac{x-1}{2}\log\frac{(s^{-1}(z)-1)^2+c^2+2c}{2c(c+1)}\\ +\sqrt{c^2+2c}&\left[\arctan\frac{s^{-1}(z)-1}{\sqrt{c^2+2c}}-\arctan\frac{y+1}{\sqrt{c^2+2c}}\right]\\ +\sqrt{c^2+2c}&\left[\arctan\sqrt{\frac{c+2}{c}}-\arctan\sqrt{\frac{c}{c+2}}\right].\end{aligned} \tag{3.31}$$

As $s^{-1}(0)=(c+1)^2$, it follows from the latter formula, that

$$\begin{aligned} U(0,0,0)= &\ \log(c+2)+\frac{1}{2}\left[\log\frac{2(c+1)(c+2)}{c^2+2c+1}-\log\frac{(c^2+2c)^2+c^2+2c}{2c(c+1)}\right] \\ &+\sqrt{c^2+2c}\cdot\arctan\sqrt{c^2+2c}=L(K). \end{aligned} \tag{3.32}$$

Here we have used the identity

$$\arctan\frac{1}{\sqrt{c^2+2c}}=2\arctan\sqrt{\frac{c+2}{c}}-\frac{\pi}{2}=\arctan\sqrt{\frac{c+2}{c}}-\arctan\sqrt{\frac{c}{c+2}}.$$

4. The proof of the LlogL inequality

The function U constructed in the previous section is the candidate for the solution to the optimal stopping problem (2.2). First we will check that it solves the free-boundary problem (3.5)–(3.12). In fact the only condition which needs checking is the first one.

Lemma 4.1. *The function U satisfies* (3.5).

Proof. The majorization is valid on the set $\{(x,y,z):(y,z)\in D_1\}$: this follows immediately from the construction. Suppose then, that $(y,z)\notin D_1$. In view of (3.4), it suffices to show $U(x,y,z)\geq G(x,y,z)$ for nonnegative x. We have, by (3.27) and (3.28),

$$U_y(x,y+,z)=1+\frac{(-1-x)(y+1)-c(2+c)}{(y+1)^2+c(2+c)}\leq 1,$$

if $y\leq c+1$, and

$$U_y(x,y+,z)=1+\frac{(1-x)(y-1)-c(2+c)}{(y-1)^2+c(2+c)}\leq 1+\frac{y-1-c(2+c)}{(y-1)^2+c(2+c)}\leq 1,$$

if $y>c+1$. Therefore, for fixed x, z, the function $y\mapsto U(x,y,z)-G(x,y,z)$, $y\in[x,s^{-1}(z)]$, is nonincreasing and

$$U(x,y,z)-G(x,y,z)\geq U(x,s^{-1}(z),z)-G(x,s^{-1}(z),z)\geq 0,$$

as $(s^{-1}(z),z)\in D_1$ and we have the majorization here. □

Lemma 4.2. *The function U satisfies* (3.6) – (3.12).

Proof. In fact, all these conditions follow immediately from the construction of U. First, (3.6) is just a part of the definition of U. The equations (3.7) and (3.8) were the key to determine the formula for $U(y,y,z)$ and $U(z,y,z)$; see steps 3°, 4° and 6° above. Hence they hold true. The condition (3.9) requires checking only in the case $(y,z)\in D_1$, $\alpha(y,z)=\beta(y,z)=\pm 1$; for example, if both α and β equal 1, then all we need is to check the inequality for the one-sided derivatives of $U(\cdot,y,z)$: we have

$$U_x(1-,y,z)=-U_x(-1+,-z,-y)=\log\frac{-z+1}{c+2},$$

$$U_x(1+,y,z)=-\log\frac{y-1}{c},$$

so $U_x(1-,y,z)\geq U_x(1+,y,z)$ follows at once from $(y,z)\in D_1$. The equation (3.10) is a consequence of (3.13), (3.15) and (3.17), which constitute the definition of U. The validity of (3.11) and (3.12) is guaranteed by step 3°. □

We are ready to prove that U is the solution to the optimal stopping problem (2.2).

Theorem 4.1. *We have $C = C_0$, $D = D_0$ and $U = U_0$.*

Proof. First we approximate U by a sequence of smooth functions. For any $\varepsilon > 0$, there is $U^\varepsilon : \mathcal{A} \to \mathbb{R}$ of class C^2 such that $U = U_\varepsilon$ on $\{(x,y,z) : y-x < \varepsilon \text{ or } x-z < \varepsilon\}$, for any y, z $U^\varepsilon(\cdot, y, z)$ is concave and $U_\varepsilon \leq U \leq U_\varepsilon + \varepsilon$. By Itô's formula applied to U^ε, we obtain

$$\begin{aligned} U^\varepsilon(B_t, B_t^*, B_{t*}) &= U^\varepsilon(B_0, B_0^*, B_{0*}) + \int_0^t U_x^\varepsilon(B_s, B_s^*, B_{s*})dB_s \\ &\quad + \int_0^t U_y^\varepsilon(B_s, B_s^*, B_{s*})dB_s^* + \int_0^t U_z^\varepsilon(B_s, B_s^*, B_{s*})dB_{s*} \\ &\quad + \frac{1}{2}\int_0^t U_{xx}^\varepsilon(B_s, B_s^*, B_{s*})ds. \end{aligned} \tag{4.1}$$

Note that the second integral vanishes. Indeed, the process B_s^* changes only on the plane $x = y$ and U_y^ε equals zero there, by (3.7) and the fact that $U^\varepsilon = U$ on some neighborhood of the plane. Similarly, the third integral is equal to 0. Furthermore, the last integral is nonpositive, which is a consequence of the concavity of $U^\varepsilon(\cdot, y, z)$ for any (y, z).

Now let τ be a stopping time of B satisfying $\mathbb{E}\tau^{p/2} < \infty$ for some $p > 1$. By the construction of U^ε and (3.5), (3.6), we have $U^\varepsilon + \varepsilon \geq U \geq G$, and hence (4.1) yields

$$\mathbb{E}_{x,y,z} G(B_{\tau\wedge t}, B_{\tau\wedge t}^*, B_{\tau\wedge t*}) - \varepsilon \leq U^\varepsilon(x,y,z) \leq U(x,y,z).$$

Now we infer from the Burkholder-Davis-Gundy inequality that $\mathbb{E}_{x,y,z}B_\tau^*$, $\mathbb{E}_{x,y,z}B_{\tau*}$ and $\mathbb{E}_{x,y,z}\sup_{s\leq\tau}|B_s|\log^+|B_s|$ are finite. Hence, it suffices to let $t \to \infty$ and use the fact that τ and $\varepsilon > 0$ were arbitrary, to get $U \geq U_0$.

To prove the reverse estimate, we will use the stopping time τ_D. Clearly, we have $U = G \leq U_0$ on D, so it remains to prove the inequality on the set C. We will show that

$$\mathbb{E}_{x,y,z}\tau_D^{p/2} < \infty \quad \text{ if } p < K \text{ and } (x,y,z) \in C. \tag{4.2}$$

Consider the stopping time

$$\begin{aligned} \eta &= \inf\left\{ t : B_t + (c+1)^2 \leq \frac{K-1}{K}\big(B_t^* - (c+1)^2\big) \right\} \\ &= \inf\left\{ t : B_t + a \leq \frac{K-1}{K}\big(B_t^* + a\big) \right\}, \end{aligned}$$

where $a = (2K-1)(c+1)^2$. By the results of Wang [12], we have $\mathbb{E}_{x,y,z}\eta^{p/2} < \infty$ if $p < K$. To relate this stopping time to τ_D, observe first, that $\eta \geq \tau_1$. Indeed, we have

$$B_\eta^* \geq (c+1)^2 = s^{-1}(0) \geq s^{-1}(B_{\eta*})$$

or

$$B_{\eta*} \leq B_\eta \leq -(c+1)^2 \leq s(0) \leq s(B_\eta^*)$$

and it suffices to use (3.21). Now, note that on $\{B_{\tau_1} = B^*_{\tau_1}\}$, we have

$$\begin{aligned} B_\eta &= \frac{K-1}{K}(B^*_\eta - (c+1)^2) - (c+1)^2 \\ &\leq -I_{\{B^*_\eta < c+1\}} + I_{\{c+1 \leq B^*_\eta \leq K/(K-1)\}} + \frac{K-1}{K} B^*_\eta I_{\{B^*_\eta > K/(K-1)\}} \\ &= \alpha(B^*_\eta, B_{\eta *}), \end{aligned}$$

which implies that $\tau_D \leq \eta$. Consequently, if $p < K$, $\mathbb{E}_{x,y,z}\tau_D^{p/2} I_{\{B_{\tau_1} = B^*_{\tau_1}\}} \leq \mathbb{E}_{x,y,z}\eta^{p/2} < \infty$. It can be proved similarly, using a ,,symmetric" stopping time

$$\eta = \inf\left\{t : B_t - (c+1)^2 \geq \frac{K-1}{K}\left(B_{t*} + (c+1)^2\right)\right\},$$

that $\mathbb{E}_{x,y,z}\tau_D{}^{p/2} I_{\{B_{\tau_1} = B_{\tau_1 *}\}} < \infty$, thus establishing (4.2).

The next observation is that the process $(B_{\tau_D \wedge t}, B^*_{\tau_D \wedge t}, B_{\tau_D \wedge t*})$ moves on the set C, where the function U is of class C^2. Hence, by Itô's formula, for any t,

$$\mathbb{E}_{x,y,z} U(B_{\tau_D \wedge t}, B^*_{\tau_D \wedge t}, B_{\tau_D \wedge t*}) = U(x,y,z). \tag{4.3}$$

By Burkholder-Davis-Gundy inequality, we have that $B^*_{\tau_D}$, $B_{\tau_D *}$ belong to L^p, $p < K$. In consequence, they are both integrable along with $\sup_{0 \leq s \leq \tau_D} |B_s| \log^+ |B_s|$. Hence, by Lebesgue's dominated convergence theorem, letting $t \to \infty$ yields

$$\mathbb{E}_{x,y,z} G(B_{\tau_D}, B^*_{\tau_D}, B_{\tau_D *}) = \mathbb{E}_{x,y,z} U(B_{\tau_D}, B^*_{\tau_D}, B_{\tau_D *}) = U(x,y,z),$$

that is, $U_0 \geq U$. This completes the proof. □

5. The weak type estimate

Now we deal with inequality (1.8). By homogeneity, it suffices to show that

$$\mathbb{P}(B^*_\tau - B_{\tau *} \geq 1) \leq \frac{p+2}{2}\, \mathbb{E}|B_\tau|^p$$

for any stopping time τ, for which $\mathbb{E}\tau^{p/2} < \infty$. Therefore we are forced to consider the following optimal stopping problem

$$V(x,y,z) = \sup \mathbb{E}_{x,y,z} H(B_\tau, B^*_\tau, B_{\tau *}), \qquad (x,y,z) \in \mathcal{A}, \tag{5.1}$$

where supremum is taken over all stopping times τ of B as above and

$$H(x,y,z) = I_{\{y-z \geq 1\}} - \frac{p+2}{2}|x|^p.$$

It turns out that if $p \in [1,2]$, then the optimal strategy is to wait until the diameter reaches 1: $\overline{\tau} = \inf\{t : B^*_t - B_{t*} \geq 1\}$ Since $\overline{\tau} \leq \inf\{t : \sup_{0 \leq s \leq t} |B_s| = 1/2\}$, we infer that $\overline{\tau}$ is exponentially integrable and hence may be taken into account in (5.1).

Lemma 5.1. *Let $v : \mathcal{A} \to \mathbb{R}$ be given by $v(x,y,z) = \mathbb{E}_{x,y,z}|B_{\overline{\tau}}|^p$.*

(i) We have

$$\begin{aligned} v(x,y,z) = (1-y)^p(y-x) &+ (z+1)^p(x-z) \\ &- \frac{x}{p+1}[(1+z)^{p+1} + (-z)^{p+1} - y^{p+1} - (1-y)^{p+1}] \\ &+ \frac{1}{p+2}[(z+z)^{p+2} - (-z)^{p+2} - y^{p+2} + (1-y)^{p+2}] \end{aligned} \tag{5.2}$$

if $y-z<1$, and $v(x,y,z)=|x|^p$ for $y-z\geq 1$.

(ii) We have the majorization

$$v(x,y,z)\leq \frac{2}{p+2}I_{\{y-z<1\}}+|x|^p. \tag{5.3}$$

Proof. (i) Fix $(x,y,z)\in\mathcal{A}$. If $y-z\geq 1$, then $\overline{\tau}=0$, so $v(x,y,z)=|x|^p$, as claimed. Suppose that $y-z<1$; then (5.2) can be verified with the use of Itô's formula. Indeed, if we denote the right-hand side by $f(x,y,z)$, then

$$|B_{\overline{\tau}}|^p=f(B_{\overline{\tau}},B^*_{\overline{\tau}},B_{\overline{\tau}*})=f(x,y,z)+\int_0^{\overline{\tau}}\frac{\partial f(B_u,B_u^*,B_{u*})}{\partial x}dB_u,$$

since the remaining integrals are zero (we have $f_{xx}=0$, $f_y(y,y,z)=0$ and $f_z(z,y,z)=0$ for all x, y, z). Hence, it remains to take the expectation of both sides and (5.2) follows.

We would like, however, to provide some arguments which have led us to the right-hand side of (5.2). Suppose that $y-z<1$. We see that $B_{\overline{\tau}}$ takes values in the set $[y-1,z]\cup[y,z+1]$ and has at most two atoms, at $y-1$ and $z+1$. Indeed, $B_{\overline{\tau}}=y-1$ if and only if B reaches $y-1$ before y, hence $\mathbb{P}(B_{\overline{\tau}}=y-1)=y-x$. Similarly, $\mathbb{P}(B_{\overline{\tau}}=z+1)=x-z$. To find the density of the continuous part of $B_{\overline{\tau}}$, we will use the following intuitive argument (which can be easily made precise by approximation by a symmetric random walk). For $w\in[y,z+1)$, we see that $B_{\overline{\tau}}=w$ if the following two-step procedure takes place: first, B visits $w-1$ before w; then, it does not drop below $w-1$ before it reaches w. As the probability of the second step does not depend on w, we infer that the density of $B_{\overline{\tau}}$ on $[y,z+1]$ is of the form $g(w)=\kappa(w-x)$, where κ is a certain constant. Similarly, we see that the density of $B_{\overline{\tau}}$ on $[y-1,z]$ is equal to $g(w)=\kappa(x-w+1)$ (with the same constant κ). The integral of g must be equal to $1-(y-z)$, which yields $\kappa=1$. Having determined the distribution of $B_{\overline{\tau}}$, one easily verifies (5.2).

(ii) It suffices to prove (5.3) for $y-z<1$. Since $v(x,y,z)=v(-x,-z,-y)$, we may assume that x is nonnegative. We have

$$\begin{aligned}v_z(x,y,z)&=x[-(1+z)^p+(-z)^p+p(z+1)^{p-1}]\\&\quad+(1+z)^pz+(-z)^{p+1}-p(z+1)^{p-1}z\\&\geq x\cdot(-z)^p+(-z)^{p+1}\geq 0,\end{aligned}$$

which implies that it is enough to show the majorization for $z=0$. Now we have

$$v_y(x,y,0)=x[p(1-y)^{p-1}+y^p-(1-y)^p]-p(1-y)^{p-1}y+(1-y)^py-y^{p+1}.$$

Since $v_y(y,y,0)=0$ and the expression in the square brackets is nonnegative, we infer that $v_y(x,y,0)\leq 0$ and therefore we reduce the proof of (5.3) to the case $x=y\geq 0$, $z=0$. The inequality takes form

$$x-\frac{x}{p+1}[1-x^{p+1}-(1-x)^{p+1}]+\frac{1}{p+2}[1-x^{p+2}+(1-x)^{p+2}]-\frac{2}{p+2}-x^p\leq 0$$

and is to be valid for $x\in[0,1]$. Denoting the left hand side by $F(x)$, we have $F(0)=0$ and hence it suffices to show that F is nonincreasing. We derive that

$$F'(x)=\frac{p}{p+1}[1-(1-x)^{p+1}]+\frac{x^{p+1}}{p+1}-px^{p-1}-x(1-x)^p.$$

If $x \leq 1/2$, then, using $1-(1-x)^{p+1} \leq (p+1)x$ and $x^{p+1} \leq x(1-x)^p$ we obtain

$$F'(x) \leq p(x - x^{p-1}) - \frac{p}{p+1} x(1-x)^p < 0,$$

as $p \in [1,2]$. If $x > 1/2$, then

$$F'(x) \leq \frac{p}{p+1} + \frac{1}{p+1} - px^{p-1} \leq 1 - \frac{p}{2^{p-1}} \leq 0,$$

where in the last passage we again used $p \in [1,2]$. This completes the proof. □

Remark 1. *Observe that inequality* (5.3) *fails to hold if* $p > 2$. *Therefore* $\overline{\tau}$ *is* not *optimal in this case.*

Now we turn to the optimal stopping problem (5.1).

Theorem 5.1. *We have*

$$V(x,y,z) = \begin{cases} 1 - \frac{p+2}{2} v(x,y,z) & \text{if } y - z < 1, \\ 1 - \frac{p+2}{2} |x|^p & \text{if } y - z \geq 1. \end{cases} \tag{5.4}$$

Proof. We start from the observation that $V(x,y,z)$ is not smaller than the right hand side of (5.4). Indeed, this follows from the fact the latter equals $H(B_{\overline{\tau}}, B^*_{\overline{\tau}}, B_{\overline{\tau}*})$ and that $\overline{\tau}$ has the necessary integrablility property. It remains to establish the reverse estimate. If $y-z>1$, then for any $\tau \in L^{p/2}$ we have, by Burkholder-Davis-Gundy inequality, $\mathbb{E}_{x,y,z} \sup_{s \leq \tau} |B_s|^p < \infty$ and $|x|^p \leq \lim_{t \to \infty} \mathbb{E}_{x,y,z} |B_{\tau \wedge t}|^p = \mathbb{E}_{x,y,z} |B_\tau|^p$. Hence

$$\mathbb{E}_{x,y,z} H(B_\tau, B^*_\tau, B_{\tau*}) \leq 1 - \frac{p+2}{2} \mathbb{E}_{x,y,z} |B_\tau|^p \leq 1 - \frac{p+2}{2} |x|^p.$$

Finally, suppose $y - z < 1$. We have that $V \geq H$. We easily check that the function V satisfies

$$V_y(y,y,z) = V_z(z,x,z) = 0 \qquad \text{for all } y,\ z \tag{5.5}$$

and

$$V(\cdot, y, z) \text{ is concave for all } y,\ z. \tag{5.6}$$

Therefore it follows from Itô's formula and approximation argument that

$$V(B_t, B^*_t, B_{t*}) \leq V(B_0, B^*_0, B_{0*}) + \int_0^t V_x(B_s, B^*_s, B_{s*}) dB_s,$$

Hence, for any stopping time $\tau \in L^{p/2}$ of B, we have

$$\mathbb{E}_{x,y,z} H(B_{\tau\wedge t}, B^*_{\tau \wedge t}, B_{\tau\wedge t*}) \leq \mathbb{E}_{x,y,z} V(B_{\tau\wedge t}, B^*_{\tau \wedge t}, B_{\tau\wedge t*}) \leq V(x,y,z).$$

Now we let $t \to \infty$. By Burkholder-Davis-Gundy inequality, $\sup_{0 \leq s \leq \tau} |B_s$ belongs to L^p, hence Lebesgue's dominated convergence theorem together with the fact that τ is arbitrary proves the claim. □

Proof of Theorem 1.3. This immediately follows from the theorem above and the fact that $V(0,0,0) = 0$. □

There is a very interesting question about the sharp weak-type bound and the optimal stopping strategy in the case $p > 2$. Summarizing, we see that the

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