ON THE OPERATOR Λ^* AND THE BEURLING-AHLFORS TRANSFORM ON RADIAL FUNCTIONS

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ABSTRACT. We study an operator Λ^* which arises in the study of the action of the Beurling-Ahlfors transform on the class of radial functions. Using a novel estimate for pure-jump martingales, we provide a new proof of the weak-type $(1, 1)$ estimate for Λ^*, originally established by J. Gill by completely different techniques.

1. INTRODUCTION

The purpose of this paper is to study an operator closely related to the Beurling-Ahlfors transform on the complex plane. We recall that the latter is the singular integral operator acting on $L^p(\mathbb{C})$, defined by the formula

$$Bf(z) = -\frac{1}{\pi}\text{p.v.}\int_{\mathbb{C}} \frac{f(w)}{(z-w)^2}\mathrm{d}w,$$

where p.v. means the principal value and the integration is with respect to the Lebesgue measure on the complex plane $\mathbb{C}$. This operator plays a fundamental role in the study of quasiconformal mappings and partial differential equations (see [1] and references therein for an overview and applications). An important and interesting problem concerns the precise value of the L^p norms of this operator. This question has gained considerable interest in the literature and the long standing conjecture of T. Iwaniec [9] states that

$$||B||_{L^p(\mathbb{C})\to L^p(\mathbb{C})} = p^* - 1,$$

where $p^* = \max\{p, p/(p-1)\}$. While the lower bound of $p^* - 1$ was obtained by Lehto [11], the question about the upper bound remains open. Thus far, the best results in this direction is the inequality $||B||_{L^p(\mathbb{C})\to L^p(\mathbb{C})} \leq 1.575(p^* - 1)$, established in [2] and the bound $||B||_{L^p(\mathbb{C})\to L^p(\mathbb{C})} \leq 1.4(p^* - 1)$ for $p \geq 1000$, proved in [5]. Both of these statements were shown by obtaining a martingale representation of the operator B and applying the probabilistic techniques of Burkholder [6], [7].

As a Calderón–Zygmund singular integral, the Beurling-Ahlfors operator is also of weak-type $(1, 1)$, see [14]. That is, it maps $L^1(\mathbb{C})$ into weak–$L^1(\mathbb{C})$. A problem of interest also is to determine the best constant in the weak-type $(1, 1)$ inequality. In [3], Bañuelos and Janakiraman studied the action of B on the space of real-valued radial functions on $\mathbb{C}$. Consider the Hardy-type operator given by the formula

$$\Lambda f(x) = \frac{1}{x}\int_0^x f(y)\mathrm{d}y - f(x), \qquad x > 0.$$

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It was shown in [3] that Λ is an isometry on $L^2(0,\infty)$ and that, in addition, if $f \in L^1(0,\infty)$ and F is the associated radial function given by $F(z) = f(|z|^2)$ for $z \in \mathbb{C}$, then

$$BF(z) = \frac{\bar{z}}{z}\Lambda f(|z|^2).$$

Using this representation, it is proved in [3] that

$$m(\{z \in \mathbb{C} : |BF(z)| \geq 1\}) \leq \frac{1}{\log 2}||F||_{L^p(\mathbb{C})} \tag{1.1}$$

for any real-valued radial function F, and that the constant $1/\log 2$ cannot be improved (here and below, m denotes the Lebesgue measure). In particular, this result implies that the weak-type constant of B is at least $1/\log 2$. The proof of this estimate is analytic and rests on a careful study of the operator Λ. There is an alternative probabilistic approach to (1.1), invented by the authors in [4], which is based on martingale inequalities. These estimates are of independent interesting on their own right and, remarkably, enable one to deduce the corresponding sharp weak-type (p,p) inequalities for Λ in the range $1 \leq p \leq 2$ and to obtain these for *vector-valued* radial functions. See [4] for details.

There is an interesting dual problem which is studied by Gill in [10]. Consider the operator Λ^*, acting on $L^p(0,\infty)$ by the formula

$$\Lambda^* f(x) = \int_x^\infty \frac{f(y)}{y}\mathrm{d}y - f(x), \qquad x > 0.$$

It can be shown that this operator is the formal adjoint of Λ. Moreover, this is also related to Beurling-Ahlfors transform by the following formula: if $f \in L^p(\mathbb{R}_+,\mathbb{R})$ and $F(z) = \frac{\bar{z}}{z}f(|z|^2)$ for $z \in \mathbb{C}$, then

$$BF(z) = -\Lambda^* f(|z|^2).$$

As previously, one can ask about the weak-type constant of Λ^* and one of the principal goals of [10] is to provide the answer to this question. Quite unexpectedly, Gill proved that the weak-type $(1,1)$ constant for Λ^* is again $1/\log 2$. This gives an alternative proof of the fact that the weak-type constant of B is at least $1/\log 2$. Gill establishes the weak-type bound using purely analytic arguments and a clever study of Λ^*. Motivated by the aforementioned results, one may wonder whether there is a probabilistic proof of Gill's estimate. The main objective of this paper is to provide a positive answer to this question. As in the case of Λ, we will again exploit the theory of martingales. There are two surprising issues which are worth mentioning here. First, we will *not* use the martingale inequalities of [4]. As it turns out, the study of Λ^* seems to require an entirely different framework than that presented in [4]. Second, we will accomplish our goal by considering a very special class of martingales, the so-called pure-jump ones.

We will study the problem in the vector-valued setting. Let $\mathcal{H}$ be a separable Hilbert space over $\mathbb{R}$ with norm $|\cdot|$ and scalar product $\langle\cdot,\cdot\rangle$. With no loss of generality of the results, this space will be assumed to be ℓ^2. For any p-integrable function $f = (f_1, f_2, \ldots) : \mathbb{C} \to \mathcal{H}$ we define Bf coordinatewise: that is, we set

$$Bf = (Bf_1, Bf_2, \ldots) \in \ell^2_{\mathbb{C}}.$$

Similarly, we define the action of Λ^* on $L^p(\mathbb{C},\mathcal{H})$ by

$$\Lambda^* f = (\Lambda^* f_1, \Lambda^* f_2, \ldots) \in \ell^2.$$

We shall establish the following theorem.

Theorem 1.1. *Suppose that $f:(0,\infty)\to\mathcal{H}$ is an integrable function. Then*

$$m(\{x\in(0,\infty):|\Lambda^*f(x)|\geq 1\})\leq\frac{1}{\log 2}||f||_{L^1(\mathbb{C},\mathcal{H})}. \tag{1.2}$$

The constant is the best possible.

In particular, this yields the following result for Beurling-Ahlfors transform (see [3] or [4] for details). Suppose that $f:(0,\infty)\to\mathcal{H}$ is an integrable function and let $F(z)=\frac{\bar{z}}{z}f(|z|^2)$ for $z\in\mathbb{C}$. Then

$$m(\{z\in\mathbb{C}:|BF(z)|\geq 1\})\leq\frac{1}{\log 2}||F||_1,$$

and the constant $1/\log 2$ cannot be improved.

Unfortunately, we have not been able to establish the sharp weak-type inequality for $1<p<2$, as in the case of Λ in [4], and this remains an interesting open problem.

We have organized this paper as follows. The key ingredient of the proof of (1.2) is a certain special function on $[0,\infty)^2$. In the next section we introduce this object and study its properties. Section 3 is devoted to the probabilistic version of Theorem 1.1. Finally, in Section 4, we deduce the estimate (1.2) and present an example which shows the optimality of $1/\log 2$.

2. Some special functions and their properties

Let U, $V:[0,\infty)\times[0,\infty)\to\mathbb{R}$ be given by

$$U(x,y)=\begin{cases}1_{\{x\geq 1\}}-\dfrac{y}{\log 2} & \text{if } y>0,\\ 1+\dfrac{(x+1)}{\log 2}\log\dfrac{x+1}{2} & \text{if } y=0\end{cases}$$

and $V(x,y)=1_{\{x\geq 1\}}-y/\log 2$. We start with some simple observations concerning these functions.

Lemma 2.1. *The functions U, V have the following properties.*

(i) $U(\cdot,0)$ is of class C^2 on $(0,\infty)$.

(ii) We have $U(0,0)=0$.

(iii) We have the majorization

$$U(x,y)\geq V(x,y)\qquad \textit{for all } x,\ y\geq 0. \tag{2.1}$$

Proof. (i), (ii) Evident.

(iii) Of course, it suffices to show the inequality for $y=0$. This is equivalent to

$$1_{\{x<1\}}+\frac{x+1}{\log 2}\log\frac{x+1}{2}\geq 0.$$

However, the function $t\mapsto\frac{t+1}{\log 2}\log\frac{t+1}{2}$ is increasing on $[0,\infty)$, it equals -1 for $t=0$ and vanishes for $t=1$. This proves the claim. □

The next lemma concerns the key concavity property of U. We will write $x'=x/|x|$ if $x\in\mathcal{H}\setminus\{0\}$ and $x'=0$ for $x=0$. Furthermore, with a slight abuse of notation, U_x denotes the partial derivative of U with respect to the first variable.

Lemma 2.2. *Assume that $x\in\mathcal{H}$ is a vector of norm smaller than 1. Then for any $d\in\mathcal{H}$ we have*

$$U(|x+d|,|d|)\leq U(|x|,0)+U_x(|x|,0)\langle x',d\rangle. \tag{2.2}$$

When $x = 0$*, then* $U_x(|x|, 0)$ *is understood as the one-sided derivative.*

Proof. It is convenient to split the reasoning into a few intermediate steps.

Step 1. Assume first that $|x + d| < 1$. Then the inequality is equivalent to

$$\log 2 + (|x| + 1) \log \frac{|x| + 1}{2} + |d| + \left(\log \frac{|x| + 1}{2} + 1 \right) \langle x', d \rangle \geq 0.$$

However, we have $\log \frac{|x|+1}{2} + 1 \geq 0$ and $\langle x', d \rangle \geq -|d|$, so the left hand side is not smaller than

$$\log 2 + (|x| + 1) \log \frac{|x| + 1}{2} - \log \frac{|x| + 1}{2} |d|.$$

Now observe that

$$(|x| + 1) \log \frac{|x| + 1}{2} \geq -\log 2 \qquad \text{and} \qquad \log \frac{|x| + 1}{2} |d| \leq 0$$

to get the desired bound.

Step 2. Next, assume that x and d are linearly dependent and that $|x + d| \geq 1$. Then the estimate can be rewritten in the form

$$(|x| + 1) \log \frac{|x| + 1}{2} + |d| + \left(\log \frac{|x| + 1}{2} + 1 \right) \langle x', d \rangle \geq 0. \tag{2.3}$$

If $\langle x', d \rangle \leq 0$, then the left-hand side equals

$$(|x| + 1) \log \frac{|x| + 1}{2} - |d| \log \frac{|x| + 1}{2},$$

which is nonnegative: we have $\log \frac{|x|+1}{2} \leq 0$ and the assumptions $|x| < 1$, $|x + d| \geq 1$ imply $|d| \geq 1 + |x|$. On the other hand, if $\langle x', d \rangle > 0$, then (2.3) becomes

$$(|x| + 1) \log \frac{|x| + 1}{2} + \left(\log \frac{|x| + 1}{2} + 2 \right) |d| \geq 0.$$

But $|d| \geq |x + d| - |x| \geq 1 - |x|$, so the left-hand side is not smaller than

$$2 \log \frac{|x| + 1}{2} + 2(1 - |x|),$$

which is nonnegative for $|x| < 1$.

Step 3. Finally, assume that $|x + d| \geq 1$ and x, d are not necessarily linearly dependent. We carry out the following optimization. If we add to d a vector h which is orthogonal to x, then the product $\langle x', d \rangle$ does not change. On the other hand, both $|d|$ and $|x+d|$ do change. Now choose h so that the norm $|d|$ is minimized (but the condition $|x + d| \geq 1$ still holds). This procedure shows that it suffices to consider the case in which x and d are linearly dependent, or when we have $|x + d| = 1$. The first possibility has been already analyzed in Step 2, so let us focus on the second one. Then we have $\langle x', d \rangle = (1 - |x|^2 - |d|^2)/2$ and (2.2) reads

$$(|x| + 1) \log \frac{|x| + 1}{2} + |d| + \left(\log \frac{|x| + 1}{2} + 1 \right) \frac{1 - |x|^2 - |d|^2}{2} \geq 0.$$

However, $|d| \in [1 - |x|, 1 + |x|]$ and the left-hand side is a concave function of $|d|$. Thus it suffices to check the estimate for $|d| = 1 - |x|$ or $|d| = 1 + |x|$. In both these cases x and d are linearly dependent, which takes us back to Step 2 above and completes the proof. □

3. A MARTINGALE INEQUALITY

As we have already announced in the Introduction, the heart of the matter lies in proving an appropriate martingale inequality. Let us start with introducing the necessary background. Suppose that $(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space, filtered by $(\mathcal{F}_t)_{t\geq 0}$, a nondecreasing family of sub-σ-fields of $\mathcal{F}$ such that $\mathcal{F}_0$ contains all the events of probability 0. Let $X = (X_t)_{t\geq 0}$ be an adapted continuous-time martingale, taking values in the Hilbert space $\mathcal{H}$, whose trajectories are right-continuous and have limits from the left. We will denote by ΔX_t the jump of X at time t: $\Delta X_t = X_t - X_{t-}$, $t > 0$. Furthermore, $X^* = \sup_{t\geq 0} |X_t|$ and $||X||_1 = \sup_{t\geq 0} ||X_t||_1$ will stand for the maximal function and the first moment of a process X, respectively. We impose the following crucial, "structural" assumptions on X.

1° The martingale is a pure-jump process. That is, its continuous part X^c is 0 (see Chapter II in Protter [13]),

2° With probability 1, X has at most one non-zero jump.

In particular, this means that the process X immediately stops when the jumps occur. We are ready to formulate the main result of this section.

Theorem 3.1. *For any martingale X satisfying 1° and 2° we have*

$$\mathbb{P}(X^* \geq 1) \leq \mathbb{E}U(|X_0|, 0) + \frac{1}{\log 2}||(\Delta X)^*||_1. \tag{3.1}$$

The constant is the best possible.

In particular, if X is assumed to start from 0, then (3.1) takes the nicer form

$$\mathbb{P}(X^* \geq 1) \leq \frac{1}{\log 2}||(\Delta X)^*||_1,$$

in which $1/\log 2$ is also the best. This inequality can be regarded as a "dual-type" to the sharp estimate from [4]

$$\mathbb{P}((\Delta X)^* \geq 1) \leq \frac{1}{\log 2}||X||_1,$$

valid for arbitrary martingales (i.e., not necessarily satisfying 1° and 2°). This was shown in the discrete-time setting by Cox and Kemperman in [8], consult also [4] for the vector-valued setting. It should be stressed here that the requirements 1° and 2° are absolutely necessary in (3.1). For instance, the inequality fails to hold for continuous-path martingales (which explains the importance of 1°), as well as for random walks with sufficiently small jumps (this is why we assume that at most one jump occurs).

With the functions U and V from Section 2 at hand, our approach is similar to that used in [4]. It exploits the properties of these special functions, and can be regarded as yet another extension of Burkholder's technique (for the detailed explanation of the method, see Burkholder [7], Wang [15] and the recent monograph by the second-named author [12]).

Proof of (3.1). By standard approximation argument, we may assume that $\mathcal{H}$ is finite-dimensional: $\mathcal{H} = \mathbb{R}^d$ for some $d \geq 1$. Fix a martingale X as in the statement. Obviously, we may restrict ourselves to those X, which are bounded in L^1, since otherwise there is nothing to prove. Furthermore, we may assume that X is bounded away from 0. Indeed, having proved the claim in this special class, we pick an arbitrary X, a small positive number a, apply the estimate to an $\mathcal{H} \times \mathbb{R}$-valued martingale (X, a) (which is bounded away from 0), and let a go to 0 at the end. Introduce the stopping time

$$\tau = \inf\{s \geq 0 : |X_s| \geq 1 \text{ or } (\Delta X)^*_s > 0\}.$$

The function $U(\cdot,0)$ is of class C^2 on $(0,\infty)$; therefore, the function $x\mapsto U(|x|,0)$ is of class C^2 on $\mathcal{H}\setminus\{0\}$. An application of Itô's formula for processes with jumps (see Chapter II in Protter [13]) yields

$$U(|X_{\tau\wedge t}|,(\Delta X_{\tau\wedge t})^*)=I_0+I_1+I_2, \tag{3.2}$$

where

$$\begin{aligned}
I_0&=U(|X_0|,0),\\
I_1&=\int_{0+}^{\tau\wedge t}U_x(|X_{s-}|,0)\langle X'_{s-},\mathrm{d}X_s\rangle,\\
I_2&=\sum_{0<s\leq\tau\wedge t}\Big[U(|X_s|,|\Delta X_s|)-U(|X_{s-}|,0)-U_x(|X_{s-}|,0)\langle X_{s-},\Delta X_s\rangle\Big].
\end{aligned}$$

The term I_1 has zero expectation, by the properties of stochastic integrals. In addition, there is at most one summand in I_2, and it is nonpositive, by virtue of (2.2). Consequently, integrating both sides of (3.2) gives $\mathbb{E}U(|X_{\tau\wedge t}|,(\Delta X_{\tau\wedge t})^*)\leq\mathbb{E}U(|X_0|,0)$, and combining this with (2.1) yields

$$\mathbb{P}(|X_{\tau\wedge t}|\geq 1)\leq\mathbb{E}U(|X_0|,0)+\frac{1}{\log 2}\mathbb{E}(\Delta X_{\tau\wedge t})^*\leq\mathbb{E}U(|X_0|,0)+\frac{1}{\log 2}||(\Delta X)^*||_1.$$

Next, for each $\varepsilon>0$ we have

$$\{X^*\geq 1\}\subset\bigcup_{t\geq 0}\{|X_{\tau\wedge t}|\geq 1-\varepsilon\}$$

and the events on the right are increasing. Consequently, applying the preceding inequality to the martingale $X/(1-\varepsilon)$ and letting $t\to\infty$ gives

$$\mathbb{P}(X^*\geq 1)\leq\mathbb{E}U(|X_0|,0)+\frac{1}{(1-\varepsilon)\log 2}||(\Delta X)^*||_1.$$

Since $\varepsilon>0$ was arbitrary, the result follows.

The sharpness. This will be clear by the example considered in the next section. □

4. The weak-type bound for Λ^*

Proof of (1.2). By standard approximation, it suffices to prove the assertion for a bounded function $f:[0,\infty)\to\mathcal{H}$, supported on a finite interval $[0,M]$. Furthermore, modifying f in a small neighborhood of 0 if necessary, we may restrict ourselves to those functions, which have a finite limit at 0. For such f's, we have (from [10]) that $\Lambda\Lambda^*f=f$. Next, consider the probability space

$$(\Omega,\mathcal{F},\mathbb{P})=([0,M],\mathcal{B}([0,M]),m(\cdot)/M),$$

where as above, m denotes the Lebesgue measure. For any $t\in[0,M]$, let $\mathcal{F}_t$ be the smallest complete σ-field which contains the interval $[0,M-t]$ and all Borel subsets of $[M-t,M]$; for $t>M$, put $\mathcal{F}_t=\mathcal{F}$. Obviously, $(\mathcal{F}_t)_{t\geq 0}$ is a filtration and a function $g=\Lambda^*f\in L^1((0,\infty))$, restricted to $[0,M]$, can be regarded as an integrable random variable. Note that this random variable has mean zero: indeed,

$$\frac{1}{M}\int_0^M g(s)\mathrm{d}s=\frac{1}{M}\int_0^M\int_s^M\frac{f(y)}{y}\mathrm{d}y-f(s)\mathrm{d}s=0, \tag{4.1}$$

by Fubini's theorem. The function g gives rise to the martingale

$$X=(X_t)_{t\geq 0}=\left(\mathbb{E}(g|\mathcal{F}_t)\right)_{t\geq 0}.$$

Let us analyze the properties of this process. First, we easily check that for almost all $\omega \in \Omega$,

$$X_t(\omega) = \begin{cases} g(\omega) & \text{if } t \geq M - \omega, \\ \frac{1}{M-t}\int_0^{M-t} g(s)\mathrm{d}s & \text{if } t < M - \omega, \end{cases} \tag{4.2}$$

and hence the martingale X is pure-jump; actually, it has at most one jump almost surely (for a given $\omega \in \Omega$, X is continuous on $[0, M-\omega)$ and $(M-\omega, \infty)$). Furthermore, by (4.1) and (4.2) we have $X_0 = \frac{1}{M}\int_0^M g(s)\mathrm{d}s = 0$ and hence $U(|X_0|, 0) = 0$ (see Lemma 2.1 (ii)). Next, recall that $\Lambda g = \Lambda\Lambda^* f = f$, so

$$(\Delta X)^*(\omega) = |\Delta X_{M-\omega}| = \left|\frac{1}{\omega}\int_0^\omega g(s)\mathrm{d}s - g(\omega)\right| = |\Lambda g(\omega)| = f(\omega).$$

Consequently, by (3.1),

$$\begin{aligned} m(\{x \in (0,M) : |\Lambda^* f(x)| \geq 1\}) &= m(\{x \in (0,M) : |g(x)| \geq 1\}) \\ &\leq M\mathbb{P}(X^* \geq 1) \\ &\leq \frac{1}{\log 2} M||(\Delta X)^*||_1 = \frac{1}{\log 2}\int_0^M |f(x)|\mathrm{d}x. \end{aligned}$$

Letting $M \to \infty$ gives

$$m(\{x \in (0,\infty) : |\Lambda^* f(x)| \geq 1\}) \leq \frac{1}{\log 2}||f||_{L^1([0,\infty))},$$

which is the claim. □

Sharpness. We use the example constructed by Gill [10]. Put $f(x) = \frac{\chi_{(1/2,1]}(x)}{x}$; then $\Lambda^* f(x) = \chi_{[0,1/2]}(x) - \chi_{(1/2,1]}(x)$ and hence

$$m(|\Lambda^* f| \geq 1) = 1 \qquad \text{and} \qquad ||f||_1 = \log 2.$$

In consequence, we also obtain that the inequality (3.1) is sharp, otherwise it would be possible to improve the constant in (1.2). □

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