SHARP INEQUALITIES FOR MARTINGALES WITH VALUES IN ℓ_∞^N

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Abstract. The objective of the paper is to study sharp inequalities for transforms of martingales taking values in ℓ_∞^N. Using Burkholder's method combined with an intrinsic duality argument, we identify, for each $N \geq 2$, the best constant C_N such that the following holds. If f is a martingale with values in ℓ_∞^N and g is its transform by a sequence of signs, then

$$||g||_1 \leq C_N ||f||_\infty.$$

This is closely related to the characterization of UMD spaces in terms of the so-called η-convexity, studied in the eighties by Burkholder and Lee.

1. Introduction

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, filtered by $(\mathcal{F}_n)_{n\geq 0}$, a non-decreasing sequence of sub-σ-fields of $\mathcal{F}$. Let $(\mathbb{B}, ||\cdot||)$ be a separable Banach space and let $f = (f_n)_{n\geq 0}$ be an adapted martingale taking values in $\mathbb{B}$. We define $df = (df_n)_{n\geq 0}$, the difference sequence of f, by $df_0 = f_0$ and $df_n = f_n - f_{n-1}$, $n \geq 1$. A Banach space $\mathbb{B}$ is called a UMD space if for some $1 < p < \infty$ (equivalently, for all $1 < p < \infty$) there is a finite constant $\beta = \beta_p$ with the following property: for any deterministic sequence ε_0, ε_1, ε_2, ... with values in $\{-1, 1\}$ and any f as above,

$$\left|\left|\sum_{k=0}^n \varepsilon_k df_k\right|\right|_p \leq \beta_p \left|\left|\sum_{k=0}^n df_k\right|\right|_p, \qquad n = 0, 1, 2, \ldots.$$

Here and below, we will write $||\cdot||_p$ instead of $||\cdot||_{L^p(\Omega;\mathbb{B})}$, if it is clear which Banach space $\mathbb{B}$ we work with. For given p and $\mathbb{B}$, let $\beta_{p,\mathbb{B}}$ denote the smallest possible value of the constant β_p allowed above. Then, as shown by Burkholder [2, 4], we have $\beta_{p,\mathbb{R}} = p^* - 1$, where $p^* = \max\{p, p/(p-1)\}$; in fact, the equality holds true if $\mathbb{R}$ is replaced by any separable Hilbert space $\mathcal{H}$. By Fubini's theorem, this yields $\beta_{p,\ell_p^N(\mathcal{H})} = p^* - 1$ for any integer N. For the other choices of p and $\mathbb{B}$, the values of the corresponding constants $\beta_{p,\mathbb{B}}$ are not known.

There is a beautiful geometrical characterization of UMD spaces, which is due to Burkholder. A function $\zeta : \mathbb{B} \times \mathbb{B} \to \mathbb{R}$ is called biconvex, if for any $z \in \mathbb{B}$, the functions $x \mapsto \zeta(x, z)$ and $y \mapsto \zeta(z, y)$ are convex. One of principal results of [1] states that $\mathbb{B}$ is UMD if and only if there is a biconvex function ζ satisfying

$$\zeta(0, 0) > 0 \tag{1.1}$$

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and

$$\zeta(x,y) \leq ||x+y|| \qquad \text{if } ||x|| = ||y|| = 1. \tag{1.2}$$

The existence of such a function is strictly related to the validity of the weak-type estimate

$$\mathbb{P}\left(\sup_n \left|\left|\sum_{k=0}^n \varepsilon_k df_k\right|\right| \geq 1\right) \leq C \left|\left|\sum_{k=0}^n df_k\right|\right|_1, \qquad n = 0, 1, 2, \ldots, \tag{1.3}$$

for some constant C depending only on $\mathbb{B}$. In fact, if there is ζ satisfying (1.1) and (1.2), then (1.3) holds with $C = 2/\zeta(0,0)$. Then, using classical extrapolation arguments (see Burkholder and Gundy [5]), it can be shown that

$$\left|\left|\sum_{k=0}^n \varepsilon_k df_k\right|\right|_p \leq \frac{72}{\zeta(0,0)} \cdot \frac{(p+1)^2}{p-1} \left|\left|\sum_{k=0}^n \varepsilon_k df_k\right|\right|_p, \qquad n \geq 0, \quad 1 < p < \infty. \tag{1.4}$$

In general, if $\mathbb{B}$ is UMD, then the class of all biconvex functions ζ satisfying (1.1) and (1.2) is infinite. However, it can be shown that there is the largest element in this class, i.e., the function $\bar{\zeta}$ such that $\bar{\zeta}(x,y) = \sup_\zeta \zeta(x,y)$ for all $x, y \in \mathbb{B}$ (see [1], [3]). This extremal element yields the optimal constant $2/\bar{\zeta}(0,0)$ in (1.3) and a tight one in (1.4). Thus, for a given UMD space $\mathbb{B}$, it would be desirable to find such a function $\bar{\zeta}$, or at least the value $\bar{\zeta}(0,0)$; unfortunately, this is a very difficult task and, so far, it has been successfully tackled only in the case when $\mathbb{B}$ is a Hilbert space. Namely, Burkholder [3] showed that

$$\bar{\zeta}(x,y) = \begin{cases} \left[1 + 2\langle x,y\rangle + ||x||^2||y||^2\right]^{1/2} & \text{if } ||x|| \vee ||y|| \leq 1, \\ ||x+y|| & \text{if } ||x+y|| > 1, \end{cases}$$

where $\langle \cdot, \cdot \rangle$ denotes the scalar product in $\mathbb{B}$. In view of the above remarks, this function shows that the weak-type constant for transforms of Hilbert-space-valued martingales equals 2.

In this paper we will be concerned with a different, dual geometrical characterization of UMD due to Lee [7]. Let S denote the set $\{(x,y) \in \mathbb{B} \times \mathbb{B} : ||x-y|| \leq 2\}$. One of the main results of [7] is as follows: a Banach space $\mathbb{B}$ is UMD if and only if there is a biconcave function $\eta : S \to \mathbb{R}$ satisfying

$$\eta(x,y) \geq ||x+y|| \qquad \text{for all } (x,y) \in S. \tag{1.5}$$

As we have stressed above, the existence of ζ is closely related to the validity of (1.3); a similar phenomenon occurs for η, which is strictly connected to the following martingale inequality (see p. 304 in [7]). For any martingale f and any deterministic sequence $\varepsilon_0, \varepsilon_1, \varepsilon_2, \ldots$ of signs,

$$\left|\left|\sum_{k=0}^n \varepsilon_k df_k\right|\right|_1 \leq c \left|\left|\sum_{k=0}^n df_k\right|\right|_\infty, \qquad n = 0, 1, 2, \ldots. \tag{1.6}$$

More precisely, if η satisfies (1.5), then the above bound holds with $c = \eta(0,0)/2$. This statement can be extracted from the works of Burkholder [3] and Lee [7] (another convenient reference on the subject is the paper [6] by Geiss). In analogy with the previous setting, if $\mathbb{B}$ is a UMD space, then there are many possible biconcave functions η; however, this class of functions contains *the least* element

$\bar{\eta}$, and the corresponding constant $\bar{\eta}(0,0)/2$ is optimal in (1.6). In the case when $\mathbb{B}=\mathbb{R}$, Burkholder [2] identified $\bar{\eta}$. This function is given by the symmetry property

$$\bar{\eta}(x,y)=\bar{\eta}(y,x)=\bar{\eta}(-x,-y), \qquad (x,y)\in S,$$

and the equality

$$\bar{\eta}(x,y)=\begin{cases} x+y+(y-x+2)e^{-y} & \text{if } 0\leq y\leq x\leq y+2,\\ 2(1+y)-(y-x+2)\log(1+y) & \text{if } -1<y\leq 0,\ -y\leq x\leq y+2.\end{cases}$$

This has been pushed further by Lee [7], who proved that if the dimension of $\mathbb{B}$ over $\mathbb{R}$ is at least two, then

$$\bar{\eta}(x,y)=2\sqrt{1+\langle x,y\rangle}.$$

In both cases we have $\eta(0,0)=2$ and thus the optimal constant in (1.6) (for Hilbert spaces) is equal to 1. Of course, this can also be proved directly, simply by inserting the identity $||\sum_{k=0}^n \varepsilon_k df_k||_2=||\sum_{k=0}^n df_k||_2$ in the middle of the estimate.

The purpose of this paper is to study sharp version of (1.6) for a different class of Banach spaces, namely, for $\mathbb{B}=\ell_\infty^N(\mathcal{H})$, where $\mathcal{H}$ is a Hilbert space and N is an integer larger than 1. To gain some initial insight into the size of the constants involved, let us exploit the following well-known argument. Namely, we have

$$\begin{aligned}\left|\left|\sum_{k=0}^n \varepsilon_k df_k\right|\right|_{L^1(\Omega;\ell_\infty^N(\mathcal{H}))} &\leq \left|\left|\sum_{k=0}^n \varepsilon_k df_k\right|\right|_{L^p(\Omega;\ell_\infty^N(\mathcal{H}))}\\ &\leq \left|\left|\sum_{k=0}^n \varepsilon_k df_k\right|\right|_{L^p(\Omega;\ell_p^N(\mathcal{H}))}\\ &\leq (p^*-1)\left|\left|\sum_{k=0}^n df_k\right|\right|_{L^p(\Omega;\ell_p^N(\mathcal{H}))}\\ &\leq (p^*-1)N^{1/p}\left|\left|\sum_{k=0}^n df_k\right|\right|_{L^p(\Omega;\ell_\infty^N(\mathcal{H}))}\\ &\leq (p^*-1)N^{1/p}\left|\left|\sum_{k=0}^n df_k\right|\right|_{L^\infty(\Omega;\ell_\infty^N(\mathcal{H}))},\end{aligned}$$

for $n=0,1,2,\ldots$. Here in the third inequality we have used the fact that $\beta_{p,\ell_p^N(\mathcal{H})}=p^*-1$, which was mentioned at the beginning. Hence, assuming $N>e^2$ and taking $p=\log N$, we get that (1.6) holds with the constant $e(\log N-1)$.

Actually, we will study a slightly more general setting in which the transforming sequence $(\varepsilon_n)_{n\geq 1}$ may depend on coordinates of $\ell_\infty^N(\mathcal{H})$. That is, we allow deterministic "multisigns" $\varepsilon_n=(\varepsilon_n^1,\varepsilon_n^2,\ldots,\varepsilon_n^N)\in\{-1,1\}^N$, for which we put $(\varepsilon_n df_n)_{n\geq 0}:=\big((\varepsilon_n^1 df_n^1,\varepsilon_n^2 df_n^2,\ldots,\varepsilon_n^N df_n^N)\big)_{n\geq 0}$. Of course, this is again a martingale difference sequence.

Our main result can be stated as follows.

Theorem 1.1. *Suppose that f is a martingale taking values in $\ell_\infty^N(\mathcal{H})$ and let ε_0, ε_1, ε_2, ... be a deterministic sequence with values in $\{-1,1\}^N$. Then*

$$\left|\left|\sum_{k=0}^n \varepsilon_k df_k\right|\right|_1 \leq C_N \left|\left|\sum_{k=0}^n df_k\right|\right|_\infty, \qquad n=0,\,1,\,2,\,\ldots, \tag{1.7}$$

where

$$C_N = \begin{cases} \sqrt{N} & \text{if } N \leq 4,\\ 2+\log(N/4) & \text{if } N \geq 5.\end{cases}$$

The inequality is sharp.

By duality, this leads to an analogous statement for $\ell_1^N(\mathcal{H})$ spaces.

Theorem 1.2. *Suppose that f is a martingale taking values in $\ell_1^N(\mathcal{H})$ and let ε_0, ε_1, ε_2, ... be a deterministic sequence with values in $\{-1,1\}^N$. Then*

$$\left|\left|\sum_{k=0}^n \varepsilon_k df_k\right|\right|_1 \leq C_N \left|\left|\sum_{k=0}^n df_k\right|\right|_\infty, \qquad n=0,\,1,\,2,\,\ldots, \tag{1.8}$$

and the inequality is sharp.

The novelty lies in the fact that, to the best of our knowledge, this is the first result in the literature in which the best constant for transforms of (non-Hilbert) Banach-space-valued martingales has been found. It would be very interesting if the reasoning could be modified to yield sharp bounds for other class of Banach spaces, for instance for ℓ_p spaces, $1<p<\infty$. Unfortunately, so far this seems to be hopeless.

Before we proceed, let us mention here two interesting corollaries.

Theorem 1.3. *Let $\mathbb{B}=\ell_\infty^N(\mathcal{H})$. Then*

$$\bar{\eta}(0,0) \leq \begin{cases} 2\sqrt{N} & \text{if } N \leq 4,\\ 4+2\log(N/4) & \text{if } N \geq 5.\end{cases} \tag{1.9}$$

We do not know whether equality takes place here; in other words, we do not know if the passage from signs to multisigns increases the constant in (1.7). Unfortunately, in our proof of the sharpness of this estimate, we strongly exploit the fact that the transforming sequence does depend on coordinates of ℓ_∞^N.

The second corollary provides a lower bound for the constant $\beta_{p,\ell_\infty^N(\mathcal{H})}$.

Theorem 1.4. *We have $\beta_{p,\ell_\infty^N(\mathcal{H})} \geq C_N$ for any $N\geq 1$ and any $1<p<\infty$.*

We have organized the paper as follows. In Section 2 we study an auxiliary bound for Hilbert-space-valued martingales; this is accomplished with the use of Burkholder's method combined with an intrinsic duality argument. The next two sections are the most complicated parts of the paper: we construct there appropriate examples, which prove that the constant C_N cannot be improved in (1.7). Quite surprisingly, we require completely different arguments for $N\leq 3$ and $N\geq 4$. The first case is slightly easier and is studied in Section 3; the final part addresses the sharpness of (1.7) for $N\geq 4$.

2. A sharp inequality for $\mathcal{H}$-valued martingales

Let us begin by showing that Theorems 1.1 and 1.2 are equivalent. To see that (1.7) implies (1.8), pick a bounded martingale $f = (f^1, f^2, \ldots, f^N)$ with values in $\ell_1^N(\mathcal{H})$, a multisign ε and observe that

$$\left|\left|\sum_{k=0}^n \varepsilon_k df_k\right|\right|_{L^1(\Omega;\ell_1^N(\mathcal{H}))} = \mathbb{E}\sum_{j=1}^N \left|\sum_{k=0}^n \varepsilon_k^j df_k^j\right| = \sup \mathbb{E}\sum_{j=1}^N g^j \sum_{k=0}^n \varepsilon_k^j df_k^j,$$

where the supremum is taken over all random variables $g = (g^1, g^2, \ldots, g^N)$ taking values in the unit ball of $\ell_\infty^N(\mathcal{H})$. Let $(g_n)_{n\geq 0} = (\mathbb{E}(g|\mathcal{F}_n))_{n\geq 0}$ denote the associated $\ell_\infty^N(\mathcal{H})$-valued martingale. Note that $(g_n)_{n\geq 0}$ is bounded by 1, since g has this property. By the orthogonality of martingale difference sequences, for any $n \geq 1$ we have

$$\begin{aligned}
\left|\left|\sum_{k=0}^n \varepsilon_k df_k\right|\right|_{L^1(\Omega;\ell_1^N(\mathcal{H}))} &= \sup_g \mathbb{E}\sum_{j=1}^N \left(\sum_{k=0}^n dg_k^j\right)\left(\sum_{k=0}^n \varepsilon_k^j df_k^j\right)\\
&= \sup_g \mathbb{E}\sum_{j=1}^N \sum_{k=0}^n \varepsilon_k^j df_k^j dg_k^j\\
&= \sup_g \mathbb{E}\sum_{j=1}^N \left(\sum_{k=0}^n \varepsilon_k^j dg_k^j\right)\left(\sum_{k=0}^n df_k^j\right)\\
&\leq \left|\left|\sum_{k=0}^n df_k\right|\right|_{L^\infty(\Omega;\ell_1^N(\mathcal{H}))} \left|\left|\sum_{k=0}^n \varepsilon_k dg_k\right|\right|_{L^1(\Omega;\ell_\infty^N(\mathcal{H}))}\\
&\leq C_N \left|\left|\sum_{k=0}^n df_k\right|\right|_{L^\infty(\Omega;\ell_1^N(\mathcal{H}))},
\end{aligned}$$

where the latter bound follows from (1.7), applied to $(g_n)_{n\geq 0}$. This establishes (1.8); the proof of the implication (1.8)$\Rightarrow$(1.7) goes along the same lines.

Thus, from now on, we will focus on Theorem 1.1. For notational convenience, the norm in ℓ_∞^N will be denoted by $||\cdot||$, while the norm in the Hilbert space $\mathcal{H}$ will be denoted by $|\cdot|$. Recall that a Banach-space-valued martingale f is called simple, if for any n the random variable f_n takes only a finite number of values and there is a deterministic number m such that $f_m = f_{m+1} = f_{m+2} = \ldots = f_\infty$. By straightforward approximation, it suffices to study (1.7) for simple martingales f, g only. Furthermore, we may restrict ourselves to those f, g, which satisfy $f_0 = g_0 = 0$. Indeed, if this is not the case, we consider an independent Rademacher variable θ and new martingales $(0, \theta f_0, \theta f_1, \theta f_2, \ldots)$, $(0, \theta g_0, \theta g_1, \theta g_2, \ldots)$. These two do start from 0, the latter is a transform of the former, and they have the same norms as f and g. Thus, by homogeneity, the assertion of Theorem 1.3 is equivalent to saying that

$$\begin{aligned}
C_N = \sup\big\{||g_\infty||_1\ :\ & f_0 = g_0 = 0,\ ||f||_\infty \leq 1,\ g \text{ is a transform of } f\\
&\text{by a deterministic sequence of multisigns}\big\}.
\end{aligned}\tag{2.1}$$

Before we proceed, let us mention here that the above supremum is closely related to the value $\bar{\eta}(0,0)$. As proved by Lee [7], we have

$$\begin{aligned}\bar{\eta}(0,0) = \sup\big\{2||g_\infty||_1\ : f_0 = g_0 = 0,\ ||f||_\infty \leq 1,\ g \text{ is a transform of } f \\ \text{by a deterministic sequence of signs}\big\}.\end{aligned} \tag{2.2}$$

This formula immediately shows how to deduce (1.9) from (1.7).

We turn to the analysis of the right-hand side of (2.1). Assume that $(f,g) = \big((f^1,f^2,\ldots,f^N),(g^1,g^2,\ldots,g^N)\big)$ is a martingale pair as appearing there. Then for each j, f^j is a Hilbert-space valued martingale bounded by 1 and g^j is its transform by a certain deterministic sequence of signs. Furthermore, there is a splitting of Ω into pairwise disjoint events A_1, A_2, …, A_N such that $A_j \subseteq \{||g_\infty|| = |g^j_\infty|\}$, and thus we may write

$$||g_\infty||_1 = \sum_{j=1}^N \mathbb{E}|g^j_\infty|1_{A_j}. \tag{2.3}$$

This suggests to analyze carefully each term under the above sum. To do this, fix $t \in [0,1]$ and put

$$V(t) = \sup\big\{\mathbb{E}|G_\infty|1_A\big\}, \tag{2.4}$$

where the supremum is taken over all $A \in \mathcal{F}$ with $\mathbb{P}(A) \leq t$ and all simple $\mathcal{H}$-valued martingales F, G starting from 0 such that F is bounded by 1 and G is a transform of F by a deterministic sequence of signs. Here we allow the filtration is to vary, as well as the probability space, unless it is assumed to be nonatomic. We have the following.

Lemma 2.1. *The function V is concave.*

Proof. This is straightforward. We may assume that the probability space is the interval $[0,1]$ equipped with its Borel subsets and Lebesgue's measure. Pick t_1, $t_2 \in [0,1]$, a weight $\alpha \in (0,1)$. Take two events A^1, A^2 and two pairs (F^1,G^1), (F^2,G^2) of simple martingales as in the definition of $V(t_1)$ and $V(t_2)$. We splice these two events into one set A, and the two pairs into one martingale pair (F,G), by the following formulas:

$$A = \alpha A^1 + (\alpha + (1-\alpha)A^2)$$

and, for $n = 0, 1, 2, \ldots$,

$$(F_{2n},G_{2n})(\omega) = \begin{cases}(F^1_n,G^1_n)(\omega/\alpha) & \text{if } 0 \leq \omega \leq \alpha, \\ (F^2_n,G^2_n)\big((\omega-\alpha)/(1-\alpha)\big) & \text{if } \alpha < \omega \leq 1\end{cases}$$

and

$$(F_{2n+1},G_{2n+1})(\omega) = \begin{cases}(F^1_{n+1},G^1_{n+1})(\omega/\alpha) & \text{if } 0 \leq \omega \leq \alpha, \\ (F^2_n,G^2_n)\big((\omega-\alpha)/(1-\alpha)\big) & \text{if } \alpha < \omega \leq 1.\end{cases}$$

Then (F,G) is a simple martingale with respect to its natural filtration, we have $F_0 = G_0 = 0$ and G is a transform of F by a deterministic sequence of signs. Furthermore, we have

$$\mathbb{P}(A) = \alpha\mathbb{P}(A^1) + (1-\alpha)\mathbb{P}(A^2) \leq \alpha t_1 + (1-\alpha)t_2.$$

Therefore, by the definition of V, we may write

$$\begin{aligned} V(\alpha t_1 + (1-\alpha)t_2) &\geq \mathbb{E}|G_\infty|1_A \\ &= \mathbb{E}|G_\infty|1_{\alpha A^1} + \mathbb{E}|G_\infty|1_{\alpha+(1-\alpha)A^2} \\ &= \alpha\mathbb{E}|G^1_\infty|1_{A^1} + (1-\alpha)\mathbb{E}|G^2_\infty|1_{A^2}. \end{aligned}$$

Taking supremum over all A^i and (F^i, G^i), we obtain the desired concavity. $\square$

Coming back to (2.3), we obtain the bound

$$||g_\infty||_1 \leq \sum_{j=1}^N V(\mathbb{P}(A_j)).$$

Thus, if we denote the supremum on the left-hand side of (2.1) by S_N, we see that the concavity of Lemma 2.1 implies

$$S_N \leq NV\left(\frac{1}{N}\sum_{j=1}^N \mathbb{P}(A_j)\right) = NV(1/N). \tag{2.5}$$

Now, to obtain the proper upper bound for $V(1/N)$, we will consider a dual approach. First we prove the following fact.

Theorem 2.2. *Suppose that ξ is an $\mathcal{H}$-valued martingale and let ζ be its transform by a deterministic sequence of signs. Then for any $C \geq 1$ we have*

$$||\zeta||_1 \leq C||\xi||_1 + \frac{e^{1-C}}{4}||\xi||_\infty. \tag{2.6}$$

For each C the constant $e^{1-C}/4$ is the best possible.

This bound will be established with the use of Burkholder's method. In order to simplify the technicalities, we shall combine the technique with an "integration argument", invented in [8] (see also [9]). That is, first we introduce a simple function $v_\infty : \mathcal{H} \times \mathcal{H} \to \mathbb{R}$, for which the calculations are relatively easy; then define U by integrating this object against an appropriate nonnegative kernel. Let

$$v_\infty(x,y) = \begin{cases} 0 & \text{if } |x|+|y| \leq 1, \\ (|y|-1)^2 - |x|^2 & \text{if } |x|+|y| > 1. \end{cases}$$

We have the following fact (see Lemma 2.2 in [9] for a slightly stronger statement concerning differentially subordinated martingales).

Lemma 2.3. *Let ξ be a square integrable, $\mathcal{H}$-valued martingale and let ζ be its transform by a deterministic sequence of signs. Then we have*

$$\mathbb{E}v_\infty(\xi_n, \zeta_n) \leq 0 \qquad \textit{for any } n \geq 0.$$

Let $\mathcal{K}$ denote the unit ball of $\mathcal{H}$ and define $U : \mathcal{K} \times \mathcal{H} \to \mathbb{R}$ by the formula

$$U(x,y) = \frac{1}{2}\int_{\exp(1-C)/2}^{1/2} v_\infty(y/t, x/t)\mathrm{d}t + e^{C-1}(|y|^2 - |x|^2) + \frac{e^{1-C}}{4}$$

(note that under the integral, we have $v_\infty(y/t,x/t)$, not $v_\infty(x/t,y/t)$!). One easily computes the explicit formula for U. Namely, we have

$$U(x,y)=\begin{cases} e^{C-1}(|y|^2-|x|^2)+e^{1-C}/4 & \text{if } |x|+|y|\leq e^{1-C}/2,\\ |x|+|y|-|x|\log(2|x|+2|y|)-C|x| & \text{if } e^{1-C}/2<|x|+|y|\leq 1/2,\\ |y|^2-|x|^2+(1-C)|x|+1/4 & \text{if } |x|+|y|>1/2. \end{cases}$$

We will need the following majorization property of U.

Lemma 2.4. *For any $(x,y)\in\mathcal{K}\times\mathcal{H}$ we have*

$$U(x,y)\geq |y|-C|x|. \tag{2.7}$$

Proof. Let r be a fixed nonnegative number. Let us fix $|x|+|y|=r$ and consider both sides of (2.7) as functions of $s=|y|$. These functions are both linear and hence it suffices to establish the majorization in three extremal cases: for $x=0$, for $|x|=1$ and for $y=0$. If $x=0$ and $|y|\leq e^{1-C}/2$, the inequality is equivalent to

$$(2e^{C-1}|y|-1)^2\geq 0,$$

which is obviously true. If $x=0$ and $|y|\in(e^{1-C}/2,1/2)$, then both sides are equal. Next, if $x=0$ and $|y|\geq 1/2$, or $|x|=1$, then the majorization can be rewritten in the form $(2|y|-1)^2\geq 0$, which holds trivially. Finally, suppose that $y=0$. If $|x|\leq e^{1-C}/2$, we must prove that

$$-e^{C-1}|x|^2+e^{1-C}/4+C|x|\geq 0.$$

But this is straightforward: the left-hand side, as a function of $|x|\in[0,e^{1-C}/2)$, is increasing, and we have already verified the estimate for $x=0$. If $e^{1-C}/2<|x|\leq 1/2$, the majorization is equivalent to $1-\log(2|x|)\geq 0$, which is obvious. Finally, if $|x|>1/2$, the inequality (2.7) reads $-|x|^2+|x|+1/4\geq 0$, which is evident. □

We turn to the proof of Theorem 2.2.

Proof of (2.6). Pick ξ, ζ as in the statement. We may and do assume that ξ is bounded, since otherwise the right-hand side is infinite and there in nothing to prove. By homogeneity, it suffices to show that

$$||\zeta||_1\leq C||\xi||_1+\frac{e^{1-C}}{4}$$

under the assumption $||\xi||_\infty\leq 1$. Then in particular ξ is square integrable and hence so is ζ, since $||\zeta_n||_2=||\xi_n||_2$ for all n. Because the transforming sequence ε takes values in $\{-1,1\}$, we see that the relation of being a transform by ε is symmetric. Consequently, for any $t>0$ the martingale ξ/t is a transform of ζ/t and thus, by Lemma 2.3, we have $\mathbb{E}v_\infty(\zeta_n/t,\xi_n/t)\leq 0$ for all n. Furthermore, we have $v_\infty(x,y)\leq c(|x|^2+|y|^2+1)$ for some universal constant c, so by Fubini's theorem, we get

$$\mathbb{E}U(\xi_n,\zeta_n)=\mathbb{E}\int_{\exp(1-C)/2}^{1/2}v_\infty(\zeta_n/t,\xi_n/t)\mathrm{d}t+||\zeta_n||_2^2-||\xi_n||_2^2+\frac{e^{1-C}}{4}\leq\frac{e^{1-C}}{4}.$$

Therefore, an application of (2.7) yields

$$||\zeta_n||_1-C||\xi_n||_1\leq\mathbb{E}U(\xi_n,\zeta_n)\leq\frac{e^{1-C}}{4}$$

and it suffices to let n go to infinity. □

Theorem 2.5. *We have*

$$V(1/N) \leq \begin{cases} N^{-1/2} & \text{if } N \leq 4, \\ 2N^{-1} + N^{-1}\log(N/4) & \text{if } N \geq 5. \end{cases}$$

Proof. This statement, combined with (2.5), will yield (1.7). Furthermore, comparing (2.1) and (2.2), we will get the assertion of Theorem 1.3. Let F, G, A be as in the definition of $V(1/N)$ and let ε_0, ε_1, ε_2, $\dots$ be the transforming deterministic sequence which produces G from F. Suppose first that $N \leq 4$; in this case the proof is very simple. Namely, by Schwarz inequality, we have

$$\mathbb{E}|G_\infty|1_A \leq \sqrt{\mathbb{E}|G_\infty|^2}\sqrt{\mathbb{P}(A)} \leq \sqrt{\mathbb{E}|F_\infty|^2}\sqrt{1/N} \leq 1/\sqrt{N},$$

since $||F||_\infty \leq 1$. The case $N \geq 5$ is more involved. Introduce the random variable $\xi = 1_A G_\infty/|G_\infty|$ (with the convention $\xi = 0$ if $G_\infty = 0$) and consider the martingales $(\xi_n)_{n\geq 0} = (\mathbb{E}(\xi|\mathcal{F}_n))_{n\geq 0}$ and

$$\zeta_n = \sum_{k=0}^{n} \varepsilon_k d\xi_k, \qquad n = 0, 1, 2, \dots.$$

Clearly, $(\zeta_n)_{n\geq 0}$ is a transform of $(\xi_n)_{n\geq 0}$ by $(\varepsilon_n)_{n\geq 0}$. Consequently, we may write the following chain of expressions:

$$\begin{aligned} \mathbb{E}|G_\infty|1_A &= \mathbb{E}\langle G_\infty, \xi_\infty\rangle = \sum_{k=0}^{\infty} \mathbb{E}\big\langle dG_k, d\xi_k\big\rangle \\ &= \sum_{k=0}^{\infty} \mathbb{E}\big\langle \varepsilon_k dG_k, \varepsilon_k d\xi_k\big\rangle = \mathbb{E}\langle F_\infty, \zeta_\infty\rangle \leq \mathbb{E}|\zeta_\infty|. \end{aligned}$$

Now we apply (2.6) with $C = 1 + \log(N/4)$. The martingale $(\xi_n)_{n\geq 0}$ is bounded by 1 and $||\xi||_1 \leq \mathbb{P}(A) \leq 1/N$, so we obtain

$$\mathbb{E}|G_\infty|1_A \leq (1 + \log(N/4))N^{-1} + N^{-1} = 2N^{-1} + N^{-1}\log(N/4),$$

which is the claim. □

Remark 2.6. It is well known that in general Burkholder's function (that is, the special function leading to a given martingale inequality) is not unique, see e.g. [4]. Sometimes it is of interest to determine the optimal (that is, the least) of the possible ones, at least for $\mathcal{H} = \mathbb{R}$. Though we shall not need this, we would like to mention here that we have managed to find the least function for (2.6) in the real case. Namely, for $(x, y) \in [-1, 1] \times \mathbb{R}$, the value of this function at (x, y) equals

$$\begin{cases} e^{C-1}(y^2 - x^2) + e^{1-C}/4 & \text{if } |x| + |y| \leq e^{1-C}/2, \\ |x| + |y| - |x|\log(2|x| + 2|y|) - C|x| & \text{if } e^{1-C}/2 < |x| + |y| \leq 1/2, \\ |y| + |x|\exp(1 - 2|x| - 2|y|) - C|x| & \text{if } 1/2 - |y| \leq |x| \leq 1/2, \\ |y| + (1 - |x|)\exp(-1 - 2|y| + 2|x|) - C|x| & \text{if } 1/2 < |x| \leq |y| + 1/2, \\ |y| + 1 - |x| - (1 - |x|)\log(2 + 2|y| - 2|x|) - C|x| & \text{if } |x| > |y| + 1/2. \end{cases}$$

We omit the further details in this direction, leaving them to the interested reader.

3. Sharpness, the case $N = 2$ and $N = 3$

3.1. **Preliminary observations.** We begin by several useful remarks, which will be often exploited below. We will show that the constant C_N is already the best for the Banach space $\ell_\infty^N = \ell_\infty^N(\mathbb{R})$. To do this, it suffices, for each N and $\varepsilon > 0$, to construct a pair (f, g) of ℓ_∞^N-valued martingales such that f is bounded by 1, g is a transform of f by a sequence of multisigns and $||g||_1 > C_N - \varepsilon$. In the search for appropriate examples, we recall the following inequality for transforms of real-valued martingales, proved by Burkholder [4]. Namely, if f is bounded by 1, g is its transform by a sequence of signs and $\lambda > 1$, then we have the sharp bound

$$\mathbb{P}(|g_\infty| \geq \lambda) \leq \begin{cases} \lambda^{-2} & \text{if } 1 < \lambda \leq 2, \\ e^{2-\lambda}/4 & \text{if } \lambda > 2. \end{cases} \tag{3.1}$$

Note that if we pick $\lambda = C_N$, the above estimate becomes $\mathbb{P}(|g_\infty| \geq \lambda) \leq 1/N$. This gives a very strong indication how to proceed: if we work with ℓ_∞^N-valued martingales f, g, then

1° for each $1 \leq k \leq N$ the coordinates f^k, g^k must be the extremal in (3.1),
2° the sets $\{|g_\infty^1| \geq \lambda\}$, $\{|g_\infty^2| \geq \lambda\}$, ..., $\{|g_\infty^N| \geq \lambda\}$ must be pairwise disjoint.

Having ensured these two conditions, we are done: then the martingale f is bounded by 1, g is its transform by a certain deterministic multisign and $||g_\infty|| \geq \lambda = C_N$ with probability 1.

There is nothing special in the requirement 1°: one only has to study carefully Burkholder's examples (which are quite complicated) to get the intuition about them. However, the condition 2° turns out to be much more difficult. It is a nontrivial combinatorial problem to take N pairs of extremal martingales as in 1° and bind them together so that 2° holds. The obstacle is that the pairs (f^k, g^k) must be adapted to *the same filtration* and thus have complicated dependence structure.

3.2. **An auxiliary function.** It will be convenient to work with a certain function closely related to $\bar{\eta}$ and the supremum S_N considered in (2.1). Let N be a given positive integer. For any x, $y \in \ell_\infty^N$ such that $||x|| \leq 1$, let $\mathcal{M}(x, y)$ denote the class of all pairs (f, g) of simple ℓ_∞^N-valued martingales such that

1° f starts from x and satisfies $||f||_\infty \leq 1$,
2° g starts from y and satisfies $dg_n = \varepsilon_n df_n$ for $n \geq 1$, for some deterministic sequence ε_1, ε_2, ... of multisigns.

Here the filtration is to vary, as well as the probability space, unless it is assumed to be nonatomic.

Define $\mathcal{U} : \{(x, y) \in \ell_\infty^N \times \ell_\infty^N : ||x|| \leq 1\} \to \mathbb{R}$ by the formula

$$\mathcal{U}(x, y) = \sup\{||g_\infty||_1 : (f, g) \in \mathcal{M}(x, y)\}.$$

We will prove the following statement.

Lemma 3.1. *The function $\mathcal{U}$ satisfies the following properties.*

(a) *For any multisigns $\theta = (\theta_1, \theta_2, \ldots, \theta_N)$, $\gamma = (\gamma_1, \gamma_2, \ldots, \gamma_N)$ and x, y,*

$$\mathcal{U}(\theta x, \gamma y) = \mathcal{U}(x, y) \tag{3.2}$$

(here $\theta x = (\theta_1 x_1, \theta_2 x_2, \ldots, \theta_N x_N)$ and similarly for γy).

(b) *For any x, $y \in \ell_\infty^N$ and any permutation π of the set $\{1, 2, \ldots, N\}$,*

$$\mathcal{U}(x_\pi, y_\pi) = \mathcal{U}(x, y) \tag{3.3}$$

(here $x_\pi = (x_{\pi_1}, x_{\pi_2}, \ldots, x_{\pi_N})$ and similarly for y_π).

(c) *For any x, $y \in \ell_\infty^N$ we have the majorization*

$$\mathcal{U}(x,y) \geq ||y||. \tag{3.4}$$

(d) *The function $\mathcal{U}$ enjoys the following concavity property. For any x, $y \in \ell_\infty^N$ with $||x|| \leq 1$, any multisign θ, any t_1, $t_2 \in \ell_\infty^N$ with $||x + t_i|| \leq 1$ and any $\alpha \in (0,1)$ such that $\alpha t_1 + (1-\alpha)t_2 = 0$,*

$$\mathcal{U}(x,y) \geq \alpha\mathcal{U}(x+t_1, y+\theta t_1) + (1-\alpha)\mathcal{U}(x+t_2, y+\theta t_2). \tag{3.5}$$

Proof. The properties (a) and (b) are evident and follow at once from the very definition of $\mathcal{U}$ and the fact that the three conditions: $(f,g) \in \mathcal{M}(x,y)$, $(\theta f, \gamma g) \in \mathcal{M}(\theta x, \gamma y)$ and $(f_\pi, g_\pi) \in \mathcal{M}(x_\pi, y_\pi)$, are equivalent. The majorization (c) is also straightforward: the constant pair $(f,g) \equiv (x,y)$ belongs to $\mathcal{M}(x,y)$. The condition (d) can be easily proved using the splicing argument: see the proof of Lemma 2.1 above. $\square$

3.3. **The case $N = 2$.** We start with recalling Burkholder's extremal example for (3.1) with $\lambda = \sqrt{2}$. Consider the points

$$P_0 = P_8 = -P_4 = \left(-\frac{\sqrt{2}}{2}, 1 - \frac{\sqrt{2}}{2}\right), \qquad P_1 = -P_5 = (-1, 0),$$

$$P_2 = -P_6 = \left(\frac{\sqrt{2}}{2} - 1, \frac{\sqrt{2}}{2}\right), \qquad P_3 = -P_7 = (-1, \sqrt{2}).$$

Introduce a Markov martingale (f,g) with values in $\mathbb{R}^2$, with the distribution uniquely determined by the following requirements:

(i) We have $(f_0, g_0) = (-1/2, 1/2)$.
(ii) In its first move, it goes to P_0 or to P_2.
(iii) For $k \in \{0,1,2,3\}$, the state P_{2k} leads to P_{2k+1} or to P_{2k+2}.
(iv) All the remaining points are absorbing.

We easily check that g is a transform of f by a sequence of signs and that $|f_\infty| = 1$, $|g_\infty| = \sqrt{2}$ almost surely. Thus

$$2\mathbb{P}(|g_\infty| \geq \sqrt{2}) = ||g_\infty||_2^2 = ||f_\infty||_2^2 = 1,$$

so both sides of (3.1) are equal. To get the extremal pair of martingales with values in ℓ_∞^2, we need to complicate the above example a little bit. Namely, consider a Markov martingale (f,g) with values in $\ell_\infty^2 \times \ell_\infty^2$, with the distribution given as follows.

(i) We have $(f_0^1, f_0^2, g_0^1, g_0^2) = (-1/2, -1/2, 1/2, 1/2)$.
(ii) In its first move, it goes to (P_0, P_2) or to (P_2, P_0).
(iii) For $k \in \{0,1,2,3\}$, the state (P_{2k}, P_{2k+2}) leads to (P_{2k+1}, P_{2k+3}) or to (P_{2k+2}, P_{2k+4}) (here $P_{10} = P_2$).
(iv) For $k \in \{0,1,2,3\}$, the state (P_{2k+2}, P_{2k}) leads to (P_{2k+2}, P_{2k+4}) or to (P_{2k+1}, P_{2k+3}).
(iv) All the remaining points are absorbing.

One easily verifies that the above definition makes sense (i.e., the moves described in (iii) and (iv) are of martingale type), that the martingale g is a transform of f by a multisign and that 1°, 2° are satisfied. This implies that the inequality (1.7) is sharp for $N = 2$. However, it will be convenient to rewrite this proof in a different manner, with the use of the function $\mathcal{U}$ introduced in the previous subsection. This approach will be particularly efficient (much simpler) in the case $N=3$, in which the explicit example is extremely complicated.

By the very definition of $\mathcal{U}$, it suffices to show the inequality

$$\mathcal{U}((1/2,1/2),(1/2,1/2)) \geq \sqrt{2}. \tag{3.6}$$

Using the concavity of $\mathcal{U}$ (see (d)) and the property (3.3), we write

$$\begin{aligned}\mathcal{U}\left(\left(\frac{1}{2},\frac{1}{2}\right),\left(\frac{1}{2},\frac{1}{2}\right)\right) \geq & \frac{1}{2}\mathcal{U}\left(\left(1-\frac{\sqrt{2}}{2},\frac{\sqrt{2}}{2}\right),\left(\frac{\sqrt{2}}{2},1-\frac{\sqrt{2}}{2}\right)\right)\\ &+\frac{1}{2}\mathcal{U}\left(\left(\frac{\sqrt{2}}{2},1-\frac{\sqrt{2}}{2}\right),\left(1-\frac{\sqrt{2}}{2},\frac{\sqrt{2}}{2}\right)\right)\\ =&\,\mathcal{U}\left(\left(\frac{\sqrt{2}}{2},1-\frac{\sqrt{2}}{2}\right),\left(1-\frac{\sqrt{2}}{2},\frac{\sqrt{2}}{2}\right)\right).\end{aligned}$$

The further use of the concavity and the application of (3.2), (3.3) and (3.4) give

$$\begin{aligned}&\mathcal{U}\left(\left(\frac{\sqrt{2}}{2},1-\frac{\sqrt{2}}{2}\right),\left(1-\frac{\sqrt{2}}{2},\frac{\sqrt{2}}{2}\right)\right)\\ &\geq (\sqrt{2}-1)\mathcal{U}\left(\left(1-\frac{\sqrt{2}}{2},-\frac{\sqrt{2}}{2}\right),\left(-\frac{\sqrt{2}}{2},1-\frac{\sqrt{2}}{2}\right)\right)+(2-\sqrt{2})\mathcal{U}((1,1),(0,\sqrt{2}))\\ &= (\sqrt{2}-1)\mathcal{U}\left(\left(\frac{\sqrt{2}}{2},1-\frac{\sqrt{2}}{2}\right),\left(1-\frac{\sqrt{2}}{2},\frac{\sqrt{2}}{2}\right)\right)+(2-\sqrt{2})\cdot\sqrt{2},\end{aligned}$$

which implies

$$\mathcal{U}\left(\left(\frac{\sqrt{2}}{2},1-\frac{\sqrt{2}}{2}\right),\left(1-\frac{\sqrt{2}}{2},\frac{\sqrt{2}}{2}\right)\right)\geq\sqrt{2}$$

and thus (3.6) follows. Of course, this proof of the sharpness is the same as the previous one: the weights in Jensen inequalities exploited above correspond to the transition probabilities from (ii), (iii) and (iv), and the value points are exactly (P_i,P_j) used there (up to some changes in the signs of the coordinates).

The case $N=3$. Here the calculations are much more involved. We do not specify the extremal Markov martingale (f,g), and write the proof in the language of the function $\mathcal{U}$. It suffices to show that

$$\mathcal{U}\left(\left(\frac{1}{2},\frac{1}{2},\frac{1}{2}\right),\left(\frac{1}{2},\frac{1}{2},\frac{1}{2}\right)\right)\geq\sqrt{3}. \tag{3.7}$$

Using concavity and the conditions (3.2) and (3.3), we get

$$\begin{aligned}\mathcal{U}\left(\left(\frac{1}{2},\frac{1}{2},\frac{1}{2}\right),\left(\frac{1}{2},\frac{1}{2},\frac{1}{2}\right)\right)\leq & \frac{1}{2}\mathcal{U}\left(\left(1-\frac{\sqrt{3}}{2},\frac{1}{2},\frac{\sqrt{3}}{2}\right),\left(\frac{\sqrt{3}}{2},\frac{1}{2},1-\frac{\sqrt{3}}{2}\right)\right)\\ &+\frac{1}{2}\mathcal{U}\left(\left(\frac{\sqrt{3}}{2},\frac{1}{2},1-\frac{\sqrt{3}}{2}\right),\left(1-\frac{\sqrt{3}}{2},\frac{1}{2},\frac{\sqrt{3}}{2}\right)\right)\\ =&\,\mathcal{U}\left(\left(\frac{\sqrt{3}}{2},\frac{1}{2},1-\frac{\sqrt{3}}{2}\right),\left(1-\frac{\sqrt{3}}{2},\frac{1}{2},\frac{\sqrt{3}}{2}\right)\right).\end{aligned}$$

However, if we put $\alpha = 4(1 - 1/\sqrt{3})$, then

(3.8)
$$
\begin{aligned}
&\mathcal{U}\left(\left(\frac{\sqrt{3}}{2}, \frac{1}{2}, 1 - \frac{\sqrt{3}}{2}\right), \left(1 - \frac{\sqrt{3}}{2}, \frac{1}{2}, \frac{\sqrt{3}}{2}\right)\right) \\
&\geq \frac{1}{\alpha}\mathcal{U}\left(\left(1 - \left(1 - \frac{\sqrt{3}}{2}\right)\alpha, 1 - \frac{\alpha}{2}, 1 - \frac{\sqrt{3}}{2}\alpha\right), \left(\left(1 - \frac{\sqrt{3}}{2}\right)\alpha, \frac{\alpha}{2}, \sqrt{3} - \frac{\sqrt{3}}{2}\alpha\right)\right) \\
&\qquad\qquad + \left(1 - \frac{1}{\alpha}\right)\mathcal{U}((1,1,1),(0,0,\sqrt{3})) \\
&= \frac{1}{\alpha}\mathcal{U}\left(\left(1 - \left(1 - \frac{\sqrt{3}}{2}\right)\alpha, 1 - \frac{\alpha}{2}, 1 - \frac{\sqrt{3}}{2}\alpha\right), \left(\left(1 - \frac{\sqrt{3}}{2}\right)\alpha, \frac{\alpha}{2}, \sqrt{3} - \frac{\sqrt{3}}{2}\alpha\right)\right) \\
&\qquad\qquad + \left(1 - \frac{1}{\alpha}\right)\cdot\sqrt{3},
\end{aligned}
$$

where in the last line we have used (3.4). Denote the first term in the latter sum by $\frac{1}{\alpha}I$. By the concavity of $\mathcal{U}$,

$$
\begin{aligned}
I \geq{}& \frac{2\sqrt{3}-2}{\sqrt{3}}\mathcal{U}\left(\left(\frac{5}{2} - \sqrt{3}, 1 - \frac{\sqrt{3}}{2}, 2 - \frac{3\sqrt{3}}{2}\right), \left(\sqrt{3} - \frac{3}{2}, \frac{\sqrt{3}}{2}, 3 - \frac{3\sqrt{3}}{2}\right)\right) \\
&+ \frac{2-\sqrt{3}}{\sqrt{3}}\mathcal{U}\Big((6 - 3\sqrt{3}, 2 - \sqrt{3}, 2 - \sqrt{3}), (3\sqrt{3} - 5, \sqrt{3} - 1, 3 - 2\sqrt{3})\Big).
\end{aligned}
\tag{3.9}
$$

Similarly,

$$
\begin{aligned}
&\mathcal{U}\left(\left(\frac{5}{2} - \sqrt{3}, 1 - \frac{\sqrt{3}}{2}, 2 - \frac{3\sqrt{3}}{2}\right), \left(\sqrt{3} - \frac{3}{2}, \frac{\sqrt{3}}{2}, 3 - \frac{3\sqrt{3}}{2}\right)\right) \\
&\geq \frac{\sqrt{3}}{2+\sqrt{3}}\mathcal{U}\left(\left(\frac{1}{2}, -\frac{\sqrt{3}}{2}, \frac{\sqrt{3}}{2} - 1\right), \left(\frac{1}{2}, \frac{\sqrt{3}}{2} - 1, \frac{\sqrt{3}}{2}\right)\right) \\
&\quad + \frac{2}{2+\sqrt{3}}\mathcal{U}((1,1,-1),(0,\sqrt{3},0)) \\
&= \frac{\sqrt{3}}{2+\sqrt{3}}\mathcal{U}\left(\left(\frac{\sqrt{3}}{2}, \frac{1}{2}, 1 - \frac{\sqrt{3}}{2}\right), \left(\frac{1-\sqrt{3}}{2}, \frac{1}{2}, \frac{\sqrt{3}}{2}\right)\right) + \frac{2\sqrt{3}}{2+\sqrt{3}}
\end{aligned}
$$

and, for $\beta = 2/\sqrt{3}$,

$$
\begin{aligned}
&\mathcal{U}\Big((6 - 3\sqrt{3}, 2 - \sqrt{3}, 2 - \sqrt{3}), (3\sqrt{3} - 5, \sqrt{3} - 1, 3 - 2\sqrt{3})\Big) \\
&= \frac{1}{\beta}\mathcal{U}((1 - (3\sqrt{3} - 5)\beta, 1 - (\sqrt{3} - 1)\beta, (3 - \sqrt{3})\beta - 1), \\
&\qquad\qquad (3\sqrt{3} - 5)\beta, (\sqrt{3} - 1)\beta, (3 - \sqrt{3})\beta - \sqrt{3})) \\
&\qquad\qquad + \left(1 - \frac{1}{\beta}\right)\mathcal{U}((1,1,-1),(0,0,-\sqrt{3})).
\end{aligned}
$$

A little calculation shows that the latter expression is equal to $\frac{1}{\beta}I + (1-\frac{1}{\beta})\cdot\sqrt{3}$. Plugging the last two statements into (3.9) yields

$$\frac{\sqrt{3}}{2}I \geq \frac{2\sqrt{3}-2}{2+\sqrt{3}}\mathcal{U}\left(\left(\frac{\sqrt{3}}{2},\frac{1}{2},1-\frac{\sqrt{3}}{2}\right),\left(1-\frac{\sqrt{3}}{2},\frac{1}{2},\frac{\sqrt{3}}{2}\right)\right)+\frac{20\sqrt{3}-33}{2}$$

and combining this with (3.8) implies (3.7), the desired lower bound.

4. Sharpness, the case $N \geq 4$

Here we will use a different method, based on the explicit construction of extremal examples. For the sake of convenience, we split the reasoning into several parts.

4.1. **The case** $N = 4$**.** As we will see, the calculations in the case $N = 4$ are easy; however, it is instructive to analyze this case carefully, as similar arguments will be used later, while studying the sharpness for larger N.

As previously, the underlying idea is to keep f bounded by 1 and make $||g||$ as close to C_4 as possible. The construction is as follows. As in the preceding section, it suffices to find a martingale pair (f,g) which starts from the point $((-1/2,-1/2,-1/2,-1/2),(1/2,1/2,1/2,1/2))$. The first step is to split the pair into two: (f^1,g^1) and $(f^2,f^3,f^4,g^2,g^3,g^4)$. We determine the distributions of the variables (f_1^1,g_1^1) and $(f_1^2,f_1^3,f_1^4,g_1^2,g_1^3,g_1^4)$ by the requirements

(i) (f_1^1,g_1^1) takes values $(-1,0)$, $(1,2)$.
(ii) $f_1^2=f_1^3=f_1^4$, $g_1^2=g_1^3=g_1^4$.
(iii) (f_1^4,g_1^4) takes values $(-1/3,2/3)$, $(-1,0)$.

Note that the values listed in (i) are attained with probabilities 3/4 and 1/4, respectively; the same is true for those in (iii). Thus, merging these two variables appropriately, we obtain (f_1,g_1) such that $(f_n,g_n)_{n=0}^1$ forms a martingale with respect to *one* filtration.

We turn to the second step. The first coordinates f^1 and g^1 are kept fixed and will not be changed. On the set $\{(f_1^4,g_1^4)=(-1,0)\}$ we have $||g||=2=C_4$, so g is large, as we wanted - thus, we will not change (f,g) on this set. On the other hand, we do change the martingale on $\{(f_1^4,g_1^4)=(-1/3,2/3)\}$, and this is done as follows. We split $(f^2,f^3,f^4,g^2,g^3,g^4)$ into two pairs: (f^2,g^2) and (f^3,f^4,g^3,g^4). We determine the conditional distributions of the variables (f_2^2,g_2^2) and $(f_2^3,f_2^4,g_2^3,g_2^4)$ on $\{(f_1^1,g_1^1)\neq(1,2)\}$ by the requiring that when restricted to this set,

(i) (f_2^2,g_2^2) takes values $(-1,0)$, $(1,2)$.
(ii) $f_2^3=f_2^4$, $g_2^3=g_2^4$.
(iii) (f_2^4,g_2^4) takes values $(0,1)$, $(-1,0)$.

The values listed in (i) are attained with (conditional) probabilities 2/3 and 1/3, respectively; the same is true for those in (iii). Thus, these two variables can be appropriately glued so that $(f_n,g_n)_{n=0}^2$ forms a martingale with respect to *one* filtration.

We turn to the final step. The second coordinates f^2, g^2 are kept fixed. On the set $\{(f_2^4,g_2^4)=(-1,0)\}$ we have $||g||_2=2$, so the goal of approaching C_4 by g is achieved; thus, (f,g) will not be altered on this set. Let us restrict ourselves to the set $\{(f_2^4, g_2^4) = (0,1)\}$ and split (f^3, f^4, g^3, g^4) into two pairs: (f^3, g^3) and (f^4, g^4). We require that conditionally on this set,

(i) (f_3^3, g_3^3) takes values $(-1,0)$ and $(1,2)$.
(ii) (f_3^4, g_3^4) takes values $(1,2)$ and $(-1,0)$.

Again the values listed in (i) and (ii) are taken with conditional probability 1/2 and thus we may appropriately splice these variables, extending the martingale (f,g) to the time-set $\{0,1,2,3\}$. Note that the martingale f is bounded by 1 and for each ω, a certain coordinate of $g_3(\omega)$ is equal to C_4; thus, equality in (1.7) is attained. Furthermore, it is clear that g is a transform of f by the sequence -1, 1, 1, 1, and this completes the proof of the sharpness.

4.2. **A splitting argument.** We turn to the analysis of the case $N \geq 5$. It will be convenient to work with continuous-time processes. Throughout, δ is a fixed positive number (which will be eventually sent to 0) and we take $\lambda = C_N = 2 + \log(N/4)$. It is convenient to split the reasoning into a few intermediate parts.

Step 1. Special intervals. First let us introduce some auxiliary notation. Consider the following families $(I_k^+)_{k\geq 0}$, $(I_k^-)_{k\geq 1}$ of line segments. Let I_0^+ be a line segment with endpoints $(-1, \lambda-2)$ and $(1,\lambda)$; for $k \geq 1$, we assume that

$$\begin{aligned} I_k^+ &\text{ has endpoints } (-1, \lambda - 2 - 2k\delta) \text{ and } (\delta, \lambda - 1 - 2k\delta + \delta), \\ I_k^- &\text{ has endpoints } (0, \lambda - 1 - 2k\delta + 2\delta) \text{ and } (1, \lambda - 2 - 2k\delta + 2\delta). \end{aligned}$$

Note that the segments $I_\ell^\pm$ has the slope ± 1. See Figure 1 below.

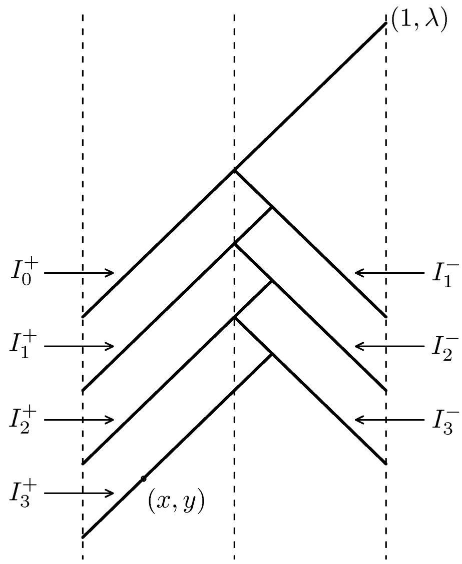

FIGURE 1. The intervals $I_\ell^\pm$.

Step 2. A family of Markov martingales. Let k be a fixed positive integer and let $(x,y) \in I_k^+$. Let B be a standard Brownian motion starting from 0 and consider the decreasing families $(\tau_j^+)_{j=0}^k$, $(\tau_j^-)_{j=0}^{k+1}$ of stopping times, given by the backward induction as follows: $\tau_{k+1}^- \equiv 0$, and

$$\begin{aligned} \tau_\ell^+ &= \inf\left\{t > \tau_{\ell+1}^- : B_t \leq -x-1 \text{ or } B_t \geq \delta - x\right\}, \qquad && \ell = k,\, k-1,\, \ldots,\, 0,\\ \tau_\ell^- &= \inf\left\{t > \tau_\ell^+ : B_t \leq -x \text{ or } B_t \geq 1 - x\right\}, && \ell = k,\, k-1,\, \ldots,\, 1. \end{aligned}$$

Now, for $t \geq 0$, define the Markov martingale (X, Y) by

$$X_t = x + B_{\tau_0^+ \wedge t} \quad \text{and} \quad Y_t = y + \int_0^t H_s \mathrm{d}X_s,$$

where

$$H_s = \begin{cases} 1 & \text{if } s \in [\tau_{\ell+1}, \tau_\ell^+) \text{ for some } \ell,\\ -1 & \text{if } s \in [\tau_\ell^+, \tau_\ell^-) \text{ for some } \ell. \end{cases}$$

To gain some intuition about the process (X, Y), let us look at the line segments $I_\ell^\pm$. The process (X, Y) starts from $(x, y) \in I_k^+$ and moves along this line segment until it reaches one of its endpoints (which occurs at time τ_k^+). If it gets to the left endpoint (i.e., lying on the line $x = -1$), it terminates; otherwise, it starts to evolve along I_k^-, until it reaches one of the endpoints of this line segment (which happens for $t = \tau_k^-$). If it gets to the right endpoint (that is, lying on the line $x = 1$), it stops; if this is not the case, it starts moving along I_{k-1}^+, until it reaches one of its endpoints, etc. The pattern of the movement is then repeated. We see that the terminal variable $X_\infty = X_{\tau_0^+}$ takes values ± 1 with probability 1, while $Y_\infty = Y_{\tau_0^+}$ is concentrated on the set

$$\{\lambda, \lambda - 2, \lambda - 2 - 2\delta, \lambda - 2 - 4\delta, \lambda - 2 - 6\delta, \ldots, \lambda - 2 - 2k\delta\}.$$

Note that $(X_\infty, Y_\infty) = (1, \lambda)$ if and only if (X, Y) leaves I_ℓ^+'s through their right endpoints and I_ℓ^-'s through their left endpoints. Consequently, we easily see that

$$p(x, y) = \mathbb{P}((X_\infty, Y_\infty) = (1, \lambda)) = \frac{1 + x}{1 + \delta} \cdot (1 - \delta)^k \cdot \frac{1}{(1 + \delta)^{k-1}} = \frac{1 + x}{2} \cdot \left(\frac{1 - \delta}{1 + \delta}\right)^k$$

and this probability is a continuous function of (x, y). Since $(1 - \delta)/(1 + \delta) \leq e^{-2\delta}$, we have

$$p(x, y) \leq \frac{1 + x}{2} e^{-2k\delta} = \frac{1 + x}{2} e^{y - x + 1 - \lambda}. \tag{4.1}$$

A similar calculation can be carried out if the starting point (x, y) belongs to one of the "negative" intervals I_k^-.

Step 3. A stopping procedure. Now we will appropriately stop the process (X, Y) and use its Markov property. We start with the following crucial observation: if $(x, y) \in I_k^+$ and $s < p(x, y)$, then on each $I_\ell^\pm$, $\ell \leq k$, we may choose points $P_\ell^\pm$ such that $p(P_\ell^\pm) = s$. That is, on each $I_\ell^\pm$ we may choose such a starting point, that the probability of reaching $(1, \lambda)$ is equal to s. A similar statement can be formulated if $(x, y) \in I_k^-$ (but then the interval I_k^+ is not taken into account). Now, let $\mathcal{P} = \mathcal{P}^s = \{P_\ell^\pm\}$ denote the collection of the chosen points and define

$$\tau_s = \inf\left\{t : (X_t, Y_t) \in \mathcal{P}\right\},$$

with the convention $\inf \emptyset = \infty$. We may write

$$\begin{aligned} p(x,y) = \mathbb{P}\big((X_\infty, Y_\infty) = (1,\lambda)\big) &= \mathbb{P}\big((X_\infty, Y_\infty) = (1,\lambda)|\tau_s < \infty\big)\mathbb{P}(\tau_s < \infty) \\ &\quad + \mathbb{P}\big((X_\infty, Y_\infty) = (1,\lambda)|\tau_s = \infty\big)\mathbb{P}(\tau_s = \infty) \\ &= s(1 - \mathbb{P}(\tau_s = \infty)) + \mathbb{P}(\tau_s = \infty) \end{aligned}$$

and hence the probability that the stopped process (X^{τ_s}, Y^{τ_s}) ever reaches $(1,\lambda)$ equals $(p(x,y) - s)/(1-s)$ which, with a proper choice of s, can be equal to any arbitrary number from the interval $(0, p(x,y)]$. A similar argumentation can be repeated if the starting point (x,y) belongs to one of the "negative" intervals I_k^-.

Step 4. Discretization. The stopped process (X^{τ_s}, Y^{τ_s}) can be represented by a pair $(f_n, g_n)_{n=0}^M$ of finite discrete-time martingales, starting from (x,y) and satisfying $df_n \equiv \pm dg_n$ for each n (here by representation we mean that the distribution of (X_{τ_s}, Y_{τ_s}) coincides with that of the terminal value (f_M, g_M)). Again, we will describe it in detail when $(x,y) \in I_k^+$. This is straightforward: we put $M = 2k+1$, $(f_0, g_0) \equiv (x,y)$ and for each $n = 1, 2, \ldots, k$,

$$(f_{2n-1}, g_{2n-1}) = \left(X_{\tau_s \wedge \tau^+_{k+1-n}}, Y_{\tau_s \wedge \tau^+_{k+1-n}}\right)$$

and

$$(f_{2n}, g_{2n}) = \left(X_{\tau_s \wedge \tau^-_{k+1-n}}, Y_{\tau_s \wedge \tau^-_{k+1-n}}\right).$$

Finally, we set

$$(f_{2k+1}, g_{2k+1}) = \left(X_{\tau_s \wedge \tau^+_0}, Y_{\tau_s \wedge \tau^+_0}\right).$$

Directly from the construction we check that the condition $df_n = (-1)^{n+1} dg_n$ is satisfied. Note that

$$\mathbb{P}((f_{2k+1}, g_{2k+1}) = (1,\lambda)) = \mathbb{P}((X_{\tau_s}, Y_{\tau_s}) = (1,\lambda)) = \frac{p(x,y) - s}{1-s} \tag{4.2}$$

in light of the considerations of Step 3.

Step 5. Iteration. The whole procedure described above can be applied inductively to several values of s. Suppose that we are given a starting point (x,y) and a sequence $0 \leq s_m < s_{m-1} < \ldots < s_1 < p(x,y)$. The above reasoning gives the corresponding sets $\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_m$. Let $(f,g)_{k=0}^{M_1}$ be the finite martingale starting from (x,y), corresponding to $s = s_1$; its terminal variable takes values in $\mathcal{P}_1 \cup \{(1,\lambda)\}$. Now, on each set of the form $\{(f_{M_1}, g_{M_1}) = P_\ell\}$, where $P_\ell \in \mathcal{P}_1$, we repeat the above construction with the starting point P_ℓ and $s = s_2$, thus obtaining the extension of the martingale (f,g) to a larger time interval $\{0, 1, 2, \ldots, M_2\}$. Now the terminal variable (f_{M_2}, g_{M_2}) takes values in $\mathcal{P}_2 \cup \{(1,\lambda)\}$. We continue the reasoning, applying the above construction on each set of the form $\{(f_{M_2}, g_{M_2}) = P_\ell\}$, where this time P_ℓ is a given point from $\mathcal{P}_2$, and thus extend the martingale (f,g) to the time set $\{0, 1, 2, \ldots, M_3\}$, etc..

Step 6. A summary. Let (x,y) be a fixed starting point and consider a sequence $0 \leq s_m < s_{m-1} < \ldots < s_1 < s_0 = p(x,y)$. We have constructed a finite martingale (f,g) starting from (x,y) and satisfying $df_n \equiv \pm dg_n$ for each $n \geq 1$, and a deterministic sequence $0 = M_0 < M_1 < M_2 < \ldots < M_m$ such that the following holds:

$$\mathbb{P}((f_{M_n}, g_{M_n}) = (1,\lambda)|(f_{M_{n-1}}, g_{M_{n-1}}) \neq (1,\lambda)) = \frac{s_{n-1} - s_n}{1 - s_n}, \qquad n \geq 1,$$

or, equivalently,

$$\mathbb{P}((f_{M_n}, g_{M_n}) \neq (1,\lambda)|(f_{M_{n-1}}, g_{M_{n-1}}) \neq (1,\lambda)) = \frac{1-s_{n-1}}{1-s_n}, \qquad n \geq 1, \tag{4.3}$$

This equality follows directly from (4.2). Observe that in particular, the choice $m = 1$ and $s_1 = 0$ leads to (f, g) which is just the discretization of the process (X, Y) presented in Step 2.

4.3. **A splicing procedure.** Now we will describe another tool which will be used in our construction. Let $(x, y) \in [-1, 1] \times \mathbb{R}$ be a fixed point lying on a certain interval I_k^+ or I_k^-, $k \geq 1$. Consider the continuous-time process (X, Y) studied in Step 2 of the previous subsection. Since $p(x, y) < 1/2$, it is easy to see that there is a unique $y' > y$ such that if we put $\tau = \inf\{t : Y_t = y'\}$ (again, $\inf \emptyset = \infty$), then

$$\mathbb{P}(\tau = \infty) = p(x, y). \tag{4.4}$$

Let $(F_k, G_k)_{k=0}^K$ be the discretization of the process (X^τ, Y^τ): we repeat the formulas from Step 4, with τ_s replaced by τ. Decreasing K if necessary, we may assume that it is equal to the length of (F, G) (i.e., for each $0 \leq m < K$ we have $G_m \neq G_{m+1}$).

For $k = 0, 1, 2, \ldots, K-1$, put

$$p_k = \mathbb{P}(dG_{k+1} > 0|dG_k > 0).$$

In view of (4.4), we have $p_0p_1p_2 \ldots p_{K-1} = \mathbb{P}(G_M = y') = 1 - p(x, y)$. Define a sequence $s_k = 1 - p_mp_{m+1}p_{m+2} \ldots p_{K-1}$, $k = 0, 1, 2, \ldots, K-1$, and put $s_K = 0$. We easily check that $0 = s_K < s_{K-1} < \ldots < s_1 < s_0 = p(x, y)$ and

$$\frac{1-s_{k-1}}{1-s_k} = p_{k-1}, \qquad k = 1, 2, \ldots, K. \tag{4.5}$$

Let (f, g) be a martingale corresponding to $(s_k)_{k=0}^K$, defined in Step 6 of the previous subsection and let $(\Omega, \mathcal{F}, (\mathcal{F}_n)_{n\geq 0}, \mathbb{P})$ be the probability space on which (f, g) was constructed. We will define a pair $(\tilde{F}, \tilde{G})$ on this probability space, closely related in distribution to (F, G). Namely, put $(\tilde{F}_0, \tilde{G}_0) = (x, y)$, and require that for all $1 \leq n \leq M$,

$$(\tilde{F}_{M_n} - \tilde{F}_{M_{n-1}}, \tilde{G}_{M_n} - \tilde{G}_{M_{n-1}}) \text{ has the same distribution as } (dF_n, dG_n)$$

and

$$\{\tilde{G}_{M_n} - \tilde{G}_{M_{n-1}} > 0\} = \{(f_{M_n}, g_{M_n}) \neq (1, \lambda)\}.$$

This is possible: the two above events have the same probability, by (4.3) and (4.5). We extend $(\tilde{F}, \tilde{G})$ to the martingale on the full time set $\{0, 1, 2, \ldots, M_{K-1}\}$ by putting $(\tilde{F}_n, \tilde{G}_n) = \mathbb{E}\big[(\tilde{F}_{M_{K-1}}, \tilde{G}_{M_{K-1}})|\mathcal{F}_n\big]$. Note that for any n we have $F_{M_n} - F_{M_{n-1}} = G_{M_n} - G_{M_{n-1}}$ with probability 1 or $F_{M_n} - F_{M_{n-1}} = -(G_{M_n} - G_{M_{n-1}})$ almost surely; this guarantees that the "finer" martingale pair $(\tilde{F}, \tilde{G})$ also satisfies $d\tilde{F}_n = \pm d\tilde{G}_n$ for each n. Observe that we have the following crucial property: on the set $\{G_{M_n} - G_{M_{n-1}} < 0\}$ we have $g_{M_n} = \lambda$.

4.4. **An extremal example.** We are ready to exhibit a martingale pair $(\tilde{f}, \tilde{g})$ with values in $\ell_\infty^N \times \ell_\infty^N$, for which the inequality in (1.7) will be almost attained. This is based on the following inductive argument. First, put

$$-\tilde{f}_0 = \tilde{g}_0 = \left(\frac{1}{2}, \frac{1}{2}, \ldots, \frac{1}{2}\right) \in \mathbb{R}^N.$$

Now let $(\tilde{F}_k, \tilde{G}_k)_{k=0}^M$, $(f_k, g_k)_{k=0}^M$ be the finite martingales corresponding to $(x_0, y_0) = (-1/2, 1/2)$, constructed in the previous subsection. Put

$$(\tilde{f}^1, \tilde{g}^1) = (f, g) \qquad \text{and} \qquad (\tilde{f}^2, \tilde{g}^2) = (\tilde{f}^3, \tilde{g}^3) = \ldots = (\tilde{f}^N, \tilde{g}^N) = (\tilde{F}, \tilde{G}).$$

The martingale $(\tilde{f}^N, \tilde{g}^N)$ either terminates in $\{-1, 1\} \times \mathbb{R}$, or at some point (x_1, y_1). These events have probabilities $p(x_0, y_0)$ and $1 - p(x_0, y_0)$, respectively. On the set where the first possibility occurs, the construction is complete. Note that in view of the last sentence of the previous subsection, we have $||g|| = \sup_{1 \leq \ell \leq N} |g_\infty^\ell| = \lambda$ on this set (actually, $g_\infty^1 = \lambda$ there). On the other hand, to continue the construction on the set $\{(\tilde{f}_\infty^N, \tilde{g}_\infty^N) = (x_1, y_1)\}$, let $(\tilde{F}_k, \tilde{G})_{k=0}^{M'}$, $(f, g)_{k=0}^{M'}$ be the finite martingales corresponding to (x_1, y_1), considered in the previous subsection. We define $(\tilde{f}_k, \tilde{g}_k)$ for $k = M+1, M+2, \ldots, M+M'$ by requiring that

(i) $(\tilde{f}^1, \tilde{g}^1) = (\tilde{f}_M^1, \tilde{g}_M^1)$ (so the first coordinates of $\tilde{f}$ and $\tilde{g}$ are not changed.
(ii) $(\tilde{f}^3, \tilde{g}^3) = (\tilde{f}^4, \tilde{g}^4) = \ldots = (\tilde{f}^N, \tilde{g}^N)$.
(iii) the conditional distribution of $(\tilde{f}^2, \tilde{g}^2)_{k=M}^{M+M'}$ on the set $\{\tilde{f}_M^N, \tilde{g}_M^N) = (x_1, y_1)\}$ is equal to that of $(\tilde{F}_k, \tilde{G}_k)_{k=0}^{M'}$.
(iv) the conditional distribution of $(\tilde{f}^3, \tilde{g}^3)_{k=M}^{M+M'}$ on $\{(\tilde{f}_M^N, \tilde{g}_M^N) = (x_1, y_1)\}$ is equal to that of $(\tilde{F}_k, \tilde{G}_k)_{k=0}^{M'}$.

The martingale $(\tilde{f}^N, \tilde{g}^N)$ constructed so far either terminates in $\{-1, 1\} \times \mathbb{R}$, or at some point (x_2, y_2). The first possibility occurs with probability $p(x_0, y_0) + (1 - p(x_0, y_0))p(x_1, y_1)$, the second has the remaining probability $(1 - p(x_0, y_0))(1 - p(x_1, y_1))$. On the set where the first possibility occurs, the construction is over; observe that $||g|| = \lambda$ on this set. To continue the construction on $\{(f_\infty^N, g_\infty^N) = (x_2, y_2)\}$, consider $(\tilde{F}, \tilde{G})$, (f, g) of the previous subsection, corresponding to (x_2, y_2), and so on.

After k steps, the probability that $(\tilde{f}_\infty^N, \tilde{g}_\infty^N)$ is equal to (x_k, y_k) (so the construction will be continued) is equal to

$$L_k = (1 - p(x_0, y_0))(1 - p(x_1, y_1)) \ldots (1 - p(x_{k-1}, y_{k-1})).$$

On the other hand, we have

$$(1 - p(x_0, y_0))(1 - p(x_1, y_1)) \ldots (1 - p(x_{k-1}, y_{k-1}))p(x_k, y_k) = p(x_0, y_0).$$

Indeed, the left-hand side is the probability that $(\tilde{f}^{k+1}, \tilde{g}^{k+1})$ ever reaches $(1, \lambda)$, which is $p(x_0, y_0)$, since for all ℓ, $(\tilde{f}^\ell, \tilde{g}^\ell)$ are discretized versions of the process (X, Y) (in distribution). In consequence, we have

$$L_k = 1 - kp(x_0, y_0)$$

and thus

$$p(x_k, y_k) = \frac{p(x_0, y_0)}{L_k} = \left(\frac{1}{p(x_0, y_0)} - k\right)^{-1} < \frac{1}{N-k},$$

because, by (4.1), $p(x_0, y_0) < e^{2-\lambda}/4 = 1/N$. Thus, the above procedure can be repeated $N - 1$ times (as we have noted at the beginning of §4.3, we require

$p(x,y) < 1/2$ to proceed, so the $N-1$-st step is allowed). Now, if δ is sufficiently small, then $p(x_0, y_0)$ can be made arbitrarily close to $1/N$. By the above reasoning, we get that $p(x_{N-1}, y_{N-1})$ is close to 1 which implies that y_{N-1} is as close to λ as we wish.

Summarizing, we have constructed a pair $(\tilde{f}, \tilde{g})$ which has the following properties:

(i) $-f_0 = g_0 \equiv (1/2, 1/2, \ldots, 1/2)$.
(ii) $\tilde{g}$ is a transform of $\tilde{f}$ by a certain sequence of multisigns in $\mathbb{R}^N$.
(iii) $||f||_\infty \leq 1$.
(iv) With probability 1, either $g_\infty^\ell = \lambda$ for some $1 \leq \ell \leq N-1$, or g_∞^N is larger than $\lambda - \varepsilon$ (where ε is an arbitrary positive number).

This implies that the best constant in the inequality (1.7) cannot be smaller than $\lambda = C_N$. The proof is complete.

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