INEQUALITIES FOR NONCOMMUTATIVE DIFFERENTIALLY SUBORDINATE MARTINGALES

YONG JIAO, ADAM OSĘKOWSKI, AND LIAN WU

ABSTRACT. The classical differential subordination of martingales, introduced by Burkholder in the eighties, is generalized to the noncommutative setting. Working under this domination, we establish the strong-type inequalities with the constants of optimal order as $p \to 1$ and $p \to \infty$, and the corresponding endpoint weak-type (1,1) estimate. In contrast to the classical case, we need to introduce two different versions of noncommutative differential subordination, depending on the range of the exponents. For the L^p-estimate, $2 \leq p < \infty$, a certain weaker version is sufficient; on the other hand, the strong-type (p, p) inequality, $1 < p < 2$, and the weak-type (1,1) estimate require a stronger version. As an application, we present a new proof of noncommutative Burkholder-Gundy inequalities. The main technical advance is a noncommutative version of the good λ-inequality and a certain summation argument. We expect that these techniques will be useful in other situation as well.

1. INTRODUCTION

Martingale inequalities play a distinguished role in probability theory and have profound applications in many areas of mathematics, including stochastic analysis, harmonic analysis, functional analysis and geometry of Banach spaces; we refer to two recent monographs [20, 40] for the classical martingale theory. The purpose of this paper is to study certain estimates arising in the context of noncommutative (or quantum) probability. This branch of martingale theory has become a very active area of research in the recent twenty years and many important inequalities from the classical case have been successfully transferred to the noncommutative setting, often revealing certain unexpected phenomena. The literature on the subject is very extensive and we will briefly mention here several results closely related to those obtained in this work. The paper [41] of Pisier and Xu is fundamental to the whole theory. The authors introduced there the abstract noncommutative setup, formulated an appropriate form of Burkholder-Gundy inequalities and established a noncommutative analogue of Stein's inequality. The Doob maximal estimate was generalized to the noncommutative context by Junge [26] and the appropriate versions of Burkholder/Rosenthal inequalities were studied by Junge and Xu in [27, 30]. The non-classical analogue of Gundy's decomposition of a martingale was obtained by Parcet and Randrianantoanina in [38], and the noncommutative version of Davis' decomposition was given by Perrin in [39]. We should mention here the important works on the weak-type versions of the estimates above, obtained by Randrianantoanina in [43, 44, 45]. We also refer the interested reader to the writings [5, 6, 7, 14, 15, 16, 17, 21, 22, 46, 47] for martingale inequalities in the context of various noncommutative symmetric spaces, to the articles [8, 12] for certain non-classical atomic decompositions,

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to the works [19, 29] for the noncommutative versions of John-Nirenberg inequalities, and to the papers [23, 24, 25] for the noncommutative analogs of Johnson-Schechtman inequalities.

In this paper, we will continue the above line of research and study noncommutative analogs of classical martingales inequalities under the assumption of the so-called differential subordination. Let us briefly recall the basic definitions in the classical case. Suppose that $(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space equipped with a discrete-time filtration $(\mathcal{F}_n)_{n\geq 0}$. Let $x = (x_n)_{n\geq 0}$, $y = (y_n)_{n\geq 0}$ be two adapted martingales taking values in some real or complex Hilbert space $(\mathcal{H}, |\cdot|)$. We define the associated difference sequences $dx = (dx_n)_{n\geq 0}$ and $dy = (dy_n)_{n\geq 0}$ by the formulae $dx_0 = x_0$ and $dx_n = x_n - x_{n-1}$ for $n \geq 1$, and analogously for dy. Following Burkholder [11], we say that y is *differentially subordinate* to x, if for any $n \geq 0$ we have the inequality $|dy_n| \leq |dx_n|$ almost surely. There are two important examples, which link the notion of differential subordination to other basic operators arising in the martingale theory. First, suppose that $x = (x_n)_{n\geq 0}$ is an arbitrary martingale, $\varepsilon = (\varepsilon_n)_{n\geq 0}$ is a deterministic sequence of signs and let y be the martingale transform of x by ε, i.e.,

$$y_n = \sum_{k=0}^{n} \varepsilon_k dx_k, \qquad n = 0, 1, 2, \cdots .$$

Then y is a martingale and we have $|dy_n| = |dx_n|$ for all n, so the differential subordination holds. To describe the second example, suppose that $x = (x_n)_{n\geq 0}$ is an arbitrary martingale taking values in some Hilbert space $(\mathcal{H}, |\cdot|)$. Then we may treat x as a process taking values in a larger Hilbert space $\ell_2(\mathcal{H})$, simply embedding it onto the first coordinate (i.e., identifying $x \in \mathcal{H}$ with $(x, 0, 0, \cdots) \in \ell_2(\mathcal{H})$). Consider another $\ell_2(\mathcal{H})$-valued martingale given by $y_n = (dx_0, dx_1, dx_2, \cdots, dx_n, 0, 0, \cdots)$. Obviously, we have $|dy_n|_{\ell_2(\mathcal{H})} = |dx_n|_{\ell_2(\mathcal{H})}$ for each n, so the differential subordination is satisfied (actually, in both directions: the martingales x and y are differentially subordinate to each other). On the other hand, we have $|y_n|_{\ell_2(\mathcal{H})} = \left(\sum_{k=0}^n |dx_k|^2\right)^{1/2}$ and hence any general estimate between differentially subordinate martingales immediately yields the corresponding analogue for square functions.

The differential subordination implies many interesting inequalities between the martingales involved. Furthermore, as Burkholder showed in [11], there is a general powerful technique, sometimes referred to as the Bellman function method in the recent literature, which enables the identification of optimal constants in such estimates. As an application of this approach, Burkholder [11] proved that for any martingale x, if the martingale y is differentially subordinate to x, then we have the sharp weak-type inequality

$$||y||_{1,\infty} \leq 2||x||_1$$

and the sharp L^p-bound

$$||y||_p \leq (p^* - 1)||x||_p, \qquad 1 < p < \infty, \tag{1.1}$$

where $p^* = \max\{p, p/(p-1)\}$. These estimates can be extended in numerous directions; see the monograph [35] by the second-named author, which contains almost up-to-date exposition on the subject. See also the works of Wang [49] and Bañuelos and Wang [4] where an appropriate continuous-time version of the differential subordination was introduced. Besides the two probabilistic examples mentioned above (i.e., the square functions and martingale transforms), such estimates have important applications in harmonic analysis and the theory of quasiconformal mappings (see e.g. [1, 2, 3, 4, 9, 36, 37] and consult the references therein). This immediately gives rise to the problem of extending these results to the noncommutative case. To the best of our knowledge, this issue is almost completely open (except for [34], which contains the study of the corresponding weak-type inequality under an unnatural condition). The primary goal of this paper is to set up an appropriate framework and establish fundamental estimates in this direction, that is, to establish the weak-type (1,1) inequality and strong-type (p, p) inequality for noncommutative differentially subordinate martingales. As we shall see, our methods will require several new ideas and we hope that our approach may be useful in the study of other inequalities which arise naturally in the area.

Let us write a few words about the organization of the remaining part of this paper.

In the next section we present the necessary noncommutative background and some basic facts of the theory of martingales in this setting.

Section 3 contains the discussion on the concept of noncommutative version of differential subordination under which we will work later. Quite unexpectedly (though the form of noncommutative Burkholder-Gundy inequalities might indicate this), we need to introduce two different versions of this domination relation, a stronger and a weaker one (the reason for this will be explained in a moment).

In Section 4 we show that the stronger condition guarantees the validity of the weak-type (1,1) estimate. This is accomplished by exploiting a certain novel version of noncommutative Gundy's decomposition.

Section 5 is devoted to the L^p-inequalities for differentially subordinate martingales. We need the aforementioned stronger version of the domination to show the inequality for $1 < p < 2$; however, it turns out that for $p \geq 2$ the weaker version is sufficient. We should point out here that the study of the strong-type estimates is not a mere repetition or slight modification of well-known arguments and methods, but it requires the development of completely new ideas. In a typical situation (e.g., during the study of L^p-bounds for martingale transforms or Burkholder-Gundy inequalities), one establishes the corresponding weak-type $(1, 1)$ and strong-type $(2, 2)$ estimate and then proceeds with interpolation and duality arguments. In the context of differential subordination, neither of these tools is directly available. We overcome this difficulty by using a certain "summation argument" which can be regarded as a proper substitution of the real interpolation method; this enables us to handle the L^p-estimates in the case $1 < p < 2$. For $p \geq 2$, we exploit a certain novel approach which can be viewed as a noncommutative version of the good-λ inequality. As we shall see, the study of both these cases will rest on a very delicate technical analysis, which does not occur very often in the noncommutative case.

The final section of the paper contains an application of the results obtained earlier. As we have seen above, any estimate for classical differentially subordinate martingales implies the corresponding bound for classical square functions. It turns out that a certain non-classical version of this phenomenon also holds true, and this allows us to obtain an alternative proof of noncommutative Burkholder-Gundy inequalities obtained originally in [41].

2. Preliminaries

We start with some basic facts from the operator theory, for the detailed exposition of the subject we refer the reader to [31, 32, 48]. Throughout the paper, $\mathcal{M}$ is a von Neumann algebra equipped with a semifinite normal faithful trace τ. We assume that $\mathcal{M}$ is a subalgebra of the algebra of all bounded operators acting on some Hilbert space $\mathcal{H}$. A closed densely defined operator a on $\mathcal{H}$ is said to be affiliated with $\mathcal{M}$ if $u^*au = a$ for all unitary operators u in the commutant $\mathcal{M}'$ of $\mathcal{M}$. A closed densely defined operator a on $\mathcal{H}$ affiliated with $\mathcal{M}$ is said to be τ-measurable if for any $\varepsilon > 0$ there exists a projection e such that $e(\mathcal{H})$ is contained in the domain of x and $\tau(I - e) < \varepsilon$; here and below, I denotes the identity operator. The set of all τ-measurable operators will be denoted by $L^0(\mathcal{M}, \tau)$. The trace τ can be extended to a positive tracial functional on the positive part $L^0_+(\mathcal{M}, \tau)$ of $L^0(\mathcal{M}, \tau)$ and this extension is still denoted by

τ. For a given family $(e_i)_{i\in I}$ of projections, the symbol $\bigwedge_{i\in I} e_i$ will denote the intersection of the family, i.e., the projection onto $\bigcap_{i\in I} e_i(\mathcal{H})$. Next, suppose that a is a self-adjoint τ-measurable operator and let $a = \int_{-\infty}^{\infty} \lambda de_\lambda$ stand for its spectral decomposition. For any Borel subset B of $\mathbb{R}$, the spectral projection of a corresponding to the set B is defined by $I_B(a) = \int_{-\infty}^{\infty} \chi_B(\lambda) de_\lambda$.

For $0 < p < \infty$, we recall that the noncommutative L^p-space associated with $(\mathcal{M}, \tau)$ is defined by $L^p(\mathcal{M}, \tau) = \{x \in L^0(\mathcal{M}, \tau) : \tau(|x|^p) < \infty\}$ equipped with the (quasi-)norm $\|x\|_p = (\tau(|x|^p))^{1/p}$, where $|x| = (x^*x)^{1/2}$ is the modulus of x. For $p = \infty$, the space $L^p(\mathcal{M}, \tau)$ coincides with $\mathcal{M}$ with its usual operator norm. We refer to the survey [42] and the references therein for more details.

We now turn our attention to the general setup of noncommutative martingales. Suppose that $(\mathcal{M}_n)_{n\geq 0}$ is a filtration, i.e., a nondecreasing sequence of von Neumann subalgebras of $\mathcal{M}$ whose union is weak*-dense in $\mathcal{M}$. Then for any $n \geq 0$ there is a normal conditional expectation $\mathcal{E}_n$ from $\mathcal{M}$ onto $\mathcal{M}_n$, satisfying

(i) $\mathcal{E}_n(axb) = a\mathcal{E}_n(x)b$ for all a, $b \in \mathcal{M}_n$ and $x \in \mathcal{M}$;
(ii) $\tau \circ \mathcal{E}_n = \tau$.

It is straightforward to check that the conditional expectations satisfy the tower property $\mathcal{E}_m\mathcal{E}_n = \mathcal{E}_n\mathcal{E}_m = \mathcal{E}_{\min(m,n)}$ for all nonnegative integers m and n. Furthermore, since $\mathcal{E}_n$ is trace preserving, it can be extended to a contractive projection from $L^p(\mathcal{M}, \tau)$ onto $L^p(\mathcal{M}_n, \tau_n)$ for all $1 \leq p \leq \infty$, where τ_n is the restriction of τ to $\mathcal{M}_n$.

A sequence $x = (x_n)_{n\geq 0}$ in $L^1(\mathcal{M})$ is called a noncommutative martingale (with respect, or adapted to $(\mathcal{M}_n)_{n\geq 0}$), if for any $n \geq 0$ we have the equality

$$\mathcal{E}_n(x_{n+1}) = x_n.$$

The associated difference sequence is defined as in the commutative case, with the use of the formulae $dx_0 = x_0$ and $dx_n = x_n - x_{n-1}$ for $n \geq 1$. If for some given $1 \leq p \leq \infty$ we have $x = (x_n)_{n\geq 0} \subset L^p(\mathcal{M})$ and

$$\|x\|_p = \sup_{n\geq 0} \|x_n\|_p < \infty,$$

then x is said to be a bounded L^p-martingale. An important identification is in order. Suppose that $1 \leq p < \infty$ and $x = (x_n)_{n\geq 0}$ is a martingale given by $x_n = \mathcal{E}_n(x_\infty)$ for some operator $x_\infty \in L^p(\mathcal{M})$. Then x is a bounded L^p-martingale and $\|x\|_p = \|x_\infty\|_p$. Conversely, if $1 < p < \infty$, then every bounded L^p-martingale converges in $L^p(\mathcal{M})$, and so is given by some operator x_∞ as previously. Consequently, one can identify the space of bounded L^p-martingales with the space $L^p(\mathcal{M})$ in the case $1 < p < \infty$, with the identification given by $x \mapsto x_\infty$.

Finally, let us briefly discuss the context of square functions and Hardy spaces associated with noncommutative martingales. We follow the presentation given by Pisier and Xu in [41]. Let $1 \leq p < \infty$. For a given finite sequence $a = (a_n)_{n\geq 0} \subset L^p(\mathcal{M})$, we define

$$\|a\|_{L^p(\mathcal{M},\ell_c^2)} = \left\| \left(\sum_{n\geq 0} |a_n|^2 \right)^{1/2} \right\|_p, \qquad \|a\|_{L^p(\mathcal{M},\ell_r^2)} = \left\| \left(\sum_{n\geq 0} |a_n^*|^2 \right)^{1/2} \right\|_p.$$

Then $\|\cdot\|_{L^p(\mathcal{M},\ell_c^2)}$ and $\|\cdot\|_{L^p(\mathcal{M},\ell_r^2)}$ are norms on the family of all finite sequences of $L^p(\mathcal{M})$. Furthermore, the corresponding completions $L^p(\mathcal{M}, \ell_c^2)$, $L^p(\mathcal{M}, \ell_r^2)$ form Banach spaces which can be regarded as the column and row subspaces of $L^p(\mathcal{M}\bar{\otimes}B(\ell^2), \tau \otimes \mathrm{tr})$.

For a given noncommutative martingale $x = (x_n)_{n\geq 0}$, we define its column and row square functions respectively by

$$S_{c,n}(x) = \left(\sum_{k=0}^{n} |dx_k|^2\right)^{1/2} \quad \text{and} \quad S_{r,n}(x) = \left(\sum_{k=0}^{n} |dx_k^*|^2\right)^{1/2}.$$

Then, in the language of the column and row spaces, dx belongs to $L^p(\mathcal{M}, \ell_c^2)$ (resp. $L^p(\mathcal{M}, \ell_r^2)$) if and only if $(S_{c,n}(x))_{n\geq 0}$ (resp. $(S_{r,n}(x))_{n\geq 0}$) is a bounded sequence in $L^p(\mathcal{M})$. In such a case, the ''full" sum

$$S_c(x) = \left(\sum_{k=0}^{\infty} |dx_k|^2\right)^{1/2} \quad \left(\text{resp. } S_r(x) = \left(\sum_{k=0}^{\infty} |dx_k^*|^2\right)^{1/2}\right)$$

belongs to $L^p(\mathcal{M})$. These operators are the noncommutative versions of the square functions from the classical theory of martingales.

Now, we introduce the noncommutative martingale Hardy spaces. Define $H_c^p(\mathcal{M})$ and $H_r^p(\mathcal{M})$ to be the spaces of all L^p-martingales x such that $dx \in L^p(\mathcal{M}, \ell_c^2)$ and $dx \in L^p(\mathcal{M}, \ell_r^2)$, respectively. Then, equipped with the norms

$$\|x\|_{H_c^p(\mathcal{M})} = \|dx\|_{L^p(\mathcal{M},\ell_c^2)} = \|S_c(x)\|_p$$

and

$$\|x\|_{H_r^p(\mathcal{M})} = \|dx\|_{L^p(\mathcal{M},\ell_r^2)} = \|S_r(x)\|_p,$$

the spaces $H_c^p(\mathcal{M})$, $H_r^p(\mathcal{M})$ become Banach spaces. If $1 \leq p < 2$, we define

$$H^p(\mathcal{M}) = H_c^p(\mathcal{M}) + H_r^p(\mathcal{M})$$

equipped with the norm

$$\|x\|_{H^p(\mathcal{M})} = \inf\left\{\|y\|_{H_c^p(\mathcal{M})} + \|z\|_{H_r^p(\mathcal{M})}\right\},$$

where the infimum is taken over all decompositions $x = y + z$ such that $y \in H_c^p(\mathcal{M})$, $z \in H_r^p(\mathcal{M})$. If $2 \leq p \leq \infty$, we set

$$H^p(\mathcal{M}) = H_c^p(\mathcal{M}) \cap H_r^p(\mathcal{M})$$

equipped with the intersection norm

$$\|x\|_{H^p(\mathcal{M})} = \max\left\{\|x\|_{H_c^p(\mathcal{M})}, \|x\|_{H_r^p(\mathcal{M})}\right\}.$$

The noncommutative Burkholder-Gundy inequalities can now be stated as follows.

Theorem 2.1 (Pisier and Xu [41]). *For $1 < p < \infty$, there exist finite constants c_p, C_p such that*

$$c_p^{-1}\|x\|_{H^p(\mathcal{M})} \leq \|x\|_p \leq C_p\|x\|_{H^p(\mathcal{M})}.$$

3. Noncommutative differential subordination

Our main task of this section is to find a proper extension of the notion of differential subordination in noncommutative setting. We start with the following domination relation which was proposed by the second-named author in [34].

Definition 3.1. Let x, y be two self-adjoint L^2-bounded martingales. We say that y is *differentially subordinate* to x if the following two conditions are satisfied:

(i) for any $n \geq 0$ and any projection $R \in \mathcal{M}_n$, we have

$$\tau(Rdy_nRdy_nR) \leq \tau(Rdx_nRdx_nR);$$

(ii) for any $n \geq 0$ and any orthogonal projections R, $S \in \mathcal{M}_n$ such that $R+S \in \mathcal{M}_{n-1}$, we have

$$\tau(Rdy_nSdy_nR) \leq \tau(Rdx_nSdx_nR).$$

Here and below, if $n=0$, then the phrase "$R+S \in \mathcal{M}_{n-1}$" means that $R+S=I$. As in the commutative case, an important example is that of martingale transforms. Indeed, assume that x is an arbitrary L^2-bounded martingale and $\varepsilon = (\varepsilon_n)_{n\geq 0}$ is a sequence of signs. Define y by setting $dy_n = \varepsilon_n dx_n$, $n = 0, 1, 2, \cdots$. Then, obviously, y is differentially subordinate to x.

However, it should be pointed out that the above definition has two severe deficiencies. The first issue is the assumption on the square-integrability of the martingales, without which some traces in (i) or (ii) might become infinite; nothing of this type seems to arise in the commutative context. The second problem is the complexity of the domination, which makes it very difficult to be applied.

Our first contribution is the correction of both defects above. We propose a different definition, which is much simpler, does not require the square integrability of martingales and is actually weaker than that above. Furthermore, as we will see later, it enforces the appropriate weak- and strong-type martingale inequalities.

Definition 3.2. Let x, y be two self-adjoint martingales. We say that y is *weakly differentially subordinate* to x if for any $n \geq 0$ and any projection $R \in \mathcal{M}_{n-1}$, we have

$$Rdy_nRdy_nR \leq Rdx_nRdx_nR. \tag{3.1}$$

We say that y is *very weakly differentially subordinate* to x if for any $n \geq 0$, we have

$$dy_n^2 \leq dx_n^2. \tag{3.2}$$

Let us compare the definitions above. Obviously, in the commutative case, all the dominations are equivalent to saying that $|dy_n| \leq |dx_n|$ for all $n \geq 0$, which is the usual differential subordination introduced by Burkholder [11]. Clearly, the weak differential subordination implies the very weak differential subordination and the implication cannot be reversed. What about the connection with the domination of Definition 3.1? The following lemma clarifies this issue.

Lemma 3.3. *The differential subordination implies the weak differential subordination. Furthermore, the reverse implication is not true in general.*

Proof. Let x, y be two self-adjoint martingales. Suppose that y is differentially subordinate to x. Fix $n \geq 0$ and an arbitrary projection $R \in \mathcal{M}_{n-1}$, we define

$$T = I_{(0,\infty)}(Rdy_nRdy_nR - Rdx_nRdx_nR).$$

Then $T \in \mathcal{M}_n$ is a sub-projection of R, so by (i) and (ii) of Definition 3.1, we have

$$\tau(Tdy_nTdy_nT) \leq \tau(Tdx_nTdx_nT)$$

and

$$\tau\Big(Tdy_n(R-T)dy_nT\Big) \leq \tau\Big(Tdx_n(R-T)dx_nT\Big).$$

Adding the two estimates above, we get an inequality equivalent to

$$\tau(Tdy_nRdy_nT - Tdx_nRdx_nT) \leq 0,$$

or

$$\tau\Big(T(Rdy_nRdy_nR - Rdx_nRdx_nR)\Big) \leq 0.$$

However, the operator under the latter trace is nonnegative, by the very definition of T. Thus we obtain $T = 0$ and hence the weak differential subordination holds.

We turn our attention to the second part of the lemma. We will construct an appropriate example. Let $\mathcal{M}$ be the algebra of 2×2 matrices equipped with the normalized trace τ. Consider the filtration $(\mathcal{M}_n)_{n\geq 0}$ given by $\mathcal{M}_0 = \{aI : a \in \mathbb{R}\}$ and $\mathcal{M}_1 = \mathcal{M}_2 = \cdots = \mathcal{M}$. Then the associated conditional expectations $\mathcal{E}_n : \mathcal{M} \to \mathcal{M}_n$ act as follows: for $n \geq 1$, $\mathcal{E}_n$ is just the identity, and

$$\mathcal{E}_0\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \frac{a+d}{2}I.$$

Finally, consider the adapted self-adjoint sequences $x = (x_n)_{n\geq 0}$, $y = (y_n)_{n\geq 0}$ with the differences given by

$$dx_1 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \qquad dy_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

and, for remaining n, $dx_n = dy_n = 0$. It is evident that $(dx_n)_{n\geq 0}$, $(dy_n)_{n\geq 0}$ are martingale differences, directly from the above description of the conditional expectations. Next, it is easy to see that y is weakly differentially subordinate to x. To this end, it is enough to check that for any projection $R \in \mathcal{M}_0$ we have

$$Rdy_1Rdy_1R \leq Rdx_1Rdx_1R.$$

However, there is only one nontrivial projection in $\mathcal{M}_0$, the identity operator. For $R = I$, the above estimate becomes $dy_1^2 \leq dx_1^2$, which holds true: both sides are actually equal to I. On the other hand, the condition (i) of Definition 3.1 is not satisfied. Consider the projection

$$R = \begin{bmatrix} 1/2 & -1/2 \\ -1/2 & 1/2 \end{bmatrix} \in \mathcal{M}_1$$

onto the subspace spanned by the vector $(1,-1)$. We compute that $Rdy_1Rdy_1R = R$ and $Rdx_1Rdx_1R = 0$, so the differential subordination is indeed violated. □

Definition 3.2 gives us two alternative noncommutative versions of the differential subordination. As we have announced in the introductory section, the stronger condition (3.1) will be needed to establish the weak- and L^p-bounds for $1 < p < 2$, while the weaker condition (3.2) will be sufficient for the case $p \geq 2$. Let us present an important construction which illustrates the necessity of introducing the stronger of the requirements.

Example 3.4. We will prove now that the condition (3.2) is in general too weak to imply the weak- and L^p-estimate for $1 < p < 2$. Fix a large positive even integer N and let ε_0, ε_1, ε_2, $\cdots$, ε_N be independent Rademacher variables on some (classical) probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Suppose that for each $n \geq 0$, $\mathcal{F}_n$ is the σ-algebra generated by ε_0, ε_1, ε_2, $\cdots$, ε_n (with the convention $\mathcal{F}_{-1} = \{\emptyset, \Omega\}$ and $\mathcal{F}_n = \mathcal{F}$ if $n > N$). Consider the algebra $\mathcal{M} = L^\infty(\Omega, \mathcal{F}, \mathbb{P})\overline{\otimes}\mathbb{M}_{N+2}$ equipped with the tensor product trace (here $\mathbb{M}_{N+2}$ is the algebra of $(N+2)\times(N+2)$ matrices with the usual trace) and the filtration $\mathcal{M}_n = L^\infty(\Omega, \mathcal{F}_n, \mathbb{P})\overline{\otimes}\mathbb{M}_{N+2}$, $n = 0, 1, 2, \cdots$. Let $\mathcal{E}_n = \mathbb{E}_n \otimes I : \mathcal{M} \to \mathcal{M}_n$ be the corresponding conditional expectation.

Finally, consider the sequences $dx = (dx_n)_{n\geq 0}$, $dy = (dy_n)_{n\geq 0}$ given by $dx_n = \varepsilon_n \otimes (e_{1,n+2} + e_{n+2,1})$ and $dy_n = \varepsilon_n \otimes (e_{1,1} + e_{n+2,n+2})$, $n = 0, 1, 2, \cdots, N$; for remaining n, set $dx_n = dy_n = 0$ (here and below, the symbol $e_{i,j}$ denotes the matrix which has 1 at the place lying in i-th column and j-th row, and zeros elsewhere). It is obvious that dx and dy are martingale differences. Furthermore, we compute that

$$dx_n^2 = dy_n^2 = \begin{cases} 1 \otimes (e_{1,1} + e_{n+2,n+2}) & \text{if } 1 \leq n \leq N, \\ 0 & \text{otherwise,} \end{cases}$$

so the very weak subordination is satisfied. On the other hand, we have

$$y_N = \begin{bmatrix} \varepsilon_0 + \varepsilon_1 + \cdots + \varepsilon_N & 0 & 0 & \cdots & 0 \\ 0 & \varepsilon_0 & 0 & \cdots & 0 \\ 0 & 0 & \varepsilon_1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & \varepsilon_N \end{bmatrix},$$

which implies

$$|y_N| = \begin{bmatrix} |\varepsilon_0 + \varepsilon_1 + \cdots + \varepsilon_N| & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}$$

and $\tau\left(I_{[1,\infty)}(|y_N|)\right) = N + 2$ (here we use the assumption that N is even: this guarantees that the entry in the upper-left corner of $|y_N|$ is at least 1). On the other hand, we have

$$x_N = \begin{bmatrix} 0 & \varepsilon_0 & \varepsilon_1 & \cdots & \varepsilon_N \\ \varepsilon_0 & 0 & 0 & \cdots & 0 \\ \varepsilon_1 & 0 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \varepsilon_N & 0 & 0 & \cdots & 0 \end{bmatrix}.$$

To derive the trace of $|x_N|$, note that

$$x_N^2 = \begin{bmatrix} N+1 & 0 & 0 & \cdots & 0 \\ 0 & \varepsilon_0^2 & \varepsilon_0\varepsilon_1 & \cdots & \varepsilon_0\varepsilon_N \\ 0 & \varepsilon_1\varepsilon_0 & \varepsilon_1^2 & \cdots & \varepsilon_1\varepsilon_N \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \varepsilon_N\varepsilon_0 & \varepsilon_N\varepsilon_1 & \cdots & \varepsilon_N^2 \end{bmatrix} = (N+1)(P_1 + P_\varepsilon),$$

where P_1, P_ε are the projections onto the one-dimensional spaces spanned by $(1, 0, 0, \cdots, 0)$ and $(0, \varepsilon_0, \varepsilon_1, \cdots, \varepsilon_N)$, respectively. These spaces are orthogonal, so $|x_N| = \sqrt{N+1}(P_1 + P_\varepsilon)$ and hence $\tau(|x_N|) = 2\sqrt{N+1}$. We have thus obtained that

$$\tau\left(I_{[1,\infty)}(|y_N|)\right) / \tau(|x_N|) = \frac{N+2}{2\sqrt{N+1}},$$

so the weak-type (1,1) estimate cannot hold with any finite universal constant.

The above example also provides some information in the context of L^p-estimates. Indeed, for x, y as above, we have $||y_N||_p \geq (N+2)^{1/p}$ and $||x_N||_p = 2^{1/p}\sqrt{N+1}$. Consequently, the condition (3.2) cannot guarantee in general the strong-type (p,p) estimate

$$||y_N||_p \leq C_p||x_N||_p, \qquad 1 < p < 2,$$

with any finite universal constant C_p. On the contrary, when $p \geq 2$, then (3.2) does imply the validity of the L^p-bound. This follows directly from noncommutative Burkholder-Gundy inequalities (Theorem 2.1):

$$||y_N||_p \leq C_p \left\| \Big(\sum_{n=0}^N dy_n^2 \Big)^{1/2} \right\|_p \leq C_p \left\| \Big(\sum_{n=0}^N dx_n^2 \Big)^{1/2} \right\|_p \leq C_p' ||x_N||_p.$$

However, the resulting constant is of order $O(p^2)$ as $p \to \infty$, which is much more than one might expect from the commutative case. We will improve this order to the optimal $O(p)$, which is already the best possible in the commutative setting; see (1.1).

4. Weak-type estimate

The principal statement of this section is the weak-type (1,1) estimate for weakly differentially subordinate martingales. By Lemma 3.3, it strengthens the main result [34, Theorem 1]. It should be emphasized that our approach is completely different from that used in [34], where the noncommutative Burkholder operators play an important role; while our proof is based on a certain novel version of noncommutative Gundy's decomposition (see (4.4) below), which should be compared with the one given by Parcet and Randrianantoanina in [38].

We now state the precise formulation.

Theorem 4.1. *Suppose that x, y are self-adjoint martingales such that y is weakly differentially subordinate to x. Then for any $\lambda > 0$ and any $N \geq 0$ we have*

$$\lambda\tau\left(I_{[\lambda,\infty)}(|y_N|)\right) \leq 36||x||_1.$$

The proof is a combination of several separate intermediate estimates. A key ingredient of our reasoning is the following family of projections associated with a given self-adjoint martingale $x = (x_n)_{n\geq 0}$. Set $R_{-1} = I$ and for each $n \geq 0$ define, inductively,

$$\begin{aligned} R_n &= R_{n-1}I_{(-1,1)}(R_{n-1}x_nR_{n-1}),\\ U_n &= R_{n-1}I_{(-\infty,-1]}(R_{n-1}x_nR_{n-1}),\\ D_n &= R_{n-1}I_{[1,\infty)}(R_{n-1}x_nR_{n-1}). \end{aligned}$$

The family $(R_n)_{n\geq -1}$ can be regarded as two-sided, extended version of the classical projections introduced by Cuculescu in [13]. In what follows, we will need the following properties of these objects.

Lemma 4.2. *Let $x = (x_n)_{n\geq 0}$ be an L^1-bounded self-adjoint martingale. Then the following statements hold true:*

(i) *for each $n \geq 0$ the projections R_n, U_n and D_n belong to $\mathcal{M}_n$ and $R_n + U_n + D_n = R_{n-1}$;*
(ii) *for each $n \geq 0$, the projections R_n, U_n and D_n commute with $R_{n-1}x_nR_{n-1}$;*
(iii) *for each $n \geq 0$ we have*

$$-R_n \leq R_nx_nR_n \leq R_n, \quad U_nx_nU_n \leq -U_n, \quad D_nx_nD_n \geq D_n;$$

(iv) *for any $0 \leq n \leq N$, we have*

$$-R_N \leq R_Nx_nR_N \leq R_N \quad \text{and} \quad \tau(I - R_N) \leq \tau((I - R_N)|x_N|) \leq ||x||_1.$$

Proof. It is clear that (i), (ii) and (iii) are satisfied. Furthermore,

$$R_N x_n R_N = R_N R_n x_n R_n R_N \leq R_N R_n R_N = R_N$$

and similarly $R_N x_n R_N \geq -R_N$. To prove the trace estimate from (iv), note that for each $N \geq 0$,

$$\begin{aligned}
\tau(I - R_N) &= \sum_{n=0}^{N} \tau(R_{n-1} - R_n) = \sum_{n=0}^{N} \tau(U_n + D_n) \\
&\leq \sum_{n=0}^{N} \tau\left(-U_n x_n U_n + D_n x_n D_n\right) \\
&= \sum_{n=0}^{N} \tau\left(-U_n x_N U_n + D_n x_N D_n\right) \\
&= \tau\left(\left(\sum_{n=0}^{N} U_n\right)(-x_N) + \left(\sum_{n=0}^{N} D_n\right) x_N\right) \\
&\leq \tau\left(\left(\sum_{n=0}^{N} U_n\right)|x_N| + \left(\sum_{n=0}^{N} D_n\right)|x_N|\right) \\
&\leq \tau\Big((I - R_N)|x_N|\Big) \\
&\leq \tau(|x_N|).
\end{aligned}$$

This yields the assertion. □

Later on, we will exploit the following estimate.

Lemma 4.3. *For any L^1-bounded self-adjoint martingale $x = (x_n)_{n\geq 0}$ and any $N \geq 0$ we have*

$$\tau\left(\sum_{n=0}^{N} R_n dx_n R_{n-1} dx_n R_n\right) \leq 2||x||_1.$$

Proof. Observe that for any $0 \leq n \leq N$,

$$\begin{aligned}
\tau(R_n dx_n R_{n-1} dx_n R_n) &= \tau\Big(R_n(x_n - x_{n-1})R_{n-1}(x_n - x_{n-1})R_n\Big) \\
&= \tau\Big(R_n x_n R_{n-1} x_n R_n + R_n x_{n-1} R_{n-1} x_{n-1} R_n \\
&\qquad - R_n x_{n-1} R_{n-1} x_n R_n - R_n x_n R_{n-1} x_{n-1} R_n\Big)
\end{aligned} \tag{4.1}$$

(if $n = 0$, we interpret x_{n-1} to be zero). Let us transform each of the four terms in the latter expression. First, applying the commuting property of R_n, we see that

$$\tau(R_n x_n R_{n-1} x_n R_n) = \tau(R_n x_n R_n x_n R_n).$$

Next, by the tracial property of τ and the estimate $R_n \leq R_{n-1}$, we get

$$\begin{aligned}
\tau(R_n x_{n-1} R_{n-1} x_{n-1} R_n) &= \tau(R_{n-1} x_{n-1} R_n x_{n-1} R_{n-1}) \\
&\leq \tau(R_{n-1} x_{n-1} R_{n-1} x_{n-1} R_{n-1}).
\end{aligned}$$

Finally, observe that

$$\tau(R_n x_{n-1} R_{n-1} x_n R_n) = \tau(R_{n-1} x_{n-1} R_{n-1} x_n R_n) = \tau(R_{n-1} x_{n-1} R_n x_n R_n)$$

and similarly

$$\tau(R_n x_n R_{n-1} x_{n-1} R_n) = \tau(R_{n-1} x_{n-1} R_n x_n R_n).$$

Thus we have proved that

$$\tau(R_n dx_n R_{n-1} dx_n R_n) \leq \tau\Big(R_n x_n R_n x_n R_n + R_{n-1} x_{n-1} R_{n-1} x_{n-1} R_{n-1} - 2R_{n-1} x_{n-1} R_n x_n R_n\Big),$$

which can be rewritten in the form

$$\begin{aligned}\tau(R_n dx_n R_{n-1} dx_n R_n) \leq\ & \tau(R_n x_n R_n x_n R_n) - \tau(R_{n-1} x_{n-1} R_{n-1} x_{n-1} R_{n-1}) \\ & + 2\tau\Big(R_{n-1} x_{n-1} R_{n-1}(R_{n-1} x_{n-1} R_{n-1} - R_n x_n R_n)\Big).\end{aligned}$$

By the martingale property of x, we have the equality

$$\tau(R_{n-1} x_{n-1} R_{n-1} x_{n-1} R_{n-1}) = \tau(R_{n-1} x_{n-1} R_{n-1} x_n R_{n-1}).$$

Therefore, by the definition of R_n, U_n, D_n and their commuting properties, we easily check that

$$\begin{aligned}&\tau\Big(R_{n-1} x_{n-1} R_{n-1}(R_{n-1} x_{n-1} R_{n-1} - R_n x_n R_n)\Big) \\ &\quad = \tau\Big(R_{n-1} x_{n-1} R_{n-1}(R_{n-1} x_n R_{n-1} - R_n x_n R_n)\Big) \\ &\quad = \tau\Big(R_{n-1} x_{n-1} R_{n-1}(R_{n-1} - R_n) R_{n-1} x_n R_{n-1}\Big) \\ &\quad = \tau(R_{n-1} x_{n-1} R_{n-1} U_n R_{n-1} x_n R_{n-1}) + \tau(R_{n-1} x_{n-1} R_{n-1} D_n R_{n-1} x_n R_{n-1}) \\ &\quad = \tau(R_{n-1} x_{n-1} R_{n-1} U_n x_n U_n) + \tau(R_{n-1} x_{n-1} R_{n-1} D_n x_n D_n).\end{aligned}$$

Now, by Lemma 4.2 (iii), the operator $U_n x_n U_n$ is nonpositive and $R_{n-1}(x_{n-1}+I)R_{n-1}$ is nonnegative. Consequently, we have

$$\begin{aligned}\tau(R_{n-1} x_{n-1} R_{n-1} U_n x_n U_n) &= \tau\Big(R_{n-1}(x_{n-1}+I) R_{n-1} U_n x_n U_n\Big) - \tau(U_n x_n U_n) \\ &\leq -\tau(U_n x_n) = -\tau(U_n x_N) \leq \tau(U_n |x_N|).\end{aligned}$$

Similarly, one may prove that

$$\tau(R_{n-1} x_{n-1} R_{n-1} D_n x_n D_n) \leq \tau(D_n |x_N|).$$

Putting all the above facts together, we obtain that

$$\begin{aligned}\tau(R_n dx_n R_{n-1} dx_n R_n) \leq\ & \tau(R_n x_n R_n x_n R_n) - \tau(R_{n-1} x_{n-1} R_{n-1} x_{n-1} R_{n-1}) \\ & + 2\tau\Big((U_n + D_n)|x_N|\Big),\end{aligned}$$

which yields

$$\tau\left(\sum_{n=0}^{N} R_n dx_n R_{n-1} dx_n R_n\right) \leq \tau(R_N x_N R_N x_N R_N) + 2\tau\left(\sum_{n=0}^{N} (U_n + D_n)|x_N|\right). \tag{4.2}$$

By Lemma 4.2 (i) , we have $\sum_{n=0}^{N}(U_n + D_n) = I - R_N$ and hence the second term on the right equals $2\tau\big((I - R_N)|x_N|\big)$. To handle the first term, introduce the auxiliary projections

$$R_N^+ = R_{N-1} I_{[0,1)}(R_{N-1} x_N R_{N-1}), \qquad R_N^- = R_{N-1} I_{(-1,0)}(R_{N-1} x_N R_{N-1}).$$

We have $R_N = R_N^+ + R_N^-$ and both $R_N^\pm$ commute with $R_{N-1}x_N R_{N-1}$. Therefore,

$$\begin{aligned}\tau(R_N x_N R_N x_N R_N) &= \tau(R_N R_{N-1} x_N R_{N-1} R_N R_{N-1} x_N R_{N-1} R_N)\\ &= \tau(R_N x_N R_{N-1} x_N R_{N-1})\\ &= \tau\Big((R_N^+ + R_N^-) x_N R_{N-1} x_N R_{N-1}\Big)\\ &= \tau(R_N^+ x_N R_N^+ x_N R_N^+) + \tau(R_N^- x_N R_N^- x_N R_N^-).\end{aligned}$$

The operator $R_N^+ x_N R_N^+$ is nonnegative and $R_N^+(x_N - I)R_N^+$ is nonpositive. This implies

$$\tau\Big(R_N^+ x_N R_N^+ (x_N - I) R_N^+\Big) \leq 0.$$

Hence

$$\tau\left(R_N^+ x_N R_N^+ x_N R_N^+\right) \leq \tau\left(R_N^+ x_N R_N^+\right) \leq \tau\left(R_N^+ |x_N| R_N^+\right) = \tau\left(R_N^+ |x_N|\right).$$

Analogous argumentation yields the estimate

$$\tau\left(R_N^- x_N R_N^- x_N R_N^-\right) \leq \tau\left(R_N^- |x_N|\right).$$

Combining the above two observations gives

$$\tau(R_N x_N R_N x_N R_N) \leq \tau\Big((R_N^+ + R_N^-)|x_N|\Big) = \tau(R_N |x_N|),$$

which plugged into (4.2) yields

$$\tau\left(\sum_{n=0}^N R_n dx_n R_{n-1} dx_n R_n\right) \leq \tau(R_N|x_N|) + 2\tau\Big((I - R_N)|x_N|\Big) \leq 2\tau(|x_N|) \leq 2||x||_1.$$

This is the desired assertion. □

We will also use the following fact.

Lemma 4.4. *For any L^1-bounded self-adjoint martingale $x = (x_n)_{n\geq 0}$ and any $N \geq 0$ we have*

$$\left\|\sum_{n=0}^N \big|(R_{n-1} - R_n) dx_n (R_{n-1} - R_n)\big|\right\|_1 \leq 2\tau((I - R_N)|x_N|) \leq 2||x||_1.$$

Proof. For any $0 \leq n \leq N$ we have

$$(R_{n-1} - R_n) dx_n (R_{n-1} - R_n) = (R_{n-1} - R_n)(x_n - x_{n-1})(R_{n-1} - R_n),$$

so by the triangle inequality,

$$\begin{aligned}\tau\Big(\big|(R_{n-1} - R_n) dx_n (R_{n-1} - R_n)\big|\Big) &\leq \tau\Big(\big|(R_{n-1} - R_n) x_n (R_{n-1} - R_n)\big|\Big)\\ &\quad + \tau\Big(\big|(R_{n-1} - R_n) x_{n-1} (R_{n-1} - R_n)\big|\Big).\end{aligned} \tag{4.3}$$

We will treat each of the two terms on the right hand side separately. By the martingale property of x and the fact that the conditional expectation is a contraction in L^1, we have

$$\begin{aligned}\tau\Big(\big|(R_{n-1} - R_n) x_n (R_{n-1} - R_n)\big|\Big) &= \tau\Big(\big|\mathcal{E}_n\big((R_{n-1} - R_n) x_N (R_{n-1} - R_n)\big)\big|\Big)\\ &\leq \tau\Big(\big|(R_{n-1} - R_n) x_N (R_{n-1} - R_n)\big|\Big)\\ &\leq \tau\Big((R_{n-1} - R_n)|x_N|(R_{n-1} - R_n)\Big).\end{aligned}$$

Consequently, after summation we obtain

$$\sum_{n=0}^{N}\tau\Big(\big|(R_{n-1}-R_n)x_n(R_{n-1}-R_n)\big|\Big)\leq\tau\Big((I-R_N)|x_N|\Big).$$

Concerning the second summand in (4.3), we note that the inequality $||R_{n-1}x_{n-1}R_{n-1}||_\infty\leq 1$ implies $\tau\Big(\big|(R_{n-1}-R_n)x_{n-1}(R_{n-1}-R_n)\big|\Big)\leq\tau(R_{n-1}-R_n)$, and hence

$$\sum_{n=0}^{N}\tau\Big(\big|(R_{n-1}-R_n)x_{n-1}(R_{n-1}-R_n)\big|\Big)\leq\tau(I-R_N).$$

Combining the above estimates with Lemma 4.2 (iv), we get the claim. □

We are ready for the proof of the weak-type estimate for weakly differentially subordinate martingales. Based on the lemmas above, we mainly employ a novel version of noncommutative Gundy-type decomposition.

Proof of Theorem 4.1. We may assume that x is L^1-bounded, since otherwise there is nothing to prove. Furthermore, by homogeneity, it is enough to establish the estimate for one value of λ: it will be convenient for us to take $\lambda=4$.

We start with an appropriate Gundy-type decomposition of the dominated martingale y. Namely, for any $n\geq 0$, set

$$dy_n=d\alpha_n+d\beta_n+d\gamma_n+d\delta_n,$$

where

$$\begin{aligned}d\alpha_n&=R_{n-1}dy_nR_n+R_ndy_nR_{n-1}-R_ndy_nR_n\\&\quad-\mathcal{E}_{n-1}(R_{n-1}dy_nR_n+R_ndy_nR_{n-1}-R_ndy_nR_n),\\d\beta_n&=\mathcal{E}_{n-1}(R_{n-1}dy_nR_n+R_ndy_nR_{n-1}-R_ndy_nR_n),\\d\gamma_n&=R_ndy_n(I-R_{n-1}),\\d\delta_n&=(I-R_n)dy_n-(R_{n-1}-R_n)dy_nR_n\end{aligned}\tag{4.4}$$

(with the convention $\mathcal{E}_{-1}a=0$ for all a). The differences $d\alpha$, $d\beta$, $d\gamma$ and $d\delta$ give rise to the associated processes α, β, γ and δ. By the well-known properties of a distribution function,

$$\tau\left(I_{[4,\infty)}(|y_N|)\right)\leq\tau\left(I_{[1,\infty)}(|\alpha_N|)\right)+\tau\left(I_{[1,\infty)}(|\beta_N|)\right)+\tau\left(I_{[1,\infty)}(|\gamma_N|)\right)+\tau\left(I_{[1,\infty)}(|\delta_N|)\right).\tag{4.5}$$

For the sake of convenience, we now study the terms on the right separately.

Step 1. The term $d\alpha_n$. It is evident that $(d\alpha_n)_{n\geq0}$ is a martingale difference sequence. Note that for any self-adjoint bounded operator a, we have

$$\tau\Big((a-\mathcal{E}_{n-1}(a))^2\Big)=\tau\Big(a^2-\mathcal{E}_{n-1}(a)^2\Big)\leq\tau(a^2).$$

Therefore,

$$\begin{aligned}\tau(d\alpha_n^2)&\leq\tau\Big((R_{n-1}dy_nR_n+R_ndy_nR_{n-1}-R_ndy_nR_n)^2\Big)\\&=2\tau(R_ndy_nR_{n-1}dy_n)-\tau(R_ndy_nR_ndy_n)\leq2\tau(R_{n-1}dy_nR_ndy_n).\end{aligned}$$

By the weak differential subordination of y to x, we have

$$R_{n-1}dy_nR_{n-1}dy_nR_{n-1}\leq R_{n-1}dx_nR_{n-1}dx_nR_{n-1}$$

and hence also $R_n dy_n R_{n-1} dy_n R_n \leq R_n dx_n R_{n-1} dx_n R_n$, since $R_n \leq R_{n-1}$. Passing to the trace, we obtain

$$\tau(R_{n-1} dy_n R_n dy_n) = \tau(R_n dy_n R_{n-1} dy_n) \leq \tau(R_n dx_n R_{n-1} dx_n R_n).$$

As we have mentioned above, the process $(\alpha_n)_{n\geq 0}$ is a martingale, so the above analysis and Lemma 4.3 give

$$||\alpha_N||_2^2 = \sum_{n=0}^{N} ||d\alpha_n||_2^2 \leq 2\tau\left(\sum_{n=0}^{N} R_n dx_n R_{n-1} dx_n R_n\right) \leq 4||x||_1. \tag{4.6}$$

Consequently, by Chebyshev's inequality, we obtain

$$\tau\left(I_{[1,\infty)}(|\alpha_N|)\right) \leq ||\alpha_N||_2^2 \leq 4||x||_1.$$

Step 2. The term $d\beta_n$. By the martingale property of y, we have

$$\begin{aligned} d\beta_n &= \mathcal{E}_{n-1}(-R_{n-1} dy_n R_{n-1} + R_{n-1} dy_n R_n + R_n dy_n R_{n-1} - R_n dy_n R_n) \\ &= -\mathcal{E}_{n-1}\Big((R_{n-1} - R_n) dy_n (R_{n-1} - R_n)\Big). \end{aligned}$$

By the weak differential subordination of y to x, namely,

$$R_{n-1} dy_n R_{n-1} dy_n R_{n-1} \leq R_{n-1} dx_n R_{n-1} dx_n R_{n-1},$$

it is immediate that

$$\begin{aligned} &(R_{n-1} - R_n) dy_n (R_{n-1} - R_n) dy_n (R_{n-1} - R_n) \\ &\leq (R_{n-1} - R_n) dy_n R_{n-1} dy_n (R_{n-1} - R_n) \\ &\leq (R_{n-1} - R_n) dx_n R_{n-1} dx_n (R_{n-1} - R_n) \\ &= (R_{n-1} - R_n)(x_n - x_{n-1}) R_{n-1} (x_n - x_{n-1})(R_{n-1} - R_n) \\ &= (R_{n-1} - R_n) x_n R_{n-1} x_n (R_{n-1} - R_n) + (R_{n-1} - R_n) x_{n-1} R_{n-1} x_{n-1} (R_{n-1} - R_n) \\ &\quad - (R_{n-1} - R_n) x_{n-1} R_{n-1} x_n (R_{n-1} - R_n) - (R_{n-1} - R_n) x_n R_{n-1} x_{n-1} (R_{n-1} - R_n). \end{aligned}$$

Since R_n commutes with $R_{n-1} x_n R_{n-1}$, the above sum is equal to

$$\begin{aligned} &(R_{n-1} - R_n) x_n (R_{n-1} - R_n) x_n (R_{n-1} - R_n) \\ &+ (R_{n-1} - R_n) x_{n-1} R_{n-1} x_{n-1} (R_{n-1} - R_n) \\ &- (R_{n-1} - R_n) x_{n-1} (R_{n-1} - R_n) x_n (R_{n-1} - R_n) \\ &- (R_{n-1} - R_n) x_n (R_{n-1} - R_n) x_{n-1} (R_{n-1} - R_n) \end{aligned}$$

(note that the second summand has not changed), which can be further transformed into

$$\begin{aligned} &(R_{n-1} - R_n) dx_n (R_{n-1} - R_n) dx_n (R_{n-1} - R_n) \\ &\qquad + (R_{n-1} - R_n) x_{n-1} R_n x_{n-1} (R_{n-1} - R_n). \end{aligned} \tag{4.7}$$

Let us handle the second term in the latter expression. Since $R_n \leq R_{n-1}$, we have

$$\begin{aligned} &(R_{n-1} - R_n) x_{n-1} R_n x_{n-1} (R_{n-1} - R_n) \\ &\qquad \leq (R_{n-1} - R_n) x_{n-1} R_{n-1} x_{n-1} (R_{n-1} - R_n). \end{aligned}$$

This is not bigger than $R_{n-1} - R_n$. Indeed, by Lemma 4.2 (iii), we have $R_{n-1} x_{n-1} R_{n-1} \leq R_{n-1}$, which yields $R_{n-1} x_{n-1} R_{n-1} x_{n-1} R_{n-1} \leq R_{n-1}$ and hence also the desired inequality. This enables us to bound the expression in (4.7) from above by a convenient square:

$$\Big(|(R_{n-1} - R_n) dx_n (R_{n-1} - R_n)| + R_{n-1} - R_n\Big)^2.$$

Putting all the above facts together, we conclude that

$$\begin{aligned}&(R_{n-1}-R_n)dy_n(R_{n-1}-R_n)dy_n(R_{n-1}-R_n)\\&\qquad\leq\Big(|(R_{n-1}-R_n)dx_n(R_{n-1}-R_n)|+R_{n-1}-R_n\Big)^2,\end{aligned}$$

which implies

$$|(R_{n-1}-R_n)dy_n(R_{n-1}-R_n)|\leq|(R_{n-1}-R_n)dx_n(R_{n-1}-R_n)|+R_{n-1}-R_n$$

and hence

$$\pm d\beta_n\leq\mathcal{E}_{n-1}\Big(|(R_{n-1}-R_n)dx_n(R_{n-1}-R_n)|+R_{n-1}-R_n\Big).$$

It follows from the properties of the conditional expectation, Lemma 4.2 (iv) and Lemma 4.4 that

$$||\beta_N||_1\leq\left|\left|\sum_{n=0}^N|(R_{n-1}-R_n)dx_n(R_{n-1}-R_n)|+I-R_N\right|\right|_1\leq3||x||_1. \tag{4.8}$$

This gives us the inequality

$$\tau\left(I_{[1,\infty)}(|\beta_N|)\right)\leq||\beta_N||_1\leq3||x||_1.$$

Step 3. The terms $d\gamma_n$ and $d\delta_n$. Here the analysis is much simpler. The right support of $d\gamma_n$ satisfies $r(d\gamma_n)\leq I-R_{n-1}\leq I-R_N$, so

$$\bigvee_{n=0}^N r(d\gamma_n)\leq I-R_N.$$

Therefore, by Lemma 4.2 (iv),

$$\tau\left(\bigvee_{n=0}^N r(d\gamma_n)\right)\leq||x||_1.$$

A similar analysis of the left support of $d\delta_n$ gives

$$\tau\left(\bigvee_{n=0}^N \ell(d\delta_n)\right)\leq||x||_1.$$

Consequently,

$$\tau\left(I_{[1,\infty)}(|\gamma_N|)\right)\leq\tau(r(\gamma_N))\leq\tau\left(\bigvee_{n=0}^N r(d\gamma_n)\right)\leq||x||_1$$

and analogously

$$\tau\left(I_{[1,\infty)}(|\delta_N|)\right)\leq\tau(\ell(\delta_N))\leq\tau\left(\bigvee_{n=0}^N \ell(d\delta_n)\right)\leq||x||_1.$$

Step 4. The final calculation. Combining the above estimates with (4.5) gives

$$\tau\left(I_{[4,\infty)}(|y_N|)\right)\leq9||x||_1.$$

This yields the desired claim. □

5. Strong-type (p,p) inequality

In this section, we deal with the strong-type (p,p) inequalities for weakly and very weakly differentially subordinate martingales. The following is our main result.

Theorem 5.1. *Suppose that x, y are self-adjoint martingales.*

(i) *If y is weakly differentially subordinate to x, then for any $1<p<2$ and any $N\geq 0$ we have*

$$\|y_N\|_p \leq c_p\|x_N\|_p,$$

where

$$c_p=\frac{4B^{p-1}}{B^{p-1}-1}\left(9B^p-3+\frac{4B^p(B^p-1)}{1-B^{p-2}}\right)^{1/p} \tag{5.1}$$

and $B>1$.

(ii) *If y is very weakly differentially subordinate to x, then for any $p\geq 2$ and any $N\geq 0$ we have*

$$\|y_N\|_p \leq c_p\|x_N\|_p,$$

where $c_2=1$ and, for $p>2$,

$$c_p=\frac{2^{1+1/p}p(1+2^{2-4/p})^{1/2}B^{(p+2)/2}}{(1-B^{2-p})^{1/2}}$$

and $B=1+1/p$.

Remark 5.2. (i) Note that c_p is of order $O((p-1)^{-1})$ as $p\to 1+$ (e.g., set $B=2$ in (5.1)) and $O(p)$ as $p\to\infty$. In the light of (1.1), this is already the best possible in the commutative setting.

(ii) As we already mentioned in the introductory section (see also Example 3.4), to show the strong-type (p,p) inequality for $p\geq 2$, the very weak differential subordination is sufficient. However, for $1<p<2$, we do need the stronger condition (3.1).

The proofs in the cases $1<p<2$ and $p\geq 2$ are quite different. For the sake of convenience and clarity, we have decided to split this section accordingly into two parts.

5.1. **The case $1<p<2$.** The proof of the L^p-estimate for this range of p will depend heavily on the weak-type estimate of the previous section and the objects introduced there. Our reasoning can be regarded as a variant of a real interpolation: see Remark 5.5 below.

First, we introduce several auxiliary objects. Let x, y be two self-adjoint martingales. For a fixed $\lambda>0$, let $(R_n^\lambda)_{n\geq 0}$ be the sequence of projections from the previous section, built on the martingale x/λ: that is, we have $R_{-1}^\lambda=I$ and, for any $n\geq 0$,

$$R_n^\lambda=R_{n-1}^\lambda I_{(-\lambda,\lambda)}(R_{n-1}^\lambda x_n R_{n-1}^\lambda).$$

Note that when $\lambda=1$, the family of projections $(R_n^1)_{n\geq 0}$ are just those we used in Section 4. In the sequel, we still use $(R_n)_{n\geq 0}$ to denote $(R_n^1)_{n\geq 0}$. Next, consider the following modification introduced by Randrianantoanina [43]. Namely, for a fixed $B>1$, $n\geq 0$ and $k\in\mathbb{Z}$, we set

$$P_n^{B^k}:=\bigwedge_{\ell\geq k}R_n^{B^\ell}. \tag{5.2}$$

The reason for the introduction of the family P is to ensure the monotonicity property with respect to both n and k. More precisely, note that for any fixed k, the projections $(R_n^{B^k})_{n\geq 0}$ are decreasing when n increases; however, there is no monotonicity if we fix n and change k. The new projections $(P_n^{B^k})_{n,k}$ have the monotonicity property "in both directions", i.e., we have $P_n^{B^\ell}\leq P_m^{B^k}$ if $n\geq m$ and $\ell\leq k$. Note that in the commutative case we have $P_n^{B^k}=R_n^{B^k}$, and thus we may regard $P_n^{B^k}$ as the "corrected" noncommutative version of the indicator function of the set $\{\max_{0\le m\le n}|x_m|<B^k\}$.

The next object, to be needed later, is a certain family of special operators. Namely, for $n\geq 0$, let

$$a_n := \sum_{k\in\mathbb{Z}} B^k(P_n^{B^{k+1}}-P_n^{B^k}). \tag{5.3}$$

These operators can be interpreted as a weak noncommutative version of the maximal function of x. Indeed, this follows immediately from the fact that in the commutative setting we have

$$a_n=\sum_{k\in\mathbb{Z}} B^k 1_{\{B^k\le \max_{0\le m\le n}|x_m|<B^{k+1}\}}.$$

The lemma below is devoted to the comparison of the L^p-norms of a and x.

Lemma 5.3. *Let $1<p<\infty$. Then $\|a_N\|_p\leq \frac{B^{p-1}}{B^{p-1}-1}\|x_N\|_p$ for any N.*

Proof. We may assume that $\|x_N\|_p<\infty$, since otherwise there is nothing to prove. It is convenient to split the reasoning into two parts.

Step 1. Inequalities for truncated versions of a_N. Consider the operators

$$a_{N,M}:=\sum_{k\le M}B^k(P_n^{B^{k+1}}-P_n^{B^k}),$$

where M is an arbitrary integer. Note that $a_{N,M}\le B^M$ and hence in particular $\|a_{N,M}\|_p<\infty$. Furthermore, it is easy to see that

$$\begin{aligned}
\tau\left(a_{N,M}^p\right) &= \tau\left(\sum_{k\le M}B^{kp}\left(P_N^{B^{k+1}}-P_N^{B^k}\right)\right)\\
&=\tau\left(\sum_{k\le M}\sum_{\ell\le k}\left(B^{\ell p}-B^{(\ell-1)p}\right)\left(P_N^{B^{k+1}}-P_N^{B^k}\right)\right)\\
&=\tau\left(\sum_{\ell\le M}\left(B^{\ell p}-B^{(\ell-1)p}\right)\sum_{k=\ell}^{M}\left(P_N^{B^{k+1}}-P_N^{B^k}\right)\right)\\
&=(1-B^{-p})\tau\left(\sum_{\ell\le M}B^{\ell p}\left(P_N^{B^{M+1}}-P_N^{B^\ell}\right)\right).
\end{aligned}\tag{5.4}$$

We have $\tau(I-P_N^{B^{M+1}})<\infty$ (see below), so $\tau\left(\sum_{\ell\le M}B^{\ell p}\left(I-P_N^{B^{M+1}}\right)\right)<\infty$ and we may write

$$\tau\left(a_{N,M}^p\right)=(1-B^{-p})\tau\left(\sum_{\ell\le M}B^{\ell p}\left(I-P_N^{B^\ell}\right)\right)-(1-B^{-p})\tau\left(\sum_{\ell\le M}B^{\ell p}\left(I-P_N^{B^{M+1}}\right)\right).$$

Denote the first term on the right by $(1-B^{-p})V_M$. Then $\tau\left(a_{N,M}^p\right)\le(1-B^{-p})V_M<\infty$ and

$$V_M=\tau\left(\sum_{\ell\le M}B^{\ell p}\left(I-R_N^{B^\ell}\right)\right)+\tau\left(\sum_{\ell\le M}B^{\ell p}\left(R_N^{B^\ell}-P_N^{B^\ell}\right)\right):=V_{M,1}+V_{M,2}.$$

From the fact that $R_N^{B^\ell} - P_N^{B^\ell}$ is a subprojection of $I - P_N^{B^{\ell+1}}$, it follows that

$$V_{M,2} \le \tau\left(\sum_{\ell\le M} B^{\ell p}\left(I - P_N^{B^{\ell+1}}\right)\right) = B^{-p}V_{M+1}.$$

Thus we have $V_M \le V_{M,1} + B^{-p}V_{M+1}$, or equivalently,

$$(1-B^{-p})V_M \le V_{M,1} + B^{-p}(V_{M+1}-V_M) = V_{M,1} + B^{Mp}\tau\left(I - P_N^{B^{M+1}}\right), \tag{5.5}$$

since V_M is finite. On the other hand, from the proof of Lemma 4.2 (iv), we know that

$$\tau\left(I - R_N^{B^\ell}\right) \le B^{-\ell}\tau\left(\left(I - R_N^{B^\ell}\right)|x_N|\right),$$

which gives

$$V_{M,1} \le \tau\left(\sum_{\ell\le M} B^{\ell(p-1)}\left(I - R_N^{B^\ell}\right)|x_N|\right) \le \tau\left(\sum_{\ell\le M} B^{\ell(p-1)}\left(I - P_N^{B^\ell}\right)|x_N|\right).$$

Arguing as in (5.4), we obtain that (all the relevant traces are easily shown to be finite)

$$\begin{aligned} V_{M,1} &\le \frac{1}{1-B^{-(p-1)}}\tau\left(\sum_{k\le M} B^{k(p-1)}\left(P_N^{B^{k+1}} - P_N^{B^k}\right)|x_N|\right) + \tau\left(\sum_{\ell\le M} B^{\ell(p-1)}(I - P_N^{B^{M+1}})|x_N|\right) \\ &= \frac{B^{p-1}}{B^{p-1}-1}\left(\tau\left(a_{N,M}^{p-1}|x_N|\right) + B^{M(p-1)}\tau\left((I - P_N^{B^{M+1}})|x_N|\right)\right). \end{aligned}$$

Now, by the Hölder inequality, we have $\tau\left(a_{N,M}^{p-1}|x_N|\right) \le \|a_{N,M}\|_p^{p-1}\|x_N\|_p$ and

$$\tau\left((I - P_N^{B^{M+1}})|x_N|\right) \le \tau\left(I - P_N^{B^{M+1}}\right)^{(p-1)/p}\|x_N\|_p,$$

which combined with the previous estimate yields

$$V_{M,1} \le \frac{B^{p-1}}{B^{p-1}-1}\left(\|a_{N,M}\|_p^{p-1} + \left(B^{Mp}\tau\left(I - P_N^{B^{M+1}}\right)\right)^{(p-1)/p}\right)\|x_N\|_p.$$

This bound, together with the inequality $\tau(a_{N,M}^p) \le (1-B^{-p})V_M$ and (5.5), implies

$$\begin{aligned} &\tau\left(a_{N,M}^p\right) \\ &\le \frac{B^{p-1}}{B^{p-1}-1}\left(\|a_{N,M}\|_p^{p-1} + \left(B^{Mp}\tau(I - P_N^{B^{M+1}})\right)^{(p-1)/p}\right)\|x_N\|_p + B^{Mp}\tau\left(I - P_N^{B^{M+1}}\right). \end{aligned} \tag{5.6}$$

Step 2. Handling the last two terms in (5.6). Observe that

$$\begin{aligned} \tau\left(I - P_N^{B^{M+1}}\right) \le \sum_{j\ge M+1}\tau\left(I - R_N^{B^j}\right) &\le \sum_{j\ge M+1} B^{-j}\tau\left((I - R_N^{B^j})|x_N|\right) \\ &\le \left(\sum_{j\ge M+1} B^{-j}\right)\tau\left((I - P_N^{B^{M+1}})|x_N|\right) \\ &\le B^{-M-1}(1-B^{-1})^{-1}\tau\left((I - P_N^{B^{M+1}})|x_N|\right), \end{aligned}$$

so in particular $\tau\left(I-P_N^{B^{M+1}}\right)\to 0$ as $M\to\infty$. Furthermore, by the Hölder inequality,

$$\tau\left(I-P_N^{B^{M+1}}\right)\leq B^{-M-1}(1-B^{-1})^{-1}\tau\left((I-P_N^{B^{M+1}}\right)^{(p-1)/p}\|(I-P_N^{B^{M+1}})|x_N|(I-P_N^{B^{M+1}})\|_p,$$

or equivalently

$$B^{Mp}\tau\left(I-P_N^{B^{M+1}}\right)\leq B^{-p}(1-B^{-1})^{-p}\|(I-P_N^{B^{M+1}})|x_N|(I-P_N^{B^{M+1}})\|_p.$$

But $\|x_N\|_p<\infty$ and $\tau\left(I-P_N^{B^{M+1}}\right)\to 0$ as $M\to\infty$, so the above estimate implies that $\lim_{M\to\infty}B^{Mp}\tau\left(I-P_N^{B^{M+1}}\right)=0$. Consequently, for any $\varepsilon\in(0,1)$ and M large enough we have $B^{Mp}\tau\left(I-P_N^{B^{M+1}}\right)\leq\varepsilon\tau(a^p_{N,M})$. Plugging this into (5.6) gives an inequality equivalent to

$$\|a_{N,M}\|_p\leq\frac{B^{p-1}(1+\varepsilon^{(p-1)/p})}{(B^{p-1}-1)(1-\varepsilon)}\|x_N\|_p.$$

We complete the proof by letting $M\to\infty$ and noting that $\varepsilon\in(0,1)$ was arbitrary. □

The proof of Theorem 5.1 is mainly based on an estimate on $\tau(|y_N|>4B^k)$ which is stated below as a lemma. Actually, this lemma is already implicit in Section 4, we include a proof for the reader's convenience.

Lemma 5.4. *For any $N\geq 0$ and $k\in\mathbb{Z}$, we have*

$$\tau\left(I_{[4B^k,\infty)}\left(|y_N|\right)\right)\leq 2B^{-2k}\tau\left(R_N^{B^k}x_NR_N^{B^k}x_NR_N^{B^k}\right)+9B^{-k}\tau\left((I-R_N^{B^k})|x_N|\right).\tag{5.7}$$

Proof. By homogeneity, it suffices to verify the lemma for $k=0$ (remember that, in this case, we use R_N to denote $R_N^{B^0}$). Set α, β, γ and δ as in the proof of Theorem 4.1. Then, combining (4.6) and (4.2), we deduce that

$$\begin{aligned}\tau\left(I_{[1,\infty)}(|\alpha_N|)\right)\leq\|\alpha_N\|_2^2&\leq 2\tau\left(\sum_{k=0}^N R_kdx_kR_{k-1}dx_kR_k\right)\\&\leq 2\tau(R_Nx_NR_Nx_NR_N)+4\tau\left(\sum_{k=0}^N(U_k+D_k)|x_N|\right)\\&=2\tau(R_Nx_NR_Nx_NR_N)+4\tau\left((I-R_N)|x_N|\right).\end{aligned}$$

For the term β, the estimate (4.8) together with Lemma 4.4 give that

$$\tau\left(I_{[1,\infty)}(|\beta_N|)\right)\leq\|\beta_N\|_1\leq 2\tau\left((I-R_N)|x_N|\right)+\tau\left(I-R_N\right)\leq 3\tau((I-R_N)|x_N|)$$

(the latter passage is due to Lemma 4.2 (iv)). As for the terms γ and δ, we have

$$\tau\left(I_{[1,\infty)}(|\gamma_N|)\right)\leq\tau(I-R_N)\leq\tau((I-R_N)|x_N|)$$

and, similarly, $\tau\left(I_{[1,\infty)}(|\delta_N|)\right)\leq\tau((I-R_N)|x_N|)$. Hence,

$$\begin{aligned}\tau\left(I_{[4,\infty)}(|y_N|)\right)&\leq\tau\left(I_{[1,\infty)}(|\alpha_N|)\right)+\tau\left(I_{[1,\infty)}(|\beta_N|)\right)+\tau\left(I_{[1,\infty)}(|\gamma_N|)\right)+\tau\left(I_{[1,\infty)}(|\delta_N|)\right)\\&\leq 2\tau(R_Nx_NR_Nx_NR_N)+9\tau\big((I-R_N)|x_N|\big).\end{aligned}$$

The proof is complete. □

We are now ready to show the L^p estimate for $1<p<2$. Before we do this, let us make an informal, but interesting observation.

Remark 5.5. The inequality (5.7) has a very nice interpretation in the language of interpolation theory. Clearly, the left-hand side, when multiplied by B^{kp} and summed over all $k \in \mathbb{Z}$, returns a p-th moment of y_N up to some multiplicative constant. The right-hand side can be regarded as a K-functional of the operator x_N, corresponding to the interpolating spaces L^1 and L^2. To make this more visible, take $k = 0$ and look at this expression in the commutative context. As we have explained above, the projection R_N corresponds to the indicator function of the set $\{\max_{0\leq n\leq N} |x_n| \leq 1\}$ and hence, roughly speaking, the operator

$$2R_N x_N R_N x_N R_N + 9(I - R_N)|x_N|$$

is equal to the quadratic term $2x_N^2$ when x_N is small and to the linear term $9|x_N|$ when x_N is large. This is precisely the intuition behind the K-functional; actually, the cut-off level (which decides whether x_N is small or large) refers to the maximal function $\max_{0\leq n\leq N} |x_n|$, so summing (5.7) over all $k \in \mathbb{Z}$ will additionally involve the weak maximal operator a_N in our considerations. However, this part will be handled by means of Lemma 5.3.

Let us proceed to the formal reasoning.

Proof of Theorem 5.1 (i). For any $N \geq 0$, we have

$$\begin{aligned} \|y_N\|_p^p &= \int_0^\infty pt^{p-1}\tau\left(I_{[t,\infty)}(|y_N|)\right)dt \\ &\leq \sum_k \int_{4B^k}^{4B^{k+1}} pt^{p-1}\tau\left(I_{[4B^k,\infty)}(|y_N|)\right)dt \qquad (5.8)\\ &= 4^p(B^p-1)\sum_k B^{kp}\tau\left(I_{[4B^k,\infty)}(|y_N|)\right) := 4^p(B^p-1)W. \end{aligned}$$

Multiplying both sides of the inequality (5.7) by B^{kp} and taking the sum over all $k \in \mathbb{Z}$, we obtain

$$W \leq 2W_1 + 9W_2,$$

where

$$W_1 = \sum_k B^{k(p-2)}\tau\left(R_N^{B^k} x_N R_N^{B^k} x_N R_N^{B^k}\right)$$

and

$$W_2 = \sum_k B^{k(p-1)}\tau\left((I - R_N^{B^k})|x_N|\right).$$

Reversing the arguments used in (5.4), we deduce that

$$\begin{aligned} W_2 &\leq \sum_k B^{k(p-1)}\tau\left((I - P_N^{B^k})|x_N|\right) \\ &= \frac{B^{p-1}}{B^{p-1}-1}\sum_k B^{k(p-1)}\tau\left((P_N^{B^{k+1}} - P_N^{B^k})|x_N|\right) \\ &= \frac{B^{p-1}}{B^{p-1}-1}\tau\left(a_N^{p-1}|x_N|\right). \end{aligned}$$

Using the Hölder inequality and Lemma 5.3, we arrive at

$$W_2 \leq \left(\frac{B^{p-1}}{B^{p-1}-1}\right)^p \|x_N\|_p^p.$$

Now we deal with W_1. Observe that

$$\tau\left(R_N^{B^k}x_NR_N^{B^k}x_NR_N^{B^k}\right)=\tau\left(P_N^{B^k}x_NP_N^{B^k}x_N\right)+2\tau\left((R_N^{B^k}-P_N^{B^k})x_NP_N^{B^k}x_N\right)$$
$$+\tau\left((R_N^{B^k}-P_N^{B^k})x_N(R_N^{B^k}-P_N^{B^k})x_N\right).$$

Since $P_N^{B^k}$ is a subprojection of $R_N^{B^k}$, it follows from Lemma 4.2 (iv) that

$$\begin{aligned}&\tau\left((R_N^{B^k}-P_N^{B^k})x_NP_N^{B^k}x_N\right)\\&=\tau\left((R_N^{B^k}-P_N^{B^k})R_N^{B^k}x_NR_N^{B^k}P_N^{B^k}R_N^{B^k}x_NR_N^{B^k}(R_N^{B^k}-P_N^{B^k})\right)\\&\le B^{2k}\tau\left(R_N^{B^k}-P_N^{B^k}\right).\end{aligned}$$

In a similar way, we may also show that

$$\tau\left((R_N^{B^k}-P_N^{B^k})x_N(R_N^{B^k}-P_N^{B^k})x_N\right)\le B^{2k}\tau\left(R_N^{B^k}-P_N^{B^k}\right).$$

Therefore,

$$\tau\left(R_N^{B^k}x_NR_N^{B^k}x_NR_N^{B^k}\right)\le\tau\left(P_N^{B^k}x_NP_N^{B^k}x_N\right)+3B^{2k}\tau\left(R_N^{B^k}-P_N^{B^k}\right).$$

Multiplying both sides of the above inequality by $B^{k(p-2)}$ and taking the sum over all $k\in\mathbb{Z}$, we obtain that

$$W_1\le\sum_k B^{k(p-2)}\tau\left(P_N^{B^k}x_NP_N^{B^k}x_N\right)+3\sum_k B^{kp}\tau\left(R_N^{B^k}-P_N^{B^k}\right):=W_{11}+3W_{12}.$$

For each $k\in\mathbb{Z}$, $R_N^{B^k}-P_N^{B^k}=R_N^{B^k}-R_N^{B^k}\wedge P_N^{B^{k+1}}$ is equivalent to a subprojection of $I-P_N^{B^{k+1}}$. Thus,

$$W_{12}\le\sum_k B^{kp}\tau\left(I-P_N^{B^{k+1}}\right)=B^{-p}\sum_k B^{kp}\tau\left(I-P_N^{B^k}\right).$$

By reversing the arguments in (5.4), we conclude that

$$W_{12}\le\frac{1}{B^p-1}\|a_N\|_p^p\le\frac{1}{B^p-1}\left(\frac{B^{p-1}}{B^{p-1}-1}\right)^p\|x_N\|_p^p.$$

As for W_{11}, note that

$$\tau\left(P_N^{B^k}x_NP_N^{B^k}x_N\right)=\sum_{r,s\le k}\tau\left((P_N^{B^r}-P_N^{B^{r-1}})x_N(P_N^{B^s}-P_N^{B^{s-1}})x_N\right).$$

By the tracial property, we conclude that

$$\begin{aligned}\tau\left(P_N^{B^k}x_NP_N^{B^k}x_N\right)&\le 2\sum_{s\le r\le k}\tau\left((P_N^{B^r}-P_N^{B^{r-1}})x_N(P_N^{B^s}-P_N^{B^{s-1}})x_N\right)\\&=2\sum_{r\le k}\tau\left((P_N^{B^r}-P_N^{B^{r-1}})x_NP_N^{B^r}x_N\right)\\&=2\sum_{r\le k}\tau\left((P_N^{B^r}-P_N^{B^{r-1}})P_N^{B^r}x_NP_N^{B^r}x_NP_N^{B^r}(P_N^{B^r}-P_N^{B^{r-1}})\right)\\&\le 2\sum_{r\le k}B^{2r}\tau\left(P_N^{B^r}-P_N^{B^{r-1}}\right),\end{aligned}$$

where the last inequality is due to the fact $||P_N^{B^r}x_NP_N^{B^r}||_\infty \leq B^r$. Thus we have

$$\begin{aligned}
W_{11} &\leq 2\sum_k \sum_{r\leq k} B^{k(p-2)+2r}\tau\left(P_N^{B^r} - P_N^{B^{r-1}}\right)\\
&= 2\sum_r B^{2r}\sum_{k\geq r} B^{k(p-2)}\tau\left(P_N^{B^r} - P_N^{B^{r-1}}\right)\\
&= \frac{2}{1-B^{p-2}}\sum B^{pr}\tau\left(P_N^{B^r} - P_N^{B^{r-1}}\right)\\
&= \frac{2B^p}{1-B^{p-2}}\tau\left(a_N^p\right)
\end{aligned}$$

and hence by Lemma 5.3 we get

$$W_{11} \leq \frac{2B^p}{1-B^{p-2}}\left(\frac{B^{p-1}}{B^{p-1}-1}\right)^p \|x_N\|_p^p.$$

Combining (5.8) with the above estimates of W_{11}, W_{12} and W_2, we conclude that

$$\|y_N\|_p^p \leq \left(\frac{4B^{p-1}}{B^{p-1}-1}\right)^p\left(9B^p - 3 + \frac{4B^p(B^p-1)}{1-B^{p-2}}\right)\|x_N\|_p^p.$$

The proof is complete. □

5.2. **The case $p > 2$.** For $p > 2$, the proof will be more involved technically. We will again use the projections R, D, U, but this time they will be built from the martingale y. To avoid confusion and the abuse of notation, we will write these letters in italics: $\mathcal{R}$, $\mathcal{D}$ and $\mathcal{U}$, respectively. More precisely, we define these projections by $\mathcal{R}_{-1} = I$ and, inductively,

$$\begin{aligned}
\mathcal{R}_n &= \mathcal{R}_{n-1}I_{(-1,1)}(\mathcal{R}_{n-1}y_n\mathcal{R}_{n-1}),\\
\mathcal{U}_n &= \mathcal{R}_{n-1}I_{(-\infty,-1]}(\mathcal{R}_{n-1}y_n\mathcal{R}_{n-1}),\\
\mathcal{D}_n &= \mathcal{R}_{n-1}I_{[1,\infty)}(\mathcal{R}_{n-1}y_n\mathcal{R}_{n-1}).
\end{aligned}$$

Then the assertions of Lemmas 4.2 and 5.3 are still valid, if x is replaced by y in appropriate places.

The proof of Theorem 5.1 (ii) rests on several ingredients. We start with a key lemma, which is of independent interest.

Lemma 5.6. *Let $2 < p < \infty$. Suppose that x, y are self-adjoint martingales such that y is very weakly subordinate to x. Then for any $N \geq 0$ we have*

$$\tau\left((I-\mathcal{R}_N)\left(y_N - \sum_{n=0}^N(\mathcal{D}_n - \mathcal{U}_n)\right)^2\right) \leq 2\tau\left((I-\mathcal{R}_N)(x_N^2+b)\right), \tag{5.9}$$

where $b = \left(\sum_{k=0}^N |dx_k|^p\right)^{2/p}$.

Proof. It is convenient to split the reasoning into a few parts.

Step 1. We will first prove that for any $n \geq 0$,

$$\begin{aligned}
&\tau\Big((\mathcal{R}_{n-1}-\mathcal{R}_n)dy_n^2(\mathcal{R}_{n-1}-\mathcal{R}_n)\Big)\\
&\geq \tau\Big(\mathcal{D}_n(y_n - I)\mathcal{R}_{n-1}(y_n - I) + \mathcal{U}_n(y_n+I)\mathcal{R}_{n-1}(y_n+I)\Big)\\
&= \tau\Big((\mathcal{R}_{n-1}-\mathcal{R}_n)(y_n - \mathcal{D}_n + \mathcal{U}_n)\mathcal{R}_{n-1}(y_n - \mathcal{D}_n + \mathcal{U}_n)\Big).
\end{aligned} \tag{5.10}$$

The second passage above is straightforward and follows from the tracial property of τ and the simple structural properties of $\mathcal{U}_n$, $\mathcal{D}_n$, $\mathcal{R}_{n-1}$ and $\mathcal{R}_n$ listed in Lemma 4.2. Thus, we only verify the inequality here. Note that both the operators $\mathcal{R}_{n-1}(-y_{n-1}-I)\mathcal{R}_{n-1}$ and $\mathcal{U}_n(y_n+I)\mathcal{U}_n$ are nonpositive. These imply that the operators

$$\mathcal{U}_n(-y_{n-1}-I)\mathcal{U}_n = \mathcal{U}_n\mathcal{R}_{n-1}(-y_{n-1}-I)\mathcal{R}_{n-1}\mathcal{U}_n$$

and

$$\mathcal{U}_n(2y_n+I-y_{n-1})\mathcal{U}_n = 2\mathcal{U}_n(y_n+I)\mathcal{U}_n + \mathcal{U}_n(-y_{n-1}-I)\mathcal{U}_n$$

are also nonpositive. Hence, we have

$$\tau\Big(\mathcal{U}_n(-y_{n-1}-I)\mathcal{U}_n(2y_n+I-y_{n-1})\mathcal{U}_n\Big) \geq 0.$$

Equivalently, this estimate can be rewritten in the form

$$\tau(\mathcal{U}_n dy_n \mathcal{U}_n dy_n \mathcal{U}_n) \geq \tau\Big(\mathcal{U}_n(y_n+I)\mathcal{U}_n(y_n+I)\mathcal{U}_n\Big).$$

Then, it follows from the commuting property of $\mathcal{U}_n$ (i.e., Lemma 4.2 (ii)) that

$$\tau(\mathcal{U}_n dy_n \mathcal{U}_n dy_n \mathcal{U}_n) \geq \tau\Big(\mathcal{U}_n(y_n+I)\mathcal{R}_{n-1}(y_n+I)\mathcal{U}_n\Big).$$

The same reasoning shows that

$$\tau(\mathcal{D}_n dy_n \mathcal{D}_n dy_n \mathcal{D}_n) \geq \tau\Big(\mathcal{D}_n(y_n-I)\mathcal{R}_{n-1}(y_n-I)\mathcal{D}_n\Big).$$

Combining the two last estimates with the observation that

$$\begin{aligned}\tau\Big((\mathcal{R}_{n-1}-\mathcal{R}_n)dy_n^2(\mathcal{R}_{n-1}-\mathcal{R}_n)\Big) &= \tau\big(\mathcal{U}_n dy_n^2 \mathcal{U}_n\big) + \tau(\mathcal{D}_n dy_n^2 \mathcal{D}_n)\\ &\geq \tau(\mathcal{U}_n dy_n \mathcal{U}_n dy_n \mathcal{U}_n) + \tau(\mathcal{D}_n dy_n \mathcal{D}_n dy_n \mathcal{D}_n),\end{aligned}$$

we obtain the inequality in (5.10).

Step 2. Now, we shall prove that for any $n \geq 0$,

$$\begin{aligned}&\tau\Big((\mathcal{R}_{n-1}-\mathcal{R}_n)(y_N-\mathcal{D}_n+\mathcal{U}_n)\mathcal{R}_{n-1}(y_N-\mathcal{D}_n+\mathcal{U}_n)\Big)\\ &\qquad\qquad \leq \tau\Big((\mathcal{R}_{n-1}-\mathcal{R}_n)(x_N^2+b)(\mathcal{R}_{n-1}-\mathcal{R}_n)\Big).\end{aligned} \tag{5.11}$$

By the very weak differential subordination, we have

$$(\mathcal{R}_{n-1}-\mathcal{R}_n)dy_n^2(\mathcal{R}_{n-1}-\mathcal{R}_n) \leq (\mathcal{R}_{n-1}-\mathcal{R}_n)dx_n^2(\mathcal{R}_{n-1}-\mathcal{R}_n),$$

and hence (5.10) implies

$$\begin{aligned}&\tau\Big((\mathcal{R}_{n-1}-\mathcal{R}_n)(y_n-\mathcal{D}_n+\mathcal{U}_n)\mathcal{R}_{n-1}(y_n-\mathcal{D}_n+\mathcal{U}_n)\Big)\\ &\qquad\qquad \leq \tau\Big((\mathcal{R}_n-\mathcal{R}_{n-1})dx_n^2(\mathcal{R}_{n-1}-\mathcal{R}_n)\Big).\end{aligned} \tag{5.12}$$

For $k > n$, again by the very weak differential subordination, we have

$$\tau\Big((\mathcal{R}_{n-1}-\mathcal{R}_n)dy_k^2(\mathcal{R}_{n-1}-\mathcal{R}_n)\Big) \leq \tau\Big((\mathcal{R}_{n-1}-\mathcal{R}_n)dx_k^2(\mathcal{R}_{n-1}-\mathcal{R}_n)\Big).$$

Then, by the martingale property of x and y, we obtain that

$$\begin{aligned}\tau\Big(\big(\mathcal{R}_{n-1}-\mathcal{R}_n)(y_N-y_n)^2(\mathcal{R}_{n-1}-\mathcal{R}_n)\Big) &\le \tau\Big((\mathcal{R}_{n-1}-\mathcal{R}_n)(x_N-x_n)^2(\mathcal{R}_{n-1}-\mathcal{R}_n)\Big)\\ &= \tau\Big((\mathcal{R}_{n-1}-\mathcal{R}_n)(x_N^2-x_n^2)(\mathcal{R}_{n-1}-\mathcal{R}_n)\Big)\\ &\le \tau\Big((\mathcal{R}_{n-1}-\mathcal{R}_n)x_N^2(\mathcal{R}_{n-1}-\mathcal{R}_n)\Big).\end{aligned}$$

Thus, we have

$$\begin{aligned}&\tau\Big((\mathcal{R}_{n-1}-\mathcal{R}_n)(y_N-y_n)\mathcal{R}_{n-1}(y_N-y_n)(\mathcal{R}_{n-1}-\mathcal{R}_n)\Big)\\ &\qquad\le \tau\Big((\mathcal{R}_{n-1}-\mathcal{R}_n)x_N^2(\mathcal{R}_{n-1}-\mathcal{R}_n)\Big).\end{aligned}$$

Adding this estimate to (5.12) and using the fact that $\mathcal{E}_n(y_N-y_n)=0$, we obtain

$$\begin{aligned}&\tau\Big((\mathcal{R}_{n-1}-\mathcal{R}_n)(y_N-\mathcal{D}_n+\mathcal{U}_n)\mathcal{R}_{n-1}(y_N-\mathcal{D}_n+\mathcal{U}_n)\Big)\\ &\qquad\le \tau\Big((\mathcal{R}_{n-1}-\mathcal{R}_n)(x_N^2+dx_n^2)(\mathcal{R}_{n-1}-\mathcal{R}_n)\Big).\end{aligned}\tag{5.13}$$

Since $p>2$ and $\sum_{n=0}^N|dx_n|^p\ge|dx_k|^p$ for each k between 0 and N, we conclude that

$$dx_k^2\le b \qquad \text{for any } k=0,\,1,\,2,\,\cdots,\,N.$$

Combining this with (5.13), we obtain the desired inequality (5.11).

Step 3. Our final step is to sum the estimates (5.11) over n. Observe that

$$\begin{aligned}\sum_{n=0}^N\tau\Big((\mathcal{R}_{n-1}-\mathcal{R}_n)y_N\mathcal{R}_{n-1}y_N\Big) &= \sum_{n=0}^N\sum_{k=n}^N\tau\Big((\mathcal{R}_{n-1}-\mathcal{R}_n)y_N(\mathcal{R}_{k-1}-\mathcal{R}_k)y_N\Big)\\ &\quad+\sum_{n=0}^N\tau\Big((\mathcal{R}_{n-1}-\mathcal{R}_n)y_N\mathcal{R}_Ny_N\Big)\\ &\ge \sum_{k=0}^N\sum_{n=0}^k\tau\Big((\mathcal{R}_{n-1}-\mathcal{R}_n)y_N(\mathcal{R}_{k-1}-\mathcal{R}_k)y_N\Big)\\ &= \sum_{k=0}^N\tau\Big((I-\mathcal{R}_k)y_N(\mathcal{R}_{k-1}-\mathcal{R}_k)y_N\Big).\end{aligned}$$

Consequently,

$$\begin{aligned}&2\sum_{n=0}^N\tau\Big((\mathcal{R}_{n-1}-\mathcal{R}_n)y_N\mathcal{R}_{n-1}y_N\Big)\\ &\ge \sum_{n=0}^N\tau\Big((\mathcal{R}_{n-1}-\mathcal{R}_n)y_N\mathcal{R}_{n-1}y_N\Big)+\sum_{k=0}^N\tau\Big((I-\mathcal{R}_k)y_N(\mathcal{R}_{k-1}-\mathcal{R}_k)y_N\Big)\\ &= \sum_{n=0}^N\tau\Big((I-\mathcal{R}_n+\mathcal{R}_{n-1})y_N(\mathcal{R}_{n-1}-\mathcal{R}_n)y_N\Big)\\ &= \tau\Big((I-\mathcal{R}_N)y_N^2\Big)+\sum_{n=0}^N\tau\Big((\mathcal{R}_{n-1}-\mathcal{R}_n)y_N(\mathcal{R}_{n-1}-\mathcal{R}_n)y_N\Big).\end{aligned}\tag{5.14}$$

Multiply both sides of (5.11) by 2 and take the sum over n. Then, combining the obtained inequality with (5.14), we get

$$\begin{aligned}\tau&\left((I-\mathcal{R}_N)\left(y_N-\sum_{n=0}^N(\mathcal{D}_n-\mathcal{U}_n)\right)^2\right)\\&\quad+\sum_{n=0}^N\tau\Big((\mathcal{R}_{n-1}-\mathcal{R}_n)(y_N-\mathcal{D}_n+\mathcal{U}_n)(\mathcal{R}_{n-1}-\mathcal{R}_n)(y_N-\mathcal{D}_n+\mathcal{U}_n)\Big)\\&\leq 2\tau\Big((I-\mathcal{R}_N)(x_N^2+b)\Big)\end{aligned}$$

which implies that

$$\tau\left((I-\mathcal{R}_N)\left(y_N-\sum_{n=0}^N(\mathcal{D}_n-\mathcal{U}_n)\right)^2\right)\leq 2\tau\Big((I-\mathcal{R}_N)(x_N^2+b)\Big).$$

This is precisely the claim. □

The inequality (5.9) gives an efficient bound for the distribution function of the dominated martingale. To state this bound, we need other auxiliary families of projections. For a fixed number $c>1$, consider the families $(S_n)_{n\geq -1}$, $(Q_n)_{n\geq 0}$ and $(T_n)_{n\geq 0}$ given by $S_{-1}=I$ and, inductively, for $n\geq 0$,

$$S_n=S_{n-1}I_{(-c,c)}(S_{n-1}y_nS_{n-1}),\quad Q_n=I_{[c,\infty)}(S_{n-1}y_nS_{n-1}),\quad T_n=S_{n-1}-S_n-Q_n.$$

Therefore, $(S_n)_{n\geq -1}$ coincides with the family $(\mathcal{R}_n^c)_{n\geq -1}$ of projections corresponding to the level c, and Q, T are the appropriate modifications of $\mathcal{D}$ and $\mathcal{U}$. However, for the notational convenience, we have decided to use different letters.

We will prove the following important estimate.

Theorem 5.7. *For any $c>1$ and any nonnegative integer N we have*

$$\tau(I-S_N)\leq 4(c-1)^{-2}\tau\Big((I-\mathcal{R}_N)(x_N^2+b)\Big). \tag{5.15}$$

Before we proceed to the proof, let us make a crucial observation.

Remark 5.8. The above result can be regarded as a variant of the so-called good-λ inequality, a powerful tool which is used widely in the commutative probability theory and harmonic analysis. This tool was introduced by Burkholder in [10]. Roughly speaking, the idea behind the approach (in the probabilistic context) is the following: in order to establish the estimate

$$\mathbb{E}(Y^p)\leq c_p\mathbb{E}(X^p) \tag{5.16}$$

for some p, some finite constant c_p and some nonnegative random variables, it suffices to establish the associated good-λ inequality

$$\mathbb{P}(Y\geq\beta\lambda,\ X\leq\delta\lambda)\leq\alpha_{\beta,\delta}\mathbb{P}(Y\geq\lambda), \tag{5.17}$$

for *some* β, δ, $\alpha_{\beta,\delta}>0$ and *all* $\lambda>0$. Clearly, this bound, if valid, implies

$$\mathbb{P}(Y\geq\beta\lambda)\leq\mathbb{P}(X>\delta\lambda)+\alpha_{\beta,\delta}\mathbb{P}(Y\geq\lambda),$$

which multiplied by λ^{p-1} and integrated over λ from 0 to ∞ yields (5.16), if only the parameters β, δ and $\alpha_{\beta,\delta}$ were chosen appropriately. This method has found many interesting and important applications, in particular it has turned out to yield best-order constants in several estimates; see e.g. [10, 18, 33]. The problem of extending good-λ inequalities to the noncommutative realm is a long-standing open problem.

Theorem 5.7 brings some new information in this direction. To explain this, let us mention here an important intrinsic feature of a general good-λ inequality (5.17) and related bounds (cf. [10]), which is key to its functionality. Namely, the left- and the right-hand side involve different level sets of the dominated random variable. This crucial phenomenon occurs also for estimate (5.15) in the classical setting: then the bound becomes

$$\mathbb{P}\Big(\max_{0\le n\le N}|y_n|\ge c\Big)\le 4(c-1)^{-2}\mathbb{E}\Big((x_N^2+b)1_{\{\max_{0\le n\le N}|y_n|\ge 1\}}\Big).$$

Thus we find reasonable to call (5.15) the *noncommutative good-λ inequality corresponding to the L^p-bounds for differentially subordinate martingales.*

Proof of Theorem 5.7. The reasoning is quite lengthy and rests on a number of separate parts.

Step 1. By the very definition of S, Q and T,

$$\tau(I-S_N)=\sum_{n=0}^N\tau(S_{n-1}-S_n)=\sum_{n=0}^N\big(\tau(Q_n)+\tau(T_n)\big).$$

Now fix $n\in\{0,1,2,\cdots,N\}$ and observe that

$$\begin{aligned}\tau(Q_n)&=\tau\left(I_{[c,\infty)}(Q_ny_nQ_n)\right)\\&=\tau\Big(I_{[c,\infty)}\Big(Q_n\big(\mathcal{R}_ny_n\mathcal{R}_n+(I-\mathcal{R}_n)y_n\mathcal{R}_n+y_n(I-\mathcal{R}_n)\big)Q_n\Big)\Big).\end{aligned}$$

By the properties of the projections $\mathcal{R}$, $\mathcal{D}$ and $\mathcal{U}$, the operator $d_n:=\mathcal{R}_ny_n\mathcal{R}_n+\sum_{k=0}^n(\mathcal{D}_k-\mathcal{U}_k)$ satisfies the double estimate $-I\le d_n\le I$. Furthermore, the projections $\mathcal{D}_0$, $\mathcal{U}_0$, $\mathcal{D}_1$, $\mathcal{U}_1$, $\cdots$, $\mathcal{D}_n$, $\mathcal{U}_n$ are mutually orthogonal and sum up to $I-\mathcal{R}_n$. Consequently, if we denote $z_n=y_n-\sum_{k=0}^n(\mathcal{D}_k-\mathcal{U}_k)$, then we may write

$$\begin{aligned}\tau(Q_n)&=\tau\Big(I_{[c,\infty)}\Big(Q_n\big(d_n+(I-\mathcal{R}_n)z_n\mathcal{R}_n+z_n(I-\mathcal{R}_n)\big)Q_n\Big)\Big)\\&\le\tau\Big(I_{[c-1,\infty)}\Big(Q_n\big((I-\mathcal{R}_n)z_n\mathcal{R}_n+z_n(I-\mathcal{R}_n)\big)Q_n\Big)\Big).\end{aligned}$$

Therefore, by Chebyshev's inequality,

$$\begin{aligned}(c-1)^2\tau(Q_n)&\le\tau\Big(\Big(Q_n\big((I-\mathcal{R}_n)z_n\mathcal{R}_n+z_n(I-\mathcal{R}_n)\big)Q_n\Big)^2\Big)\\&\le\tau\Big(Q_n\Big((I-\mathcal{R}_n)z_n\mathcal{R}_n+z_n(I-\mathcal{R}_n)\Big)^2Q_n\Big)\\&=\tau\Big(Q_nz_n(I-\mathcal{R}_n)z_nQ_n\Big)+\tau\Big(Q_n(I-\mathcal{R}_n)z_n\mathcal{R}_nz_n(I-\mathcal{R}_n)Q_n\Big)\\&=I_1+I_2.\end{aligned}\tag{5.18}$$

We will analyze the terms I_1 and I_2 separately below. Before we do that, let us record here that an analogous reasoning yields the estimate

$$(c-1)^2\tau(T_n)\le\tau\Big(T_nz_n(I-\mathcal{R}_n)z_nT_n\Big)+\tau\Big(T_n(I-\mathcal{R}_n)z_n\mathcal{R}_nz_n(I-\mathcal{R}_n)T_n\Big):=J_1+J_2.\tag{5.19}$$

(indeed, it suffices to change y into $-y$; then Q and T interchange their roles).

Step 2. Let us first deal with I_1. By the martingale property of y, we have

$$\begin{aligned}\tau\Big(Q_n y_N (I-\mathcal{R}_n) y_N Q_n\Big) &= \tau\Big(Q_n y_n (I-\mathcal{R}_n) y_n Q_n\Big) + \sum_{k=n+1}^{N} \tau\Big(Q_n dy_k (I-\mathcal{R}_n) dy_k Q_n\Big)\\ &\geq \tau\Big(Q_n y_n (I-\mathcal{R}_n) y_n Q_n\Big).\end{aligned}$$

Therefore,

$$\begin{aligned}I_1 &= \tau\bigg(Q_n\Big(y_n - \sum_{k=0}^{n}(\mathcal{D}_k-\mathcal{U}_k)\Big)(I-\mathcal{R}_n)\Big(y_n - \sum_{k=0}^{n}(\mathcal{D}_k-\mathcal{U}_k)\Big)Q_n\bigg)\\ &\leq \tau\bigg(Q_n\Big(y_N - \sum_{k=0}^{n}(\mathcal{D}_k-\mathcal{U}_k)\Big)(I-\mathcal{R}_n)\Big(y_N - \sum_{k=0}^{n}(\mathcal{D}_k-\mathcal{U}_k)\Big)Q_n\bigg).\end{aligned}$$

Note that, if $k>n$, then $\mathcal{D}_k$ and $\mathcal{U}_k$ are subprojections of $\mathcal{R}_n$ and hence are orthogonal to $I-\mathcal{R}_n$. Thus, we conclude that

$$\begin{aligned}I_1 &\leq \tau\bigg(Q_n\Big(y_N - \sum_{k=0}^{N}(\mathcal{D}_k-\mathcal{U}_k)\Big)(I-\mathcal{R}_n)\Big(y_N - \sum_{k=0}^{N}(\mathcal{D}_k-\mathcal{U}_k)\Big)Q_n\bigg)\\ &= \tau\big(Q_n z_N (I-\mathcal{R}_n) z_N Q_n\big) \leq \tau\big(Q_n z_N (I-\mathcal{R}_N) z_N Q_n\big).\end{aligned} \tag{5.20}$$

Step 3. We turn our attention to I_2, for which the calculations will be more involved. By the martingale property of y, we have, for any $k \geq n$,

$$\begin{aligned}&\tau\Big(Q_n(I-\mathcal{R}_k)y_k\mathcal{R}_k y_k(I-\mathcal{R}_k)Q_n\Big)\\ &\leq \tau\Big(Q_n(I-\mathcal{R}_k)y_k\mathcal{R}_k y_k(I-\mathcal{R}_k)Q_n\Big) + \tau\Big(Q_n(I-\mathcal{R}_k)dy_{k+1}\mathcal{R}_k dy_{k+1}(I-\mathcal{R}_k)Q_n\Big)\\ &= \tau\Big(Q_n(I-\mathcal{R}_k)y_{k+1}\mathcal{R}_k y_{k+1}(I-\mathcal{R}_k)Q_n\Big)\\ &= \tau\Big(Q_n(I-\mathcal{R}_k)y_{k+1}(\mathcal{R}_k-\mathcal{R}_{k+1})y_{k+1}(I-\mathcal{R}_k)Q_n\Big)\\ &\quad + \tau\Big(Q_n(I-\mathcal{R}_k)y_{k+1}\mathcal{R}_{k+1}y_{k+1}(I-\mathcal{R}_k)Q_n\Big).\end{aligned}$$

By the commuting property of the projection $\mathcal{R}$ described in Lemma 4.2 (ii), we have

$$(\mathcal{R}_k-\mathcal{R}_{k+1})y_{k+1}\mathcal{R}_{k+1} = \mathcal{R}_{k+1}y_{k+1}(\mathcal{R}_k-\mathcal{R}_{k+1}) = 0.$$

Combining this fact with the estimate above, we see that

$$\begin{aligned}&\tau\Big(Q_n(I-\mathcal{R}_k)y_k\mathcal{R}_k y_k(I-\mathcal{R}_k)Q_n\Big)\\ &\leq \tau\Big(Q_n(I-\mathcal{R}_k)y_{k+1}(\mathcal{R}_k-\mathcal{R}_{k+1})y_{k+1}(I-\mathcal{R}_k)Q_n\Big)\\ &\quad + \tau\Big(Q_n(I-\mathcal{R}_{k+1})y_{k+1}\mathcal{R}_{k+1}y_{k+1}(I-\mathcal{R}_{k+1})Q_n\Big)\end{aligned} \tag{5.21}$$

(i.e., in comparison to the last line of the preceding chain of inequalities, the projection $\mathcal{R}_k$ has been changed to $\mathcal{R}_{k+1}$).

Now, the projections $I-\mathcal{R}_n$ and $\mathcal{R}_n$ are orthogonal, and each $\mathcal{D}_j$, $\mathcal{U}_j$ is contained in one of them. Consequently,

$$I_2 = \tau\Big(Q_n(I-\mathcal{R}_n)z_n\mathcal{R}_n z_n(I-\mathcal{R}_n)Q_n\Big) = \tau\Big(Q_n(I-\mathcal{R}_n)y_n\mathcal{R}_n y_n(I-\mathcal{R}_n)Q_n\Big).$$

Hence, by the inductive use of (5.21), we get

$$I_2 \le \tau\Big(Q_n(I-\mathcal{R}_N)y_N\mathcal{R}_N y_N(I-\mathcal{R}_N)Q_n\Big) + \sum_{k=n}^{N-1}\tau\Big(Q_n(I-\mathcal{R}_k)y_{k+1}(\mathcal{R}_k-\mathcal{R}_{k+1})y_{k+1}(I-\mathcal{R}_k)Q_n\Big).$$

By the martingale property of y, we further get

$$I_2 \le \tau\Big(Q_n(I-\mathcal{R}_N)y_N\mathcal{R}_N y_N(I-\mathcal{R}_N)Q_n\Big) + \sum_{k=n}^{N-1}\tau\Big(Q_n(I-\mathcal{R}_k)y_N(\mathcal{R}_k-\mathcal{R}_{k+1})y_N(I-\mathcal{R}_k)Q_n\Big).$$

Finally, arguing as above, we may replace y_N by z_N. This follows at once from the fact that the pairs of projections $I-\mathcal{R}_N$; $\mathcal{R}_N$ and $I-\mathcal{R}_k$; $\mathcal{R}_k-\mathcal{R}_{k+1}$ are orthogonal and all $\mathcal{D}_k$, $\mathcal{U}_k$ are subprojections of one of the elements of each pair. Therefore we have proved that

$$\begin{aligned} I_2 &\le \tau\Big(Q_n(I-\mathcal{R}_N)z_N\mathcal{R}_N z_N(I-\mathcal{R}_N)Q_n\Big) \\ &\quad + \sum_{k=0}^{N-1}\tau\Big(Q_n(I-\mathcal{R}_k)z_N(\mathcal{R}_k-\mathcal{R}_{k+1})z_N(I-\mathcal{R}_k)Q_n\Big), \end{aligned} \tag{5.22}$$

which is the desired upper bound for I_2.

Step 4. Similar to the estimates we obtained in (5.20) and (5.22) for I_1 and I_2, we may show that

$$J_1 \le \tau\Big(T_n z_N(I-\mathcal{R}_N)z_N T_n\Big) \tag{5.23}$$

and

$$\begin{aligned} J_2 &\le \tau\Big(T_n(I-\mathcal{R}_N)z_N\mathcal{R}_N z_N(I-\mathcal{R}_N)T_n\Big) \\ &\quad + \sum_{k=0}^{N-1}\tau\Big(T_n(I-\mathcal{R}_k)z_N(\mathcal{R}_k-\mathcal{R}_{k+1})z_N(I-\mathcal{R}_k)T_n\Big). \end{aligned} \tag{5.24}$$

Step 5. Now we will plug the estimates (5.20), (5.22) into (5.18) and the estimates (5.23), (5.24) into (5.19), and then plus them and sum over n. For convenience, let us sum separately the pair of terms I_1, J_1 and the pair of terms I_2, J_2. First, note that by the tracial property,

$$\begin{aligned} \sum_{n=0}^{N}(I_1+J_1) &\le \sum_{n=0}^{N}\tau\big(Q_n z_N(I-\mathcal{R}_N)z_N Q_n\big) + \tau\big(T_n z_N(I-\mathcal{R}_N)z_N T_n\big) \\ &\le \tau\left(\left(\sum_{n=0}^{N}(Q_n+T_n)\right)z_N(I-\mathcal{R}_N)z_N\right) \\ &\le \tau\big(z_N(I-\mathcal{R}_N)z_N\big). \end{aligned} \tag{5.25}$$

Now we denote

$$M_1 := \sum_{n=0}^{N} \Big(\tau\big(Q_n(I-\mathcal{R}_N)z_N\mathcal{R}_N z_N(I-\mathcal{R}_N)Q_n\big) + \tau\big(T_n(I-\mathcal{R}_N)z_N\mathcal{R}_N z_N(I-\mathcal{R}_N)T_n\big)\Big);$$

$$\begin{aligned} M_2 := \sum_{n=0}^{N}\sum_{k=0}^{N-1} \bigg(\tau\Big(&Q_n(I-\mathcal{R}_k)z_N(\mathcal{R}_k-\mathcal{R}_{k+1})z_N(I-\mathcal{R}_k)Q_n\Big) \\ &+\tau\Big(T_n(I-\mathcal{R}_k)z_N(\mathcal{R}_k-\mathcal{R}_{k+1})z_N(I-\mathcal{R}_k)T_n\Big)\bigg). \end{aligned}$$

Using the tracial property again, we have

$$\begin{aligned} M_1 &= \tau\left(\left(\sum_{n=0}^{N}(Q_n+T_n)\right)(I-\mathcal{R}_N)z_N\mathcal{R}_N z_N(I-\mathcal{R}_N)\right) \\ &\le \tau\Big((I-\mathcal{R}_N)z_N\mathcal{R}_N z_N(I-\mathcal{R}_N)\Big). \end{aligned} \tag{5.26}$$

Similarly,

$$M_2 \le \sum_{k=0}^{N-1} \tau\big((I-\mathcal{R}_k)z_N(\mathcal{R}_k-\mathcal{R}_{k+1})z_N\big).$$

If we write $I-\mathcal{R}_k = \sum_{\ell=-1}^{k-1}(\mathcal{R}_\ell-\mathcal{R}_{\ell+1})$, plug this above and change the order of summation, we obtain that the right hand side of the above expression is equal to

$$\sum_{\ell=-1}^{N-2} \tau\big((\mathcal{R}_\ell-\mathcal{R}_{\ell+1})z_N(\mathcal{R}_{\ell+1}-\mathcal{R}_N)z_N\big).$$

Therefore, by the tracial property, we get

$$\begin{aligned} 2M_2 &\le 2\sum_{k=0}^{N-1}\tau\Big((I-\mathcal{R}_k)z_N(\mathcal{R}_k-\mathcal{R}_{k+1})z_N\Big) \\ &= \sum_{k=0}^{N-1}\tau\Big((I-\mathcal{R}_k)z_N(\mathcal{R}_k-\mathcal{R}_{k+1})z_N\Big) \\ &\qquad + \sum_{\ell=-1}^{N-2}\tau((\mathcal{R}_\ell-\mathcal{R}_{\ell+1})z_N(\mathcal{R}_{\ell+1}-\mathcal{R}_N)z_N) \\ &= \sum_{k=-1}^{N-1}\tau\Big((I-\mathcal{R}_k+\mathcal{R}_{k+1}-\mathcal{R}_N)z_N(\mathcal{R}_k-\mathcal{R}_{k+1})z_N\Big) \\ &\le \sum_{k=-1}^{N-1}\tau\Big((I-\mathcal{R}_N)z_N(\mathcal{R}_k-\mathcal{R}_{k+1})z_N\Big) \\ &= \tau\big((I-\mathcal{R}_N)z_N(I-\mathcal{R}_N)z_N\big). \end{aligned} \tag{5.27}$$

Combining the estimates (5.26) and (5.27), we conclude that

$$\sum_{n=0}^{N}(I_2+J_2)=M_1+M_2\leq\tau\big((I-\mathcal{R}_N)z_N\mathcal{R}_Nz_N(I-\mathcal{R}_N)\big)+\frac{1}{2}\tau\big((I-\mathcal{R}_N)z_N(I-\mathcal{R}_N)z_N\big).\tag{5.28}$$

Plugging (5.25) and (5.28) into (5.18) and (5.19), we obtain

$$\begin{aligned}&(c-1)^2\tau(I-S_N)\\&=(c-1)^2\sum_{n=0}^{N}\big(\tau(Q_n)+\tau(T_n)\big)=\sum_{n=0}^{N}(I_1+J_1+I_2+J_2)\\&\leq\tau\bigg(z_N(I-\mathcal{R}_N)z_N+(I-\mathcal{R}_N)z_N\mathcal{R}_Nz_N(I-\mathcal{R}_N)+\frac{1}{2}(I-\mathcal{R}_N)z_N(I-\mathcal{R}_N)z_N\bigg)\\&\leq2\tau(z_N(I-\mathcal{R}_N)z_N).\end{aligned}$$

It suffices to apply (5.9) to obtain the desired assertion. □

We now provide the proof of the strong-type estimate for the case $2\leq p<\infty$.

Proof of Theorem 5.1 (ii). In the case $p=2$ there is nothing to prove, so we assume that $p>2$. Take $B=c=1+1/p$ and, in analogy to the case $1<p<2$, let $\mathcal{R}_N^{B^k}$ be the projection of y corresponding to the level B^k. Since $B=c$, we have $S_n=\mathcal{R}_n^B$ and the inequality (5.15) becomes

$$\tau(I-\mathcal{R}_N^B)\leq4p^2\tau\big((I-\mathcal{R}_N)(x_N^2+b)\big),$$

(recall that $\mathcal{R}_N$ stands for $\mathcal{R}_N^1$). Therefore, by homogeneity, we get

$$B^{2k}\tau(I-\mathcal{R}_N^{B^{k+1}})\leq4p^2\tau\big((I-\mathcal{R}_N^{B^k})(x_N^2+b)\big)\leq4p^2\tau\big((I-P_N^{B^k})(x_N^2+b)\big),$$

where the projections P are given by (5.2). Multiply both sides by $B^{k(p-2)}$ and sum over all $k\in\mathbb{Z}$ to obtain

$$\sum_k B^{kp}\tau(I-\mathcal{R}_N^{B^{k+1}})\leq\frac{4p^2}{1-B^{2-p}}\tau(a_N^{p-2}(x_N^2+b)),\tag{5.29}$$

where the operator a is defined in (5.3). Let us now handle the right-hand side. First, note that the following estimate can be shown by interpolation (see also Junge and Xu [27]):

$$||b||_{p/2}^{1/2}=\left(\sum_{n=0}^{N}||dx_n||_p^p\right)^{1/p}\leq2^{1-2/p}||x_N||_p.\tag{5.30}$$

Applying the Hölder inequality, triangle inequality in $L_{p/2}$ and the estimate (5.30), we obtain

$$\begin{aligned}\tau\left(a_N^{p-2}(x_N^2+b)\right)&\leq||a_N||_p^{p-2}||x_N^2+b||_{p/2}\\&\leq||a_N||_p^{p-2}\Big(||x_N^2||_{p/2}+||b||_{p/2}\Big)\\&\leq(1+2^{2-4/p})||a_N||_p^{p-2}||x_N||_p^2.\end{aligned}$$

Furthermore, by the first estimate in (5.6) and the definition of V_1, we obtain

$$||a_N||_p^{p-2} \leq \left(\sum_k B^{kp}\tau(I-\mathcal{R}_N^{B^k})\right)^{1-2/p} = B^{p-2}\left(\sum_k B^{kp}\tau(I-\mathcal{R}_N^{B^{k+1}})\right)^{1-2/p}.$$

Plugging these two observations into (5.29) yields, after some simple manipulations,

$$\left(\sum_k B^{kp}\tau(I-\mathcal{R}_N^{B^{k+1}})\right)^{1/p} \leq \frac{2p(1+2^{2-4/p})^{1/2}B^{(p-2)/2}}{(1-B^{2-p})^{1/2}}||x_N||_p. \tag{5.31}$$

It remains to compare the left hand side to $||y_N||_p$. To this end, observe that $I_{[1,\infty)}(y_N)$ is equivalent to a subprojection of $I-\mathcal{R}_N$. Indeed, suppose that a nonzero vector ξ belongs to $I_{[1,\infty)}(y_N)(\mathcal{H})\cap\mathcal{R}_N(\mathcal{H})$. Then

$$||\xi||^2 \leq \langle y_N\xi,\xi\rangle = \langle y_N\mathcal{R}_N\xi,\mathcal{R}_N\xi\rangle = \langle \mathcal{R}_N y_N\mathcal{R}_N\xi,\xi\rangle < ||\xi||^2,$$

a contradiction (the last inequality is due to the very definition of $\mathcal{R}_N$). A similar argument shows that $I_{(-\infty,-1]}(y_N)$ is equivalent to a subprojection of $I-\mathcal{R}_N$. Consequently, we have $\tau\left(I_{[1,\infty)}(|y_N|)\right) \leq 2\tau(I-\mathcal{R}_N)$ and, by homogeneity,

$$\tau\left(I_{[B^k,\infty)}(|y_N|)\right) \leq 2\tau(I-\mathcal{R}_N^{B^k})$$

for each k. This implies

$$\begin{aligned}
2\sum_k B^{kp}\tau(I-\mathcal{R}_N^{B^{k+1}}) &\geq \sum_k B^{kp}\tau\left(I_{[B^{k+1},\infty)}(|y_N|)\right)\\
&= \sum_k\sum_{\ell>k} B^{kp}\tau\left(I_{[B^\ell,B^{\ell+1})}(|y_N|)\right)\\
&\geq \sum_\ell B^{(\ell-1)p}\tau\left(I_{[B^\ell,B^{\ell+1})}(|y_N|)\right)\\
&\geq B^{-2p}||y_N||_p^p.
\end{aligned}$$

Plugging this bound into (5.31) gives the claim. □

6. Application: Burkholder-Gundy inequality

In this section we will show how Theorem 5.1 can be used to obtain Burkholder-Gundy inequalities in the case $p \geq 2$. Consider an arbitrary L^p-bounded self-adjoint martingale $x = (x_n)_{n\geq 0}$ on the algebra $(\mathcal{M},\tau)$ with some filtration $(\mathcal{M}_n)_{n\geq 0}$. Fix a large positive integer N and consider the larger von Neumann algebra $\mathcal{N} = \mathbb{M}_{N+2}\overline{\otimes}\mathcal{M}$ equipped with the tensor product trace and the filtration $\mathcal{N}_n = \mathbb{M}_{N+2}\overline{\otimes}\mathcal{M}_n$, $n = 0, 1, 2, \cdots$. Finally, consider the martingales $\bar{y} = (\bar{y}_n)_{n\geq 0}$, $\bar{x} = (\bar{x}_n)_{n\geq 0}$ on the larger algebra with the difference sequences $d\bar{x} = (d\bar{x}_n)_{n\geq 0}$, $d\bar{y} = (d\bar{y}_n)_{n\geq 0}$ given by $d\bar{x}_n = (e_{1,1}+e_{n+2,n+2})\otimes dx_n$ and $d\bar{y}_n = (e_{1,n+2}+e_{n+2,1})\otimes dx_n$ for $n = 0, 1, 2, \cdots, N$; for remaining n, set $d\bar{x}_n = d\bar{y}_n = 0$. It is obvious that $d\bar{x}$ and $d\bar{y}$ are martingale differences and

$$d\bar{x}_n^2 = d\bar{y}_n^2 = (e_{1,1}+e_{n+2,n+2})\otimes dx_n^2$$

for all n. In other words, the martingales $\bar{x}$ and $\bar{y}$ are differentially subordinate to each other. Therefore, Theorem 5.1 implies

$$c_p^{-1}||\bar{x}_N||_p \leq ||\bar{y}_N||_p \leq c_p||\bar{x}_N||_p, \qquad p \geq 2. \tag{6.1}$$

It remains to relate the L^p-norms of $\bar{x}_N$ and $\bar{y}_N$ to the appropriate norms of x. First, note that

$$|\bar{x}_N| = e_{1,1} \otimes |x_N| + \sum_{n=0}^{N} e_{n+2,n+2} \otimes |dx_n|,$$

which implies that

$$||\bar{x}_N||_p = \left(||x_N||_p^p + \sum_{n=0}^{N} ||dx_k||_p^p \right)^{1/p}.$$

This expression is comparable to $||x_N||_p$: by the estimate (5.30),

$$||x_N||_p \leq ||\bar{x}_N||_p \leq (1 + 2^{1-2/p})||x_N||_p.$$

To analyze $\bar{y}_N$, we easily compute that

$$\bar{y}_N^2 = e_{1,1} \otimes S_N^2(x) + zz^*,$$

where $z = \sum_{n=0}^{N} e_{n+2,1} \otimes dx_n$. Consequently, we see that

$$||\bar{y}_N||_p = \left(\|S_N(x)\|_p^p + ||zz^*||_{p/2}^{p/2} \right)^{1/p}$$

is not smaller than $||S_N(x)||_p$ and not larger than

$$||S_N(x)||_p + ||zz^*||_{p/2}^{1/2} = ||S_N(x)||_p + ||z^*z||_{p/2}^{1/2} = 2||S_N(x)||_p.$$

Combining the observations above with (6.1), we conclude that

$$||x_N||_p \leq ||\bar{x}_N||_p \leq c_p\|\bar{y}_N\|_p \leq 2c_p||S_N(x)||_p$$

and

$$\|S_N(x)\|_p \leq \|\bar{y}_N\|_p \leq c_p\|\bar{x}_N\|_p \leq (1 + 2^{1-2/p})c_p||x_N||_p.$$

Thus we have proved the Burkholder-Gundy inequality in the range $p \geq 2$, with upper and lower constants of order $O(p)$ as $p \to \infty$ (which is optimal: see Junge and Xu [28]). The case $1 < p < 2$ can be deduced by duality (however, this time the obtained constant in the estimate

$$||x||_p \leq C_p||x||_{H^p(\mathcal{M})}$$

is of order $O((p-1)^{-1})$, which is not the best). It would be nice to have an analogous argument showing Burkholder-Gundy inequalities for $1 < p < 2$ directly from Theorem 5.1. However, we have been unable to find such a connection.

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References

[1] K. Astala, T. Iwaniec, and G. Martin, *Elliptic Partial Differential Equations and Quasiconformal Mappings in the Plane,* Princeton University Press, 2009.

[2] R. Bañuelos, A. Bielaszewski, and K. Bogdan, *Fourier multipliers for non-symmetric Lévy processes*, Marcinkiewicz centenary volume, Banach Center Publ. **95** (2011), 9–25, Polish Acad. Sci. Inst. Math., Warsaw.

[3] R. Bañuelos and A. Osękowski, *On Astala's theorem for martingales and Fourier multipliers*, Adv. in Math. **283** (2015), 275–302.

[4] R. Bañuelos and G. Wang, *Sharp inequalities for martingales with applications to the Beurling-Ahlfors and Riesz transformations*, Duke Math. J. **80** (1995), 575–600.

[5] T. Bekjan and Z. Chen, *Interpolation and Φ-moment inequalities of noncommutative martingales*, Probab. Theory Related Fields. **152** (2012), no. 1-2, 179–206.
[6] T. Bekjan, Z. Chen, P. Liu, and Y. Jiao, *Noncommutative weak Orlicz spaces and martingale inequalities*, Studia Math. **204** (2011), no. 3, 195–212.
[7] T. Bekjan, Z. Chen, and A. Osękowski, *Noncommutative maximal inequalities associated with convex functions*, Trans. Amer. Math. Soc. **369** (2017), no. 1, 409–427.
[8] T. Bekjan, Z. Chen, M. Perrin, and Z. Yin, *Atomic decomposition and interpolation for Hardy spaces of noncommutative martingales*, J. Funct. Anal. **258** (2010), no. 7, 2483–2505.
[9] A. Borichev, P. Janakiraman, and A. Volberg, *On Burkholder function for orthogonal martingales and zeros of Legendre polynomials*, Amer. J. Math. **135** (2013), 207–236.
[10] D. L. Burkholder, *Distribution function inequalities for martingales*, Ann. Probab. **1** (1973), 19–42.
[11] D. L. Burkholder, *Boundary value problems and sharp inequalities for martingale transforms*, Ann. Probab. **12** (1984), 647–702.
[12] J. M. Conde-Alonso and J. Parcet, *Atomic blocks for noncommutative martingales*, Indiana Univ. Math. J. **65** (2016), 1425–1443.
[13] I. Cuculescu, *Martingales on von Neumann algebras*, J. Multivariate Anal. **1** (1971), 17–27.
[14] S. Dirksen, *Noncommutative Boyd interpolation theorems*, Trans. Amer. Math. Soc. **367** (2015), no. 6, 4079–4110.
[15] S. Dirksen, *Weak-type interpolation for noncommutative maximal operators*, J. Operator Theory **73** (2015), no. 2, 515–532.
[16] S. Dirksen, B. de Pagter, D. Potapov, and F. Sukochev, *Rosenthal inequalities in noncommutative symmetric spaces*, J. Funct. Anal. **261** (2011), no. 10, 2890–2925.
[17] S. Dirksen and É. Ricard, *Some remarks on noncommutative Khintchine inequalities*, Bull. Lond. Math. Soc. **45** (2013), no. 3, 618–624. 46L52
[18] P. Hitczenko, *Best constants in martingale version of Rosenthal's inequality*, Ann. Probab. **18** (1990), no. 4, 1656–1668.
[19] G. Hong and T. Mei,*John-Nirenberg inequality and atomic decomposition for noncommutative martingales*, J. Funct. Anal. **263** (2012), no. 4, 1064–1097.
[20] T. Hytönen, J. van Neerven, M. Veraar, and L. Weis, Analysis in Banach spaces. Vol. I. Martingales and Littlewood-Paley theory [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics], 63. Springer, Cham, 2016.
[21] Y. Jiao, *Martingale inequalities in noncommutative symmetric spaces*, Arch. Math. (Basel) **98** (2012), no. 1, 87–97.
[22] Y. Jiao, *Burkholder's inequalities in noncommutative Lorentz spaces*, Proc. Amer. Math. Soc. **138** (2010), 2431–2441.
[23] Y. Jiao, F. Sukochev, G. Xie, and D. Zanin, *Φ-moment inequalities for independent and freely independent random variables.*, J. Funct. Anal. (270) 94 (2016), 4558–4596.
[24] Y. Jiao, F. Sukochev, and D. Zanin, *Johnson-Schechtman and Khintchine inequalities in noncommutative probability theory*, J. Lond. Math. Soc. (2) 94 (2016), 113–140.
[25] Y. Jiao, F. Sukochev, D. Zanin, and D. Zhou, *Johnson-Schechtman inequalities for noncommutative martingales*, J. Funct. Anal. **272** (2017), 976–1016.
[26] M. Junge, *Doob's inequality for non-commutative martingales*, J. Reine Angew. Math. **549** (2002), 149–190.
[27] M. Junge and Q. Xu, *Noncommutative Burkholder/Rosenthal inequalities*, Ann. Probab. **31** (2003), 948–995.
[28] M. Junge and Q. Xu, *On the best constants in some non-commutative martingale inequalities*, Bull. London Math. Soc. **37** (2005), 243–253.
[29] M. Junge and M. Musat, *A noncommutative version of the John-Nirenberg theorem*, Trans. Amer. Math. Soc. **359** (2007), no. 1, 115–142.
[30] M. Junge and Q. Xu, *Noncommutative Burkholder/Rosenthal inequalities. II. Applications*, Israel J. Math. **167** (2008), 227–282.
[31] K. V. Kadison and J. R. Ringrose, Fundamentals of the Theory of Operator Algebras. Vol. I, Elementary Theory, Academic Press, New York, 1983.
[32] R. V. Kadison and J. R. Ringrose, Fundamentals of the theory of operator algebras. Vol. II, Advanced Theory, Academic Press, Orlando, FL, 1986.
[33] B. Muckenhoupt and R. Wheeden, *Weighted norm inequalities for fractional integrals*, Trans. Amer. Math. Soc. **192** (1974), pp. 261-274.

[34] A. Osękowski, *Weak type inequality for noncommutative differentially subordinated martingales*, Probab. Theory Related Fields **140** (2008), 553 – 568.
[35] A. Osękowski, Sharp martingale and semimartingale inequalities, Monografie Matematyczne **72** (2012), Birkhäuser, 462 pp.
[36] A. Osękowski, *Logarithmic inequalities for Fourier multipliers*, Math. Z. **274** (2013), 515–530.
[37] A. Osękowski, *Weak-type inequalities for Fourier multipliers with applications to the Beurling-Ahlfors transform*, J. Math. Soc. Japan **66** (2014), 745–764.
[38] J. Parcet and N. Randrianantoanina, *Gundy's decomposition for non-commutative martingales and applications*, Proc. London Math. Soc. **93** (2006), 227–252.
[39] M. Perrin, *A noncommutative Davis' decomposition for martingales*, J. Lond. Math. Soc. (2) **80** (2009), no. 3, 627–648.
[40] G. Pisier, Martingales in Banach spaces, Cambridge University Press, 2016.
[41] G. Pisier and Q. Xu, *Non-commutative martingale inequalities*, Comm. Math. Physics **189** (1997), 667–698.
[42] G. Pisier and Q. Xu, *Non-commutative L^p-spaces*, in Hand book of the geometry of Banach spaces, Vol. 2, 1459–1517, North-Holland, Amsterdam, 2003.
[43] N. Randrianantoanina, *Non-commutative martingale transforms*, J. Funct. Anal. **194** (2002), 181–212.
[44] N. Randrianantoanina, *Square function inequalities for non-commutative martingales*, Israel J. Math. **140** (2004), 333–365.
[45] N. Randrianantoanina, *Conditioned square functions for noncommutative martingales*, Ann. Probab. **35** (2007), 1039–1070.
[46] N. Randrianantoanina and L. Wu, *Martingale inequalities in noncommutative symmetric spaces*, J. Funct. Anal. **269** (2015), 2222–2253.
[47] N. Randrianantoanina and L. Wu, *Noncommutative Burkholder/Rosenthal inequalities associated with convex functions*, Ann. Inst. Henri Poincaré Probab. Stat. **53** (2017), no. 4, 1575–1605.
[48] M. Takesaki, Theory of Operator Algebras. I, Springer-Verlag, New York, 1979.
[49] G. Wang, *Differential subordination and strong differential subordination for continuous-time martingales and related sharp inequalities*, Ann. Probab. 23 (1995), no. 2, 522–551.

School of Mathematics and Statistics, Central South University, Changsha 410085, People's Republic of China
E-mail address: jiaoyong@csu.edu.cn

Faculty of Mathematics, Informatics and Mechanics, University of Warsaw, Banacha 2, 02-097 Warsaw, Poland
E-mail address: ados@mimuw.edu.pl

School of Mathematics and Statistics, Central South University, Changsha 410085, People's Republic of China
E-mail address: wulian@csu.edu.cn