SHARP ESTIMATES FOR HOLOMORPHIC FUNCTIONS ON THE UNIT BALL OF $\mathbb{C}^n$

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Abstract. Let $\mathbb{S}^n$ denote the unit sphere in $\mathbb{C}^n$. The purpose the paper is to establish sharp $L^2 \log L$ estimates for $\mathcal{H}$ and P_+, the Hilbert transform and the Riesz projection on $\mathbb{S}^n$, respectively. The proof rests on the existence of a certain superhamonic function on $\mathbb{C}$ satisfying appropriate majorization conditions.

1. Introduction

Suppose that $f(\zeta) = \sum_{n\in\mathbb{Z}} \hat{f}(n)\zeta^n$ is a complex-valued integrable function on the unit circle $\mathbb{S}^1 = \{\zeta \in \mathbb{C} : |\zeta| = 1\}$. Here, as usual, $\hat{f}(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(e^{i\theta})e^{-in\theta}d\theta$ denotes the n-th Fourier coefficient of f. For $p \geq 1$, let $H^p(\mathbb{T},\mathbb{C})$ consist of all functions f satisfying $\hat{f}(n) = 0$ for $n < 0$. Then $H^p(\mathbb{T},\mathbb{C})$ is a closed subspace of $L^p(\mathbb{T},\mathbb{C})$ and can be identified with the space of analytic functions on the unit disc B^1. The Riesz projection (or analytic projection) $P_+ : L^p(\mathbb{S}^1,\mathbb{C}) \to H^p(\mathbb{S}^1,\mathbb{C})$, is the operator given by

$$P_+f(\zeta) = f_+(\zeta) = \sum_{n\geq 0} \hat{f}(n)\zeta^n, \qquad \zeta \in \mathbb{S}^1.$$

One defines the complementary co-analytic projection on $\mathbb{S}^1$ by $P_- = I - P_+$. These two projections are closely related to another classical operator, the Hilbert transform (conjugate function) on $\mathbb{S}^1$, which is given by

$$\mathcal{H}f(\zeta) = -i\sum_{n\in\mathbb{Z}} \mathrm{sgn}(n)\hat{f}(n)\zeta^n, \qquad \zeta \in \mathbb{S}^1.$$

Here $\mathrm{sgn}(n) = n/|n|$ for $n \neq 0$ and $\mathrm{sgn}(0) = 0$. A classical theorem of M. Riesz [16] asserts that the operator P_+ (equivalently, the Hilbert transform $\mathcal{H}$) is bounded on $L^p(\mathbb{S}^1,\mathbb{C})$ for $1 < p < \infty$. The question about the precise value of the norms of these operators has gathered some interest in the literature. For $p = 2^k$, $k = 1, 2, \ldots$, the exact values of the norms of $\mathcal{H}$ were determined by Gohberg and Krupnik [4], who showed that

$$||\mathcal{H}||_{L^p(\mathbb{S}^1,\mathbb{C})\to L^p(\mathbb{S}^1,\mathbb{C})} = \cot(\pi/(2p)).$$

For the remaining values of $1 < p < \infty$, the norms of the operator $\mathcal{H}$ acting on real L^p spaces were found by Pichorides [14] and, independently, by Cole (unpublished work, see Gamelin [3]):

$$||\mathcal{H}||_{L^p(\mathbb{S}^1,\mathbb{R})\to L^p(\mathbb{S}^1,\mathbb{R})} = \cot(\pi/(2p^*)),$$

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where $p^* = \max\{p, p/(p-1)\}$. Consult also Essén [2] and Verbitsky [20]. These norms do not change while passing to the complex L^p spaces (see e.g. Pełczyński [13]):

$$||\mathcal{H}||_{L^p(\mathbb{S}^1,\mathbb{C})\to L^p(\mathbb{S}^1,\mathbb{C})} = \cot(\pi/(2p^*)), \qquad 1 < p < \infty.$$

For the Riesz projection, Hollenbeck and Verbitsky [6] (see also [7]) proved that

$$||P_\pm||_{L^p(\mathbb{S}^1,\mathbb{C})\to L^p(\mathbb{S}^1,\mathbb{C})} = \csc(\pi/p), \qquad 1 < p < \infty,$$

solving a conjecture which was open for almost 40 years. For related weak-type bounds, consult e.g. the works of Davis [1], Janakiraman [8], Kolmogorov [9], Osękowski [11, 12] and Tomaszewski [18, 19]. For a treatment of the subject from a wider perspective, see Gohberg and Krupnik [5], Krupnik [10] and the classical monograph of Zygmund [21].

Our purpose is to study another class of sharp estimates of the above type. Actually, we will work in a slightly more general, n-dimensional setting. Let $\mathbb{C}^n$ be the n-dimensional complex space, equipped with the usual norm

$$|z| = \left(|z_1|^2 + |z_2|^2 + \ldots + |z_n|^2\right)^{1/2}.$$

Let B^n be the unit ball of $\mathbb{C}^n$ and let σ_n denote the rotation-invariant normalized Borel measure on the unit sphere $\mathbb{S}^n = \partial B^n$. For a σ_n-measurable function $f : \mathbb{S}^n \to \mathbb{C}$, we define the associated "$L^2 \log L$ norm" by the formula

$$||f||_{L^2 \log L} = \int_{\mathbb{S}^n} |f|^2 \log_+ |f| \mathrm{d}\sigma_n.$$

If a real-valued f satisfies $||f||_{L^2 \log L} < \infty$ and its Poisson integral $\mathcal{P}[f]$ (see e.g. Chapter 5 in [17]) is pluriharmonic, then we shall write $f \in H^2 \log H(\mathbb{S}^n; \mathbb{R})$. In such a case, there is a unique function $g : \mathbb{S}^n \to \mathbb{R}$ (up to set of σ_n-measure 0) satisfying $\int_{\mathbb{S}^n} g \mathrm{d}\sigma_n = 0$ and such that the Poisson extension of $f + ig$ is a holomorphic function on the unit ball. Such a g will be denoted by $\mathcal{H}f$ and called the Hilbert transform of f; furthermore, we can define the Riesz projection of f by $P_+ f = (f + i\mathcal{H}f + \int_{\mathbb{S}^n} f \mathrm{d}\sigma_n)/2$. One easily checks that for $n = 1$, these definitions of $\mathcal{H}$ and P_+ are consistent with those given at the beginning of the paper. Clearly, both $\mathcal{H}$ and P_+ can be extended to act on complex-valued functions $f \in H^2 \log H(\mathbb{S}^n; \mathbb{C})$ (i.e., those satisfying $||f||_{L^2 \log L} < \infty$ and whose Poisson extension is pluriharmonic), by setting $\mathcal{H}f = \mathcal{H}(\Re f) + i\mathcal{H}(\Im f)$ and $P_+ f = P_+(\Re f) + iP_+(\Im f)$.

Let us turn to the formulation of the main results of this paper. The first statement is a sharp bound for the Hilbert transform.

Theorem 1.1. *Assume that $f \in H^2 \log H(\mathbb{S}^n; \mathbb{C})$ satisfies $\int_{\mathbb{S}^n} f \, d\sigma_n = 0$. Then*

$$\int_{\mathbb{S}^n} |\mathcal{H}f(z)|^2 \log |\mathcal{H}f(z)| d\sigma_n(z) \leq c_1 \int_{\mathbb{S}^n} |f(z)|^2 \log \big(c_2 |f(z)|\big) d\sigma_n(z), \tag{1.1}$$

where $c_1 = 1$ and $c_2 = e^{\pi/2}$. Both c_1 and c_2 are the best possible, even if f is assumed to be real-valued.

The second theorem is a sharp $L^2 \log L$ estimate for Riesz projection. Quite interestingly, it holds true only for real-valued functions.

Theorem 1.2. *Assume that $f \in H^2 \log H(\mathbb{S}^n; \mathbb{R})$ satisfies $\int_{\mathbb{S}^n} f \, d\sigma_n = 0$. Then*

$$\int_{\mathbb{S}^n} |P_+ f(z)|^2 \log |P_+ f(z)| d\sigma_n(z) \leq c_3 \int_{\mathbb{S}^n} |f(z)|^2 \log \big(c_4 |f(z)|\big) d\sigma_n(z), \tag{1.2}$$

where $c_3 = 1/2$ and $c_4 = e^{\pi/4}/\sqrt{2}$. Both constants c_3 and c_4 are the best possible. The inequality (1.2) *does not hold with any constants c_3 and c_4 when f is allowed to take complex values.*

The change in the constants for P_+ when passing from the real to the complex case occurs also for the L^p bounds (see [6] and [20]), so the above phenomenon is perhaps not that surprising. What may be a little unexpected is that actually no finite constants work in the complex case. This is strictly related to the fact that the function $t \mapsto t^2 \log t$ is neither convex nor concave, and has negative and positive numbers in its range.

The proof will be based on the existence of a certain special superharmonic function satisfying appropriate majorization condition. Theorem 1.2, as well as the sharpness of (1.1), are shown in the next section. The final part of the paper is devoted to the proof of (1.1).

2. Proof of Theorem 1.2 and sharpness of (1.1)

Consider the function $U : \mathbb{R} \times [0, \infty) \to \mathbb{R}$, given by

$$U(x, y) = -\frac{\pi}{4} R^2 \cos(2\theta) - R^2 \log R \cos(2\theta) + R^2 \sin(2\theta) \left(\theta - \frac{\pi}{2}\right).$$

Here $x = R\cos\theta$, $y = R\sin\theta$ denote the usual polar coordinates (with $R \geq 0$ and $\theta \in [0, \pi]$). We extend the function to the whole plane $\mathbb{R}^2$ by setting $U(x, y) = U(x, -y)$. Note that U is also symmetric with respect to the y-axis: $U(x, y) = U(-x, y)$ for all x, y.

The key properties of the function U are gathered in a lemma below. Recall the constants c_3 and c_4, introduced in the statement of Theorem 1.2 above.

Lemma 2.1. *(i) We have the majorization*

$$U(x, y) \geq (x^2 + y^2) \log(x^2 + y^2)^{1/2} - 4c_3 x^2 \log(2c_4|x|), \qquad x,\, y \in \mathbb{R}. \tag{2.1}$$

(ii) The function U is superharmonic.

Proof. (i) It is enough to establish the majorization in the first quadrant, i.e. for nonnegative x and y. Then it takes the equivalent form

$$H(\theta) = -\frac{\pi}{4} \cos(2\theta) + \sin(2\theta) \left(\theta - \frac{\pi}{2}\right) + 2\cos^2\theta \log(\sqrt{2} e^{\pi/4} \cos\theta), \quad \theta \in [0, \pi/2],$$

and one computes that

$$H'(\theta) = 2\cos(2\theta) \left(\theta - \frac{\pi}{2}\right) - 2\sin(2\theta) \log(\sqrt{2}\cos\theta).$$

Now, if $\theta < \pi/4$, then $\cos(2\theta)\left(\theta - \frac{\pi}{2}\right) < 0$ and $\sin(2\theta)\log(\sqrt{2}\cos\theta) > 0$, so H' is negative. Similarly, if $\theta > \pi/4$, then $\cos(2\theta)\left(\theta - \frac{\pi}{2}\right) > 0$ and $\sin(2\theta)\log(\sqrt{2}\cos\theta) < 0$, which implies that H' is positive. Consequently, H has a global minimum at $\theta = \pi/4$; since $H(\pi/4) = 0$, the majorization (2.1) holds true.

(ii) First note that U is harmonic in the upper halfplane. This follows at once from the identity

$$U(x, y) = \Re\left(-z^2 \log(-iz) - \pi z^2/4\right),$$

where $z = x + iy$ and log is the principal branch of the complex logarithm. Therefore, U is also harmonic in the halplane $\{(x, y) : y < 0\}$. Consequently, the superharmonicity will follow if we show that $\lim_{y \downarrow 0} U_y(x, y) \leq 0$ for all $x \in \mathbb{R}$. Actually, by symmetry, it is enough to consider positive x only. We have $\lim_{y\downarrow 0} R_y = 0$ and $\lim_{y\downarrow 0}\theta_y = 1/x$, so $\lim_{y\downarrow 0} U_y(x,y) = -\pi x < 0$. This completes the proof. $\square$

Proof of Theorem 1.2. Let us start with the case $n = 1$. Pick any function f as in the statement. Since $\int_{\mathbb{S}^1} f \mathrm{d}\sigma_1 = 0$, we have $P_+ f = (f + i\mathcal{H}f)/2$ and hence $|2P_+f|^2 = |f|^2 + |\mathcal{H}f|^2$. Apply the majorization (2.1) to obtain

$$\int_{\mathbb{S}^1}\left[|2P_+f(z)|^2\log|2P_+f(z)| - 4c_3|f(z)|^2\log(2c_4|f(z)|)\right]\mathrm{d}\sigma_1(z)$$
$$\leq \int_{\mathbb{S}^1} U(f(z),\mathcal{H}f(z))\mathrm{d}\sigma_1(z).$$

However, the Poisson extension $\mathcal{P}[f + i\mathcal{H}f]$ is analytic on the unit disc and U is superharmonic on the plane. Consequently, the composition $U(\mathcal{P}[f+i\mathcal{H}f])$ is superharmonic on B^1 and hence the latter integral does not exceed $U(f(0),\mathcal{H}f(0)) = U(0,0) = 0$. Thus we have obtained

$$\int_{\mathbb{S}^1}|2P_+f(z)|^2\log|2P_+f(z)|\mathrm{d}\sigma_1(z) \leq 4c_3\int_{\mathbb{S}^1}|f(z)|^2\log(2c_4|f(z)|)\mathrm{d}\sigma_1(z).$$

It remains to apply this estimate to the function $f/2$ to get (1.1) for $n = 1$. When $n \geq 2$, we note that

$$\int_{\mathbb{S}^n}\left[|P_+f(z)|^2\log|P_+f(z)| - c_3|f(z)|^2\log(c_4|f(z)|)\right]\mathrm{d}\sigma_1(z)$$
$$= \int_{\mathbb{S}^n}\int_{\mathbb{S}^1}\left[|P_+f(wz)|^2\log|P_+f(wz)| - c_3|f(wz)|^2\log(c_4|f(wz)|)\right]\mathrm{d}\sigma_1(z)\mathrm{d}\sigma_n(w).$$

However, for any $w \in \mathbb{S}^n$ we see that the inner integral is nonpositive. This follows at once from the one-dimensional case we have just established, applied to the function $z \mapsto f(wz)$. This implies that the integral over $\mathbb{S}^n$ is also nonpositive, and this completes the proof of (1.2). $\square$

Sharpness of (1.1) *and* (1.2). We will show that the estimates are sharp in the case $n = 1$. To do this, we will use a certain modification of an example exploited by Pichorides in [14]. We will start with the estimate (1.2). Pick $\kappa < 2c_4$, let $\gamma \in (0, \pi/4)$ be a given parameter and consider the holomorphic function $F(z) = i\left(\frac{1+z}{1-z}\right)^{2\gamma/\pi} - i$ defined on the unit disc B^1. Since $z \mapsto \frac{1+z}{1-z}$ maps B^1 onto the left halfplane $\{z \in \mathbb{C} : \Re z \geq 0\}$, we see that the function $F + i$ maps B^1 onto the angle $\{z \in \mathbb{C} : |\Re z| \leq \tan\gamma \cdot \Im z\}$. In particular, we see that the unit circle is mapped onto the boundary of the angle, and hence

$$|F(e^{i\theta}) + i| = \frac{|\Re F(e^{i\theta})|}{\sin\gamma} = \frac{|\Im F(e^{i\theta}) + 1|}{\cos\gamma} \qquad \text{for } \theta \in [-\pi,\pi]. \tag{2.2}$$

Let us take a look at the difference

$$\int_{\mathbb{S}^1}|F(z)|^2\log|F(z)|\mathrm{d}\sigma_1(z) - 2\int_{\mathbb{S}^1}|\Re F(z)|^2\log\left(\kappa|\Re F(z)|\right)\mathrm{d}\sigma_1(z) = I + II,$$

where

$$I=\int_{\mathbb{S}^1}\Big[|F(z)|^2\log|F(z)|-|F(z)+i|^2\log|F(z)+i|\Big]\mathrm{d}\sigma_1(z),$$
$$II=\int_{\mathbb{S}^1}\Big[|F(z)+i|^2\log|F(z)+i|-2|\Re F(z)|^2\log\big(\kappa|\Re F(z)|\big)\Big]\mathrm{d}\sigma_1(z).$$

We will show that I is bounded and II tends to infinity as $\gamma\to\pi/4$; this will clearly imply that no constants $c<c_3$, $\kappa<c_4$ work in (1.2), since $F=P_+(\Re(2F))$. Let us start with the term I. Note that

$$I=\int_{\mathbb{S}^1}\left[|F(z)|^2\log\frac{|F(z)|}{|F(z)+i|}\right]\mathrm{d}\sigma_1(z)+\int_{\mathbb{S}^1}\big(2\Im F(z)-1\big)\log|F(z)+i|\mathrm{d}\sigma_1(z).$$

Both integrals are bounded as $\gamma\to\pi/4$: the identity $\frac{1+e^{i\theta}}{1-e^{i\theta}}=i\cot(\theta/2)$ implies $|F(z)|\leq 1+|\cot(\theta/2)|^{2\gamma/\pi}$. Thus, substituting $z=e^{i\theta}$, we see that we must guarantee that both integrands do not explode too fast in the neighborhood of $\theta=0$. However, we have

$$|F(z)|^2\log\frac{|F(z)|}{|F(z)+i|}=O(\cot(\theta/2)^{4\gamma/\pi-1})$$

and $\big(2\Im F(z)-1\big)\log|F(z)+i|=O\big(\cot(\theta/2)^{2\gamma/\pi-1}|\log|\cot(\theta/2)|\big)$ as $\theta\to 0$. Hence $\limsup_{\gamma\to\pi/4}|I|<\infty$. We turn our attention to II. By (2.2), we have

$$\begin{aligned}
II&=(1-2\sin^2\gamma)\int_{\mathbb{S}^1}|F(z)+i|^2\log|F(z)+i|\mathrm{d}\sigma_1(z)\\
&\quad-2\sin^2\gamma\log(\kappa\sin\gamma)\int_{\mathbb{S}^1}|F(z)+i|^2\mathrm{d}\sigma_1(z)\\
&=\frac{1-2\sin^2\gamma}{2\pi}\int_{-\pi}^{\pi}|\cot(\theta/2)|^{4\gamma/\pi}\log|\cot(\theta/2)|^{2\gamma/\pi}\mathrm{d}\theta\\
&\quad-\frac{\sin^2\gamma}{\pi}\int_{-\pi}^{\pi}|\cot(\theta/2)|^{4\gamma/\pi}\log\big(\kappa\sin\gamma\big)\mathrm{d}\theta\\
&=\frac{1-2\sin^2\gamma}{2\pi}\frac{8\gamma}{\pi}\int_0^\infty\frac{r^{4\gamma/\pi}\log r\mathrm{d}r}{1+r^2}-\frac{4\sin^2\gamma}{\pi}\log\big(\kappa\sin\gamma\big)\int_0^\infty\frac{r^{4\gamma/\pi}\mathrm{d}r}{1+r^2},
\end{aligned}\tag{2.3}$$

where the latter equality follows from the substitution $r=\cot(\theta/2)$. Now pick an arbitrary $\eta\in(0,1)$; then we have $(1+r^2)^{-1}\geq\eta r^{-2}$ for sufficiently large r; say, $r>N=N(\eta)$. Thus,

$$\begin{aligned}
II&\geq\frac{1-2\sin^2\gamma}{2\pi}\frac{8\gamma}{\pi}\cdot\eta\int_N^\infty r^{4\gamma/\pi-2}\log r\mathrm{d}r\\
&\quad-\frac{4\sin^2\gamma}{\pi}\log\big(\kappa\sin\gamma\big)\left[1+\int_1^\infty r^{4\gamma/\pi-2}\mathrm{d}r\right]\\
&=\frac{1-2\sin^2\gamma}{2\pi}\frac{8\gamma}{\pi}\cdot\eta\left[\frac{N^{4\gamma/\pi-1}}{1-4\gamma/\pi}\log N+\frac{N^{4\gamma/\pi-1}}{(1-4\gamma/\pi)^2}\right]\\
&\quad-\frac{4\sin^2\gamma}{\pi}\log\big(\kappa\sin\gamma\big)\left[1+\frac{1}{1-4\gamma/\pi}\right].
\end{aligned}$$

Now, if we let $\gamma \to \pi/4$, then both terms

$$\frac{1-2\sin^2\gamma}{2\pi}\frac{8\gamma}{\pi}\cdot\eta\frac{N^{4\gamma/\pi-1}}{1-4\gamma/\pi}\log N \quad \text{and} \quad \frac{4\sin^2\gamma}{\pi}\log\big(\kappa\sin\gamma\big)$$

converge to certain finite constants; on the other hand, for the remaining two terms, we can write

$$\begin{aligned}&\frac{1-2\sin^2\gamma}{2\pi}\frac{8\gamma}{\pi}\cdot\eta\frac{N^{4\gamma/\pi-1}}{(1-4\gamma/\pi)^2}-\frac{4\sin^2\gamma}{\pi}\frac{\log\big(\kappa\sin\gamma\big)}{1-4\gamma/\pi}\\&=2\left(1-\frac{4\gamma}{\pi}\right)^{-1}\left[\frac{1-2\sin^2\gamma}{\pi-4\gamma}\frac{2\gamma}{\pi}\cdot\eta N^{4\gamma/\pi-1}-\frac{2\sin^2\gamma}{\pi}\log\big(\kappa\sin\gamma\big)\right].\end{aligned}$$

Now if we let $\gamma \to \pi/4$, the expression in the square brackets converges to $\eta/4 - \log(\kappa/\sqrt{2})/\pi$. However, we have assumed at the beginning that $\kappa < 2c_4 = \sqrt{2}e^{\pi/4}$. Therefore, if η is a priori taken sufficiently close to 1, the expression in the square brackets converges to a positive constant and, consequently, $II \to \infty$. This shows that the constants c_3, c_4 are indeed the best possible in (1.2).

The reasoning for the estimate (1.1) is similar, actually it exploits the same function F as above. We pick $\kappa_0 < c_2$ and write

$$\int_{\mathbb{S}^1}\Big[|\Im F(z)|^2\log|\Im F(z)| - c_1|\Re F(z)|^2\log\left(\kappa_0|\Re F(z)|\right)\Big]\mathrm{d}\sigma_1(z) = I + II,$$

where

$$\begin{aligned}I &= \int_{\mathbb{S}^1}\Big[|\Im F(z)|^2\log|\Im F(z)| - |\Im F(z)+i|^2\log|\Im F(z)+i|\Big]\mathrm{d}\sigma_1(z),\\ II &= \int_{\mathbb{S}^1}\Big[|\Im F(z)+1|^2\log|\Im F(z)+1| - c_1|\Re F(z)|^2\log\left(\kappa_0|\Re F(z)|\right)\Big]\mathrm{d}\sigma_1(z).\end{aligned}$$

Now, as above, the term I remains bounded when we let $\gamma \to \pi/4$. To handle II, note that (2.2) implies

$$\begin{aligned}II &= (1-2\sin^2\gamma)\int_{\mathbb{S}^1}|F(z)+i|^2\log|F(z)+i|\mathrm{d}\sigma_1(z)\\&\quad+(\cos^2\log\cos\gamma-\sin^2\gamma\log(\kappa_0\sin\gamma))\int_{\mathbb{S}^1}|F(z)+i|^2\mathrm{d}\sigma_1(z)\\&\geq(1-2\sin^2\gamma)\int_{\mathbb{S}^1}|F(z)+i|^2\log|F(z)+i|\mathrm{d}\sigma_1(z)\\&\quad-\sin^2\gamma\log\kappa_0\int_{\mathbb{S}^1}|F(z)+i|^2\mathrm{d}\sigma_1(z),\end{aligned}$$

since $\cos^2\gamma\log\cos\gamma \geq \sin^2\gamma\log\cos\gamma$ for $\gamma \in [0,\pi/4]$. Now, compare this to the first expression on the right of (2.3). We have $\kappa_0 < c_2 = e^{\pi/2}$ and hence there is $\kappa < 2c_4 = \sqrt{2}e^{\pi/4}$ such that for γ sufficiently close to $\pi/4$,

$$\sin^2\gamma\log\kappa_0 < 2\sin^2\gamma\log(\kappa\sin\gamma).$$

But this implies that $II \to \infty$ as $\gamma \to \pi/4$, as we have shown above. This proves the optimality of the constants c_1 and c_2, since $\Im F = \mathcal{H}(\Re F)$. □

We conclude this section by indicating an example which shows that (1.2) does not hold for complex-valued functions, no matter what c_3 and c_4 are. To see this, fix these two constants and consider the function $f(z) = c_4^{-1}\zeta^{-1}/2$, $\zeta \in \mathbb{S}^1$, where a is a fixed constant. Then, from the very definition of P_+, we have $P_+f = 0$, so the left-hand side of (1.2) vanishes; on the other hand, we have $|f(\zeta)|\log(c_4|f(\zeta)|) < 0$ for all $\zeta \in \mathbb{S}^1$. Hence the right-hand side is negative and the inequality (1.2) fails to hold.

3. Proof of (1.1)

The reasoning will again be based on the properties of a certain special superharmonic function on the plane. Consider the holomorphic $G : \mathbb{C} \setminus [0,\infty) \to \mathbb{C}$, given by

$$G(z) = -z\log\left(-iz^{1/2}\right) - \pi z/4.$$

Here log denotes the principal branch of the complex logarithm and the root is given by $z^{1/2} = R^{1/2}e^{i\theta/2}$ for any $z = Re^{i\theta}$ with $\theta \in [0, 2\pi)$. Then $\Phi = \Re G$ is a harmonic function on $\mathbb{C}\setminus[0,\infty)$. We easily check that $\lim_{y\downarrow 0}\Phi(x,y) = \lim_{y\uparrow 0}\Phi(x,y)$ and hence Φ extends to a continuous function on $\mathbb{C}$. Actually, this extension (still denoted by Φ) is superharmonic, which follows from the estimate $\lim_{y\downarrow 0}\Phi_y(x,y) = -\lim_{y\uparrow 0}\Phi_y(x,y) < 0$ for any $x > 0$. Compare this to the analogous reasoning from the proof of Lemma 2.1.

The next step in our analysis is to establish the following majorization.

Theorem 3.1. *For any w, $z \in \mathbb{C}$ we have the estimate*

$$\Phi(wz) \geq \left|\frac{w-\bar{z}}{2}\right|^2 \log\left(\sqrt{2}\left|\frac{w-\bar{z}}{2}\right|\right) - c_1\left|\frac{w+\bar{z}}{2}\right|^2 \log\left(\sqrt{2}c_2\left|\frac{w+\bar{z}}{2}\right|\right). \tag{3.1}$$

Proof. Define $a = (w-\bar{z})/2$ and $b = (w+\bar{z})/2$, and let us try to express the left-hand side in terms of these auxiliary variables. We easily compute that $wz = (a+b)(\bar{b}-\bar{a}) = |b|^2 - |a|^2 + 2i\Im(a\bar{b})$; furthermore, a little calculation reveals that

$$\begin{aligned}-i(wz)^{1/2} &= \sqrt{\frac{|a|^2-|b|^2+\sqrt{(|b|^2-|a|^2)^2+(2\Im(a\bar{b}))^2}}{2}}\\ &\qquad - \operatorname{sgn}(\Im(a\bar{b}))\sqrt{\frac{|b|^2-|a|^2+\sqrt{(|b|^2-|a|^2)^2+(2\Im(a\bar{b}))^2}}{2}}.\end{aligned}$$

Consequently, we compute that

$$\begin{aligned}\Phi(wz) &= -(|b|^2-|a|^2)\left[\log\sqrt{(|b|^2-|a|^2)^2+(2\Im a\bar{b})^2}+\frac{\pi}{4}\right]\\ &\qquad -|2\Im(a\bar{b})|\arctan\sqrt{\frac{|b|^2-|a|^2+\sqrt{(|b|^2-|a|^2)^2+(2\Im(a\bar{b}))^2}}{|a|^2-|b|^2+\sqrt{(|b|^2-|a|^2)^2+(2\Im(a\bar{b}))^2}}}.\end{aligned}$$

By continuity, it suffices to show (3.1) for nonzero a. Let us minimize the left-hand side over a, b, keeping $|a|$ and $|b|$ fixed. To do this, set $u = |b|^2 - |a|^2 \in \mathbb{R}$, $v = 2|\Im(a\bar{b})| \geq 0$, and consider the function

$$H(v) = \Phi(wz) = -u\left[\log\sqrt{u^2+v^2}+\frac{\pi}{4}\right] - v\arctan\sqrt{\frac{u+\sqrt{u^2+v^2}}{-u+\sqrt{u^2+v^2}}}.$$

By the direct differentiation, we see that

$$H'(v) = -\frac{uv}{2(u^2+v^2)} - \arctan\sqrt{\frac{u/\sqrt{u^2+v^2}+1}{-u/\sqrt{u^2+v^2}+1}}.$$

Observe that there is $\alpha \in [0,\pi]$ such that

$$\frac{u}{\sqrt{u^2+v^2}} = \cos\alpha \qquad \text{and} \qquad \frac{v}{\sqrt{u^2+v^2}} = \sin\alpha$$

(the restriction on α is due to the inequality $v \geq 0$). Thus we have $H'(v) = -\left(\sin 2\alpha - 2\alpha + 2\pi\right)/2 \leq 0$, where the latter elementary estimate follows from a straightforward analysis of a derivative. Summarizing, we have shown that the function H is nonincreasing and hence it is enough to check (3.1) for the largest possible value of v, i.e., under the assumption $v = 2|a||b|$. However, then the majorization becomes

$$\begin{aligned}&-\left(|b|^2-|a|^2\right)\left[\log(|b|^2+|a|^2)+\frac{\pi}{4}\right] - 2|a||b|\arctan\frac{|b|}{|a|}\\ &\geq -|a|^2\log(\sqrt{2}|a|) - |b|^2\log(\sqrt{2}e^{\pi/2}|b|),\end{aligned}$$

or, after some equivalent rearranging,

$$J(s) = -\frac{s^2-1}{2}\log\frac{s^2+1}{2} - 2s\arctan s - \frac{\pi}{4}(s^2-1) + s^2\log(e^{\pi/2}s) \geq 0,$$

where $s = |b|/|a| \geq 0$. We derive that

$$J'(s) = -s\log\frac{s^2+1}{2} - 2\arctan s - \frac{\pi}{2}s - 2s\log(e^{\pi/2}s)$$

and

$$J''(s) = \log\frac{2s^2e^{\pi/2}}{s^2+1}.$$

Therefore, J is concave on $[0,s_0]$ and convex on $[s_0,\infty)$, for some positive number s_0; furthermore, we have $J(0) = -\log\sqrt{2}+\pi/4 > 0$ and $J(1) = J'(1) = 0$. Thus J has to be nonnegative on the whole halfline $[0,\infty)$ and the claim is proved. □

Proof of (1.1). As previously, the main difficulty lies in showing the bound for $n = 1$. Let $f : \mathbb{S}^1 \to \mathbb{C}$ be as in the statement and write $f = P_+f + P_-f$. Then $\mathcal{P}[P_+f]$ and $\mathcal{P}\left[\overline{P_-f}\right]$, the Poisson extensions of P_+f and $\overline{P_-f}$ to the disc B^1, are holomorphic. The function Φ is superharmonic and hence, by the well-known fact from the theory of analytic functions (see e.g. Range [15]), the composition $(w,z) \mapsto \Phi(wz)$ is plurisuperharmonic on $\mathbb{C}^2$. This, in turn, implies that the function $\Phi\left(4\mathcal{P}[P_+f]\mathcal{P}\left[\overline{P_-f}\right]\right)$ is superharmonic. Hence, by the mean-value property and (3.1),

$$\begin{aligned}0 = \Phi(0) &\geq \int_{\mathbb{S}^1}\Phi\left(4P_+f(z)\overline{P_-f(z)}\right)\mathrm{d}\sigma_1(z)\\ &\geq \int_{\mathbb{S}^1}\left[|\mathcal{H}f(z)|^2\log(\sqrt{2}|\mathcal{H}f(z)|) - c_1|f(z)|^2\log(\sqrt{2}c_2|f(z)|)\right]\mathrm{d}\sigma_1(z),\end{aligned}$$

where we have used the identities $\mathcal{H}f = i(P_-f - P_+f)$ and $f = P_+f + P_-f$. It remains to apply the above estimate to the function $f/\sqrt{2}$, and multiply both sides by 2 to get the claim in the case $n = 1$. The passage to the higher-dimensional setting is done exactly in the same manner as in the preceding section. We omit the straightforward details. □

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