KOLMOGOROV'S INEQUALITIES FOR MARTINGALES AND THE HAAR SYSTEM

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Abstract. Let $(h_k)_{k\geq 0}$ be the Haar system on $[0,1]$ and let $0<p<1$. We show that for any vectors a_k from a separable Hilbert space $\mathcal{H}$, any $\varepsilon_k \in [-1,1]$, $k=0,\,1,\,2,\,\ldots$ and any Borel subset A of $[0,1]$, we have the sharp inequality

$$\left|\left|\sum_{k=0}^{n}\varepsilon_k a_k h_k\right|\right|_{L^p(A)} \leq 2\left(\frac{2-p}{2-2p}\right)^{1/p}\left|\left|\sum_{k=0}^{n} a_k h_k\right|\right|_{L^1([0,1])}|A|^{1/p-1},$$

$n=0,\,1,\,2,\,\ldots$. The above estimate is generalized to the sharp estimate

$$||Y||_{L^p(A)} \leq 2\left(\frac{2-p}{2-2p}\right)^{1/p}||X||_{L^1(\Omega)}\cdot \mathbb{P}(A)^{1/p-1},$$

where X, Y stand for $\mathcal{H}$-valued continuous-time martingales such that Y is differentially subordinate to X. An application to Riesz system of harmonic functions is indicated.

1. Introduction

The motivation for the results obtained in this paper comes from a very basic question concerning the properties of the Haar system $(h_k)_{k\geq 0}$, an important basis for $L^p(0,1)$, $1\leq p<\infty$. A classical statement theorem of Marcinkiewicz [14], based on an earlier result of Paley [21], asserts that this basis is unconditional provided $1<p<\infty$. That is, there exists a universal constant $c_p\in(0,\infty)$ such that

$$c_p^{-1}\left|\left|\sum_{k=0}^{n} a_k h_k\right|\right|_{L^p(0,1)} \leq \left|\left|\sum_{k=0}^{n}\varepsilon_k a_k h_k\right|\right|_{L^p(0,1)} \leq c_p\left|\left|\sum_{k=0}^{n} a_k h_k\right|\right|_{L^p(0,1)} \tag{1.1}$$

for any n and any $a_k\in\mathbb{R}$, $\varepsilon_k\in\{-1,1\}$, $k=0,\,1,\,2,\ldots,n$. Among many extensions and modifications of this statement, there is an important probabilistic version, obtained by Burkholder in [5]. Let $(\Omega,\mathcal{F},\mathbb{P})$ be a probability space, filtered by $(\mathcal{F}_k)_{k\geq 0}$, a nondecreasing family of sub-σ-fields of $\mathcal{F}$. Suppose that $f=(f_k)_{k\geq 0}$ is a real-valued martingale with the difference sequence $(df_k)_{k\geq 0}$ given by $df_0=f_0$ and $df_k=f_k-f_{k-1}$ for $k\geq 1$. Let g be a transform of f by a real predictable sequence $v=(v_k)_{k\geq 0}$ bounded in absolute value by 1: that is, we have $dg_k=v_k df_k$ for all $k\geq 0$ and by predictability of v we mean that each term v_k is measurable with respect to $\mathcal{F}_{(k-1)\vee 0}$. Sometimes, when all the terms v_k are deterministic and have values ± 1, the sequence g is called the ± 1-transform of f. The aforementioned result of Burkholder states that if $1<p<\infty$, then there is an absolute constant c_p' for which

$$||g||_p \leq c_p'||f||_p. \tag{1.2}$$

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Here we have used the notation $||f||_p = \sup_n ||f_n||_p$. Let $c_p(1.1)$, $c'_p(1.2)$ denote the best (i.e., the least) constants in (1.1) and (1.2), respectively. Note that the Haar system is a martingale difference sequence with respect to its natural filtration (where the probability space is equal to the interval $(0,1]$ with its Borel subsets and Lebesgue measure), and hence so is $(a_k h_k)_{k\geq 0}$, for given fixed real numbers a_0, a_1, a_2, ... Therefore, we see that $c_p(1.1) \leq c'_p(1.2)$ for all $1 < p < \infty$. It follows from the results of Burkholder [7] and Maurey [15] that in fact the constants are the same: $c_p(1.1) = c'_p(1.2)$ for all $1 < p < \infty$. The question about the value of the sharp constant was answered by Burkholder in [8]: it turns out that $c_p(1.1)= p^* - 1$ for $1 < p < \infty$, where $p^* = \max\{p, p/(p-1)\}$. Furthermore, the constant remains the same if we allow the martingales and the coefficients a_k to take values in a separable Hilbert space $\mathcal{H}$.

Actually, in a typical situation one studies inequalities of the above form in a more general context of differential subordination. Suppose that $(\Omega, \mathcal{F}, \mathbb{P})$ is a complete probability space equipped with a continuous-time filtration $(\mathcal{F}_t)_{t\geq 0}$ such that $\mathcal{F}_0$ contains all the events of probability 0. Let X, Y be $\mathcal{H}$-valued martingales; with no loss of generality, one may assume that $\mathcal{H} = \ell^2$ - from now on, we will assume that this is the case. We impose the usual regularity assumptions on the trajectories of X and Y: they are right-continuous and possess limits from the left. Let $[X,X]$ denote the quadratic variation process (square bracket) associated with X. That is, $[X,X] = \sum_{n=1}^{\infty}[X^n, X^n]$, where X^n denotes the n-th coordinate of an ℓ^2-valued X and $[X^n, X^n]$ denotes the usual square bracket of a real-valued martingale (see Dellacherie and Meyer [13]). Following Bañuelos and Wang [4] and Wang [25], we say that Y is differentially subordinate to X, if the difference $([X,X]_t - [Y,Y]_t)_{t\geq 0}$ is nondecreasing and nonnegative as a function of t. If we treat discrete-time martingales f and g as continuous-time processes (via $X_t = f_{\lfloor t\rfloor}$, $Y_t = g_{\lfloor t\rfloor}$, $t \geq 0$), then the above condition becomes

$$|dg_n| \leq |df_n|, \qquad n = 0,\, 1,\, 2,\, \ldots,$$

which is the original definition of differential subordination due to Burkholder [8]. Clearly, the latter condition holds true if f is an arbitrary martingale and g is its transform by a predictable sequence bounded in absolute value by 1.

Thus the following statement, obtained by Wang in [25], generalizes (1.1) and (1.2). We use the notation $||X||_p = \sup_{t\geq 0} ||X_t||_p$, $0 < p < \infty$.

Theorem 1.1. *Assume that X, Y are $\mathcal{H}$-valued martingales such that Y is differentially subordinate to X. Then for any $1 < p < \infty$ we have the sharp bound*

$$||Y||_p \leq (p^* - 1)||X||_p. \tag{1.3}$$

There is a vast literature devoted to various extensions, connections and interesting applications of the above estimates, which include bounds for monotone bases in L^p [8, 12]; sharp estimates for Fourier multipliers [2, 3, 19, 20]; inequalities for Beurling-Ahlfors operator and their consequences for quasiconformal mappings [1, 4, 11]; and many more.

The purpose of this paper is to study a sharp localized version of the estimates (1.1), (1.2) and (1.3) for $0 < p < 1$. These can be referred to as "Kolmogorov inequalities", because of the evident analogies with the setting of Hilbert transform (cf. [26, Vol. 1, p.260]). While some initial information on these estimates can be found in the works of Burkholder [6] and the author [17], the full information has not been known until now. Here is the precise statement.

Theorem 1.2. *Suppose that X, Y are Hilbert-space-valued martingales such that Y is differentially subordinate to X. Then for any $0 < p < 1$ and any $A \in \mathcal{F}$ we have the sharp estimate*

$$||Y||_{L^p(A)} \leq 2\left(\frac{2-p}{2-2p}\right)^{1/p} ||X||_{L^1(\Omega)}\, \mathbb{P}(A)^{1/p-1}. \tag{1.4}$$

The constant $2\left(\frac{2-p}{2-2p}\right)^{1/p}$ is the best even in the corresponding inequality for ± 1-transforms of real-valued Haar martingales. Precisely, for any $0 < p < 1$, any $t \in (0, 1]$ and any $c < 2\left(\frac{2-p}{2-2p}\right)^{1/p}$ there exist a Borel subset A of $[0, 1]$ satisfying $|A| = t$, a finite sequence a_0, a_1, ..., a_n of real numbers and a finite sequence ε_0, ε_1, ..., ε_n of signs such that

$$\left|\left|\sum_{k=0}^{n} \varepsilon_k a_k h_k\right|\right|_{L^p(A)} > c \left|\left|\sum_{k=0}^{n} a_k h_k\right|\right|_{L^p(0,1)} t^{1/p-1}.$$

Consider the following application of the above result to Riesz systems of harmonic functions (cf. [22]). Let n be a positive integer and let D be an open connected subset of points $x = (x_0, x_1, \ldots, x_n)$ in $\mathbb{R}^{n+1}$. Let $w : D \to \mathcal{H}$ be a harmonic function. Then $F = \left(\frac{\partial w}{\partial x_0}, \frac{\partial w}{\partial x_1}, \ldots, \frac{\partial w}{\partial x_n}\right)$ is a Riesz system: the functions $w_j := \frac{\partial w}{\partial x_j}$ are harmonic and satisfy the generalized Cauchy-Riemann equations

$$\sum_{j=0}^{n} \frac{\partial w_j}{\partial x_j} = 0, \qquad \frac{\partial w_j}{\partial x_k} = \frac{\partial w_k}{\partial x_j}.$$

Suppose that there is $\xi \in D$ such that $\left|\frac{\partial w}{\partial x_0}(\xi)\right| \leq n^{1/2}\left(\sum_{k=1}^{n}\left|\frac{\partial w}{\partial x_k}(\xi)\right|^2\right)^{1/2}$ and pick $0 < p < 1$. Then for any bounded, open, connected set D_0 satisfying $\xi \in D_0 \subset D_0 \cup \partial D_0 \subset D$ and any $A \subset \partial D_0$, we have

$$\left|\left|\frac{\partial w}{\partial x_0}\right|\right|_{L^p(A,\, d\mu)} \leq 2\left(\frac{2-p}{2-2p}\right)^{1/p} n^{1/2} \left|\left|\left(\sum_{k=1}^{n}\left|\frac{\partial w}{\partial x_k}\right|^2\right)^{1/2}\right|\right|_{L^p(\partial D_0,\, d\mu)} \cdot \mu(A)^{1/p-1},$$

where μ denotes the harmonic measure on ∂D_0 with respect to the point ξ. The case when $n = 1$ and D is the unit disc on the complex plane corresponds to the classical Kolmogorov's inequalities for conjugate harmonic functions. The above inequality can be obtained by appropriate composition of the harmonic functions $n^{-1/2}\frac{\partial w}{\partial x_0}$ and $\left(\frac{\partial w}{\partial x_1}, \frac{\partial w}{\partial x_2}, \ldots, \frac{\partial w}{\partial x_n}\right)$ with Brownian motion started at ξ and stopped at the boundary of D_0. As the result of this composition, we obtain two martingales which satisfy the differential subordination, and the application of (1.4) yields the above estimate. We omit the further details, referring the interested reader to the papers [10] by Burkholder and [23] by Suh for the full explanation.

The paper is organized as follows. Theorem 1.2 will be established in the two sections below. Section 2 is devoted to the proof of the inequality (1.4). Our argument will depend heavily on Burkholder's method, and for the convenience of the reader, we will sketch the main ideas behind this technique. In the final part of the paper we address the optimality of the constant $2\left(\frac{2-p}{2-2p}\right)^{1/p}$, in the context of Haar martingales.

2. Proof of the inequality (1.4)

Our argument will be based on Burkholder's method, which allows to deduce a given martingale inequality from the existence of a certain special function, which enjoys appropriate majorization and concavity. Let us start with a brief description of the method and indicate the problems which arise when one tries to apply it directly. The idea is the following. Suppose that $V:\mathcal{H}\times\mathcal{H}\to\mathbb{R}$ is a given Borel function and assume that we want to establish the inequality

$$\mathbb{E}V(X_t,Y_t)\leq 0,\qquad t\geq 0, \tag{2.1}$$

for any martingales X, Y such that Y is differentially subordinate to X. Clearly, in general the martingales need to satisfy appropriate integrability properties so that the above expectation makes sense. However, let us not worry about this issue: it can be handled with some localizing-type arguments and does not bring anything essential to the general problem. To show (2.1), one constructs a special function U which majorizes V and for which the process $(U(X_s,Y_s))_{s\geq 0}$ is a supermartingale with $\mathbb{E}U(X_0,Y_0)\leq 0$. Clearly, the existence of such U yields the desired estimate, since we may write

$$\mathbb{E}V(X_t,Y_t)\leq \mathbb{E}U(X_t,Y_t)\leq \mathbb{E}U(X_0,Y_0)\leq 0.$$

The above supermartingale property of $(U(X_s,Y_s))_{s\geq 0}$ is usually deduced from an appropriate second-order partial differential inequality satisfied by U, which in turn links this subject with the theory of boundary value problems and Monge−Ampère equation (consult e.g. [8], [4], [24] and [25]).

The above approach has turned out to be very efficient in the study of (1.3) and its offspring. Note that this inequality follows once one proves that

$$\mathbb{E}|Y_t|^p\leq (p^*-1)^p\mathbb{E}|X_t|^p,\qquad t\geq 0.$$

The latter bound is of the form (2.1), with $V(x,y)=|y|^p-(p^*-1)^p|x|^p$. The corresponding special function was discovered by Burkholder in [9]: it is given by

$$U(x,y)=\alpha_p(|y|-(p^*-1)|x|)(|x|+|y|)^{p-1},$$

for a certain appropriate constant $\alpha_p>0$.

At the first glance, the argument described above does not work in our setting. In contrast with the preceding estimate, we see that the inequality (1.4) is *not* of the form (2.1), even if $A=\Omega$. No algebraic manipulations allow putting X and Y under one expectation. The second problem concerns the appearance of the event A, which is, in a sense, an additional variable in the estimate.

To overcome the above difficulties, we will have to take an intermediate step and study a certain auxiliary, strange-looking estimate

$$\mathbb{E}\left[|Y_t|^p-\left(1-\frac{p}{2}\right)\lambda^p\right]_+\leq \frac{p(2-p)}{1-p}\lambda^{p-1}\mathbb{E}|X_t|,\qquad t\geq 0,$$

where λ is a certain nonnegative parameter. This inequality is of the form (2.1); having established it, we will manage to insert A into the reasoning and then, by some optimization arguments, we will deduce the desired claim.

We turn our attention to the special function corresponding to the above intermediate estimate. This object will be a bit complicated and its analysis would be quite elaborate. To avoid these technicalities, we will make use of an "integration argument", invented by the author in [16]. Roughly speaking, we will first introduce a certain simple function u, for which the calculations are much easier, and then complicate it to obtain the appropriate U. Consider the following function $u : \mathcal{H} \times \mathcal{H} \to \mathbb{R}$, originally introduced by Burkholder in [8]:

$$u(x,y) = \begin{cases} |y|^2 - |x|^2 & \text{if } |x| + |y| \leq 1, \\ 1 - 2|x| & \text{if } |x| + |y| > 1. \end{cases}$$

This function enjoys the following property (see Wang [25] or the author [18]): if X, Y are $\mathcal{H}$-valued martingales such that Y is differentially subordinate to X, then for any $t \geq 0$,

$$\mathbb{E}u(X_t, Y_t) \leq 0. \tag{2.2}$$

Now we use a dilation and integration argument to modify this object and obtain a function suitable for our purposes. Namely, fix a nonnegative constant λ and set

$$U_\lambda(x,y) = \frac{p(2-p)}{2} \int_\lambda^\infty r^{p-1} u\left(\frac{x}{r}, \frac{y}{r}\right) \mathrm{d}r.$$

The property (2.2), combined with Fubini's theorem, yields

$$\mathbb{E}U_\lambda(X_t, Y_t) \leq 0 \qquad \text{for } t \geq 0. \tag{2.3}$$

To see that Fubini's theorem is applicable, observe that $|u(x,y)| \leq 2(|x| + |y|)$ for all x, $y \in \mathcal{H}$. Consequently,

$$\mathbb{E} \int_\lambda^\infty r^{p-1} \left| u\left(\frac{X_t}{r}, \frac{Y_t}{r}\right) \right| \mathrm{d}r \leq \frac{2\lambda^{p-1}}{1-p} \mathbb{E}(|X_t| + |Y_t|) < \infty$$

and hence (2.3) is indeed true.

Our next step is to establish the corresponding majorization. To this end, we need to find the explicit formula for the function U_λ first.

Lemma 2.1. *For any x, $y \in \mathcal{H}$ we have*

$$U_\lambda(x,y) = \frac{p}{2}\lambda^{p-2}(|y|^2 - |x|^2)$$

if $|x| + |y| < \lambda$, and

$$U_\lambda(x,y) = ((1-p)^{-1}|x| + |y|)(|x| + |y|)^{p-1} - \frac{p(2-p)}{1-p}\lambda^{p-1}|x| - \left(1 - \frac{p}{2}\right)\lambda^p$$

if $|x| + |y| \geq \lambda$.

Proof. This is straightforward. If $|x| + |y| \leq \lambda$, then $|x/r| + |y/r| \leq 1$ for all $r \geq \lambda$ and hence

$$U_\lambda(x,y) = (|y|^2 - |x|^2) \cdot \frac{p(2-p)}{2} \int_\lambda^\infty r^{p-3} \mathrm{d}r = \frac{p}{2}\lambda^{p-2}(|y|^2 - |x|^2).$$

On the other hand, if $|x|+|y|>\lambda$, we compute that

$$\begin{aligned}
U_\lambda(x,y) &= \frac{p(2-p)}{2}\left[\int_\lambda^{|x|+|y|} r^{p-1}u\left(\frac{x}{r},\frac{y}{r}\right)\mathrm{d}r + \int_{|x|+|y|}^\infty r^{p-1}u\left(\frac{x}{r},\frac{y}{r}\right)\mathrm{d}r\right]\\
&= \frac{p(2-p)}{2}\left[\int_\lambda^{|x|+|y|} r^{p-1}\left(1-\frac{2|x|}{r}\right)\mathrm{d}r + (|y|^2-|x|^2)\int_{|x|+|y|}^\infty r^{p-3}\mathrm{d}r\right]\\
&= \frac{p(2-p)}{2}(|x|+|y|)^{p-1}\left[\frac{|x|+|y|}{p}-\frac{2|x|}{p-1}+\frac{|y|-|x|}{2-p}\right]\\
&\quad -\frac{p(2-p)}{1-p}\lambda^{p-1}|x| - \frac{2-p}{2}\lambda^p\\
&= ((1-p)^{-1}|x|+|y|)(|x|+|y|)^{p-1} - \frac{p(2-p)}{1-p}\lambda^{p-1}|x| - \left(1-\frac{p}{2}\right)\lambda^p. \qquad \square
\end{aligned}$$

Now we will show that U_λ enjoys the appropriate majorization.

Lemma 2.2. *For any $x,\,y\in\mathcal{H}$ we have the estimate*

$$U_\lambda(x,y) \geq \left[|y|^p - \left(1-\frac{p}{2}\right)\lambda^p\right]_+ - \frac{p(2-p)}{1-p}\lambda^{p-1}|x|. \tag{2.4}$$

Proof. Clearly, we may assume that $\mathcal{H}=\mathbb{R}$ and restrict ourselves to $x\geq 0$, $y\geq 0$, since U_λ depends on x, y through $|x|$ and $|y|$ only. Observe that for any fixed y, the function $x\mapsto U_\lambda(x,y)$ is concave on $[0,\infty)$; this follows at once from the definition of U_λ and the fact that the basic function u also enjoys this property. Moreover, for a fixed y, the majorized right-hand side is linear in $x\in[0,\infty)$ and

$$\lim_{x\to\infty}\frac{\partial U_\lambda(x,y)}{\partial x} = \lim_{x\to\infty}\frac{U_\lambda(x,y)}{x} = -\frac{p(2-p)}{1-p}\lambda^{p-1},$$

which is the same as the analogous limit for the right-hand side of (2.4). Putting all the above facts together, we see that it is enough to establish the majorization for $x=0$. If $y\geq\lambda$, then $\left[|y|^p-\left(1-\frac{p}{2}\right)\lambda^p\right]_+ = y^p-\left(1-\frac{p}{2}\right)\lambda^p$ and both sides are equal. If $y\leq\left(1-\frac{p}{2}\right)^{1/p}\lambda$, then (2.4) is equivalent to $\frac{p}{2}\lambda^{p-2}|y|^2\geq 0$, which is evident. Finally, if $\left(1-\frac{p}{2}\right)^{1/p}\lambda<y<\lambda$, then the majorization reads

$$F(y) := \frac{p}{2}\lambda^{p-2}y^2 - y^p + \left(1-\frac{p}{2}\right)\lambda^p \geq 0.$$

To show this bound, it suffices to note that $F(\lambda)=0$ and $F'(y)=py(\lambda^{p-2}-y^{p-2})\leq 0$ when $y\leq\lambda$. This completes the proof. $\square$

We are ready to establish our main estimate.

Proof of (1.4). The inequality is obvious when A is an event of measure 0; thus, we may assume that $\mathbb{P}(A)>0$. Fix a nonnegative number t. Combining the inequality (2.3) with (2.4) gives

$$\mathbb{E}\left[|Y_t|^p - \left(1-\frac{p}{2}\right)\lambda^p\right]_+ \leq \frac{p(2-p)}{1-p}\lambda^{p-1}\mathbb{E}|X_t|, \tag{2.5}$$

which is our desired intermediate result mentioned previously. Now we insert the event A into our considerations. Take the splitting $A=A_-\cup A_+$, where

$$A_- = A\cap\left\{|Y_t|<\left(1-\frac{p}{2}\right)^{1/p}\lambda\right\},\qquad A_+ = A\cap\left\{|Y_t|\geq\left(1-\frac{p}{2}\right)^{1/p}\lambda\right\}.$$

Then, clearly, $\mathbb{E}\left(|Y_t|^p - \left(1-\frac{p}{2}\right)\lambda^p\right)\chi_{A_-} \leq 0$ and, by (2.5),

$$\mathbb{E}\left(|Y_t|^p - \left(1-\frac{p}{2}\right)\lambda^p\right)\chi_{A_+} \leq \frac{p(2-p)}{1-p}\lambda^{p-1}\mathbb{E}|X_t|.$$

Adding the two estimates above, we obtain an inequality which is equivalent to

$$\mathbb{E}|Y_t|^p\chi_A \leq \frac{p(2-p)}{1-p}\lambda^{p-1}\mathbb{E}|X_t| + \left(1-\frac{p}{2}\right)\lambda^p \cdot \mathbb{P}(A).$$

Now the right-hand side, considered as a function of $\lambda \geq 0$, is minimal for the choice $\lambda = 2\mathbb{E}|X_t|/\mathbb{P}(A)$. Plugging this particular λ transforms the bound into

$$\mathbb{E}|Y_t|^p\chi_A \leq 2^p \cdot \frac{2-p}{2-2p}\left(\mathbb{E}|X_t|\right)^p \cdot \mathbb{P}(A)^{1-p}.$$

It suffices to take p-th root and the supremum over all t to get the claim. $\square$

3. Sharpness

In this section we will show that the constant $2\left(\frac{2-p}{2-2p}\right)^{1/p}$ is the best possible even for discrete-time real-valued martingales f, g associated with the Haar system, as explained in the statement of Theorem 1.2. Actually, we will show that the inequality is sharp even if the martingale f is assumed to be nonnegative. One could establish this claim by constructing appropriate examples; however, we have chosen a different approach and our arguments exploit the properties of a certain abstract function.

We start with some definitions. For any $x \geq 0$ and $y \in \mathbb{R}$, consider the class $\mathcal{M}(x,y)$, which consists of all pairs (f,g) of functions on $[0,1]$ such that

- f is nonnegative and has the form $f = x + \sum_{k=1}^n a_k h_k$ for some integer n and some sequence a_1, a_2, …, a_n of real numbers,
- g is of the form $g = y + \sum_{k=1}^n \varepsilon_k a_k h_k$, where n and the coefficients a_1, a_2, …, a_n are those appearing in f, while ε_1, ε_2, …, ε_n is a sequence of signs.

Thus, we see that modulo the first term, g is a ± 1-transform of f (it is exactly a ± 1-transform if and only if $y = \pm x$). Next, consider the abstract function $\mathcal{U}$: $[0,\infty) \times \mathbb{R} \times [0,1] \to \mathbb{R}$, given by

$$\mathcal{U}(x,y,t) = \sup\left\{\int_A |g(s)|^p \mathrm{d}s \,:\, (f,g) \in \mathcal{M}(x,y),\ A \subseteq [0,1],\ |A| = t\right\}.$$

Note that the nonnegativity of f implies that $||f||_{L^1(0,1)} = x$ (provided $(f,g) \in \mathcal{M}(x,y)$). Hence the inequality (1.4) we established in the previous section gives

$$\mathcal{U}(x,x,t) \leq 2^p \cdot \frac{2-p}{2-2p}x^p t^{1-p} \tag{3.1}$$

for all $x \geq 0$ and $t \in [0,1]$. Clearly, we will obtain the desired sharpness if we manage to show that for each $t \in (0,1]$ there is a positive x for which the inequality can be reversed. To accomplish this goal, we will first study some structural conditions the function $\mathcal{U}$ possesses.

Theorem 3.1. *The function $\mathcal{U}$ enjoys the following properties.*

(i) For all $x \geq 0$, $y \in \mathbb{R}$ and $t \in [0,1]$ we have $\mathcal{U}(x,y,t) \geq |y|^p t$.

(ii) For any $x \geq 0$, $y \in \mathbb{R}$, $t \in [0,1]$ and $\lambda > 0$ we have $\mathcal{U}(\lambda x, \pm\lambda y, t) = \lambda^p \mathcal{U}(x,y,t)$.

(iii) For any $x_\pm \geq 0$, $y_\pm \in \mathbb{R}$ and $t_\pm \in [0,1]$ satisfying $|x_+ - x_-| = |y_+ - y_-|$, we have

$$\mathcal{U}\left(\frac{x_- + x_+}{2}, \frac{y_- + y_+}{2}, \frac{t_- + t_+}{2}\right) \geq \frac{1}{2}\mathcal{U}(x_-, y_-, t_-) + \frac{1}{2}\mathcal{U}(x_+, y_+, t_+).$$

Proof. (i) It suffices to consider the constant pair $(f,g) \equiv (x,y)$ in the definition of $\mathcal{U}(x,y,t)$.

(ii) This follows from the very definition of $\mathcal{U}$ and the fact that $(f,g) \in \mathcal{M}(x,y)$ if and only if $(\lambda f, \pm\lambda g) \in \mathcal{M}(\lambda x, \pm\lambda y)$.

(iii) Pick arbitrary $(f_-, g_-) \in \mathcal{M}(x_-, y_-)$ and $(f_+, g_+) \in \mathcal{M}(x_+, y_+)$. We splice these two pairs into one pair (f,g), by „squeezing" the domain of f_-, g_- to $[0,1/2]$ and the domain of f_+, g_+ to $(1/2,1]$. Precisely, we the use the formula

$$(f(s), g(s)) = \begin{cases} (f_-(2s), g_-(2s)) & \text{if } s \in [0,1/2], \\ (f_+(2s-1), g_+(2s-1)) & \text{if } s \in (1/2,1]. \end{cases}$$

A key observation is that $(f,g) \in \mathcal{M}((x_- + x_+)/2, (y_- + y_+)/2)$. To show this inclusion, note that the first terms in the Haar expansions of f and g are $\int_0^1 f = \left(\int_0^1 f_- + \int_0^1 f_+\right)/2 = (x_- + x_+)/2$ and, similarly, $\int_0^1 g = (y_- + y_+)/2$. Next, it is clear that f has the appropriate form, required in the definition of the class $\mathcal{M}((x_- + x_+)/2, (y_- + y_+)/2)$. The fact that the second coefficient of g has the required form (i.e., it has the same absolute value as the second coefficient of f), follows at once from the assumption $|x_+ - x_-| = |y_+ - y_-|$. The analogous property for the remaining coefficients of g is the consequence of the assumption $(f_\pm, g_\pm) \in \mathcal{M}(x_\pm, y_\pm)$ and the dyadic structure of the Haar system: for any $k \geq 3$, the k-th coefficients of f and g are equal to a certain pair of coefficients of f_- and g_- or f_+ and g_+, depending on whether the support of h_{k-1} is contained in $[0,1/2]$ or $[1/2,1]$. Let A_-, A_+ are arbitrary Borel subsets of $[0,1]$ of measures t_+ and t_-, and put $A = \frac{1}{2}A_- \cup \left(\frac{1}{2} + \frac{1}{2}A_+\right)$ (in the sense that $s \in A$ if and only if $2s \in A_-$ or $2s - 1 \in A_+$). Then $|A| = (t_- + t_+)/2$ and we have

$$\begin{aligned} \mathcal{U}\left(\frac{x_- + x_+}{2}, \frac{y_- + y_+}{2}, \frac{t_- + t_+}{2}\right) &\geq \int_A |g(s)|^p \mathrm{d}s \\ &= \frac{1}{2}\int_{A_-} |g_-(s)|^p \mathrm{d}s + \frac{1}{2}\int_{A_+} |g_+(s)|^p \mathrm{d}s. \end{aligned}$$

Taking the supremum over all $f_\pm$, $g_\pm$, $A_\pm$ as above, we get the desired estimate. □

We are ready to study the sharpness. For the sake of clarity, we have decided to split the reasoning into a few intermediate parts. Let $t \in (0,1]$ be a fixed number.

Step 1. Consider the function $\varphi(s) = \mathcal{U}(s,s,2s)$, $s \in [0,1/2]$. By the property (i), this function is bounded from below; furthermore, the condition (iii) implies that φ is mid-point concave. Consequently, φ is a concave function and hence in particular,

$$\begin{aligned} \mathcal{U}(t/2,t/2,t) = \varphi(t/2) &\geq (1-t)\varphi(0) + t\varphi(1/2) \\ &= (1-t)\mathcal{U}(0,0,0) + t\mathcal{U}(1/2,1/2,1) = t\mathcal{U}(1/2,1/2,1). \end{aligned}$$

Here in the last passage we have exploited the homogeneity property (ii): we have $\mathcal{U}(0,0,0) = \lambda^p\mathcal{U}(0,0,0)$ for all $\lambda > 0$ and hence $\mathcal{U}(0,0,0)$ is equal to zero.

Step 2. Now consider the function $\varphi(s) = \mathcal{U}(s, 1-s, 1)$, $s \in [0, \infty)$. By (i), this function is bounded from below, and by (iii), it is mid-point concave; hence it is concave on $[0, \infty)$. In particular, we see that both one-sided derivatives $\varphi'_-(1)$, $\varphi'_+(1)$ exist. Observe that by property (ii), if $s > 1$, then

$$\begin{aligned}\varphi(s) = \mathcal{U}(s, 1-s, 1) = \mathcal{U}(s, s-1, 1) &= (2s-1)^p \mathcal{U}\left(\frac{s}{2s-1}, \frac{s-1}{2s-1}, 1\right)\\ &= (2s-1)^p \varphi\left(\frac{s}{2s-1}\right).\end{aligned}$$

Consequently,

$$\begin{aligned}\varphi'_+(1) &= \lim_{s\downarrow 1} \frac{\varphi(s)-\varphi(1)}{s-1}\\ &= \lim_{s\downarrow 1} \frac{(2s-1)^p\varphi\left(\frac{s}{2s-1}\right)-\varphi(1)}{s-1}\\ &= -\lim_{s\downarrow 1}\left\{(2s-1)^{p-1}\cdot\frac{\varphi\left(\frac{s}{2s-1}\right)-\varphi(1)}{\frac{s}{2s-1}-1}\right\} + \varphi(1)\lim_{s\downarrow 1}\cdot\frac{(2s-1)^p-1}{s-1}\\ &= -\varphi'_-(1) + 2p\varphi(1).\end{aligned}$$

By the concavity of φ, we have $\varphi'_-(1) \geq \varphi'_+(1)$, which combined with the preceding bound implies $p\varphi(1) \leq \varphi'_-(1)$.

Step 3. Now, by the concavity of φ, we have $\varphi'_-(1) \leq \varphi(1) - \varphi(0)$. Furthermore, the condition (i) implies $\varphi(0) = \mathcal{U}(0, 1, 1) \geq 1$. This yields $\varphi'(1) \leq \varphi(1) - 1$, which combined with the bound $p\varphi(1) \leq \varphi'_-(1)$, obtained in the previous step, gives $\varphi(1) \geq (1-p)^{-1}$. Thus, using $p\varphi(1) \leq \varphi'_-(1)$ and $\varphi(0) \geq 1$ again, together with the concavity of φ, we get

$$\frac{p}{1-p} \leq p\varphi(1) \leq \varphi'_-(1) \leq 2(\varphi(1/2) - \varphi(0)) \leq 2\varphi(1/2) - 2,$$

or $\mathcal{U}(1/2, 1/2, 1) \geq (2-p)/(2-2p)$.

Step 4. Combining the final estimates of Step 1 and Step 3, we obtain

$$\mathcal{U}(t/2, t/2, t) \geq t \cdot \frac{2-p}{2-2p} = 2^p \cdot \frac{2-p}{2-2p}\left(\frac{t}{2}\right)^p t^{1-p}.$$

This is the desired reversion of the estimate (3.1): we see that actually we have

$$\mathcal{U}(t/2, t/2, t) = 2^p \cdot \frac{2-p}{2-2p}\left(\frac{t}{2}\right)^p t^{1-p}.$$

Thus, even if we consider the inequality (1.4) for Haar martingales and restrict ourselves to sets A of fixed measure $|A| = t \in (0, 1]$, the improvement of the constant $2\left(\frac{2-p}{2-2p}\right)^{1/p}$ is impossible.

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