Sharp L^p bound for holomorphic functions on the unit disc

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Abstract. For any $1 < p < \infty$ and any X, $Y \in \mathbb{R}$ satisfying $|X| \leq Y$, we determine the optimal constant $C_p(X, Y)$ such that the following holds. If F is a holomorphic function on the unit disc satisfying $\operatorname{Re} F(0) = X$ and $||\operatorname{Re} F||_{L^p(\mathbb{T})} = Y$, then

$$||F||_{L^p(\mathbb{T})} \geq C_p(X, Y).$$

This can be regarded as a reverse version of the classical estimates of Riesz and Essén. The proof rests on the exploitation of certain families of special subharmonic functions on the plane.

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1. Introduction

Let u, v be two harmonic functions on the unit disc $\mathbb{D}$, satisfying Cauchy-Riemann equations and normalized so that $v(0) = 0$. A classical problem, which interested many mathematicians at the beginning of the previous century, is the following: How is the size of v controlled by the size of u? Here the size of a function is measured, for instance, by the L^p norm on the unit circle $\mathbb{T}$ equipped with the normalized Haar measure m. In other words, for which $1 \leq p \leq \infty$ is there a finite constant C_p, depending only on p, such that

$$||v||_{L^p(\mathbb{T})} \leq C_p ||u||_{L^p(\mathbb{T})} \tag{1.1}$$

for all u, v as above? This question was answered by Riesz in [17], [18]: the above estimate holds with some $C_p < \infty$ if and only if $1 < p < \infty$. This result is of fundamental importance to harmonic and complex analysis and has been modified and extended in numerous directions (cf. [2], [4], [5], [6], [9], [11], [23], to name just a few). Moreover, the methods presented in the

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works of Riesz have led to the development of many areas of research (e.g., interpolation theory, functional analysis) and have had a profound influence on the shape of contemporary mathematics.

The question about the best value of C_p has gained some interest in the literature. It was answered partially by Gohberg and Krupnik in [8]: by the use of a clever inductive argument, it was shown there that $C_p = \cot(\pi/(2p))$ if $p = 2, 4, 8, \ldots$. The identification of C_p in the full range $p \in (1,\infty)$ is due to Pichorides [16] and Cole (unpublished: see Gamelin [7]): we have $C_p = \cot(\pi/(2p^*))$, where $p^* = \max\{p, p/(p-1)\}$. There are several papers which treat related sharp estimates for conjugate harmonic functions on the disc; see e.g. Aarão and O'Neill [1], Davis [5], Janakiraman [10], Nazarov and Treil [14], Osękowski [15], and consult references therein.

One can look at the estimate (1.1) from a slightly different perspective. Obviously, u can be regarded as the real part of the holomorphic function $u+iv$ on the unit disc. Consequently, by the triangle inequality, the inequality (1.1) is equivalent to the following statement: if F is a holomorphic function on $\mathbb{D}$ satisfying the normalization condition $\operatorname{Im} F(0) = 0$, then

$$||F||_{L^p(\mathbb{T})} \leq E_p || \operatorname{Re} F||_{L^p(\mathbb{T})}, \qquad 1 < p < \infty, \tag{1.2}$$

for some finite E_p depending only on p. Actually, as Essén [6] proved, the choice $E_p = \sin^{-1}(\pi/(2p^*))$ is optimal. See also Tomaszewski [21] for the sharp weak-type counterpart of this estimate.

The purpose of this paper is to study a certain reverse version of (1.2). Clearly, if F is a holomorphic function on the unit disc (with no additional assumptions on $\operatorname{Im} F(0)$), then we have

$$||F||_{L^p(\mathbb{T})} \geq || \operatorname{Re} F||_{L^p(\mathbb{T})}. \tag{1.3}$$

Of course, this bound is sharp: equality holds for constant real functions. However, one can study the following more sophisticated version of this problem: namely, find the sharp analogue of (1.3) subject to the restrictions

$$\operatorname{Re} F(0) = X \quad \text{ and } \quad || \operatorname{Re} F||_{L^p(\mathbb{T})} = Y. \tag{1.4}$$

Clearly, the answer to this question provides us with more detailed information on the behavior of the operator $F \mapsto \operatorname{Re} F$. Such a type of problems appears in many places in the literature, in the study of other classical operators and objects in harmonic analysis. See e.g. Melas [12], Melas, Nikolidakis and Stavropoulos [13] and Slavin, Stokolos and Vasyunin [19] for related problems concerning the dyadic maximal operators; consult Burkholder [3] for related results for martingale transforms and the Haar system on $[0,1]$; Vasyunin [22] studied similar questions for A_p-weights on the real line; Slavin and Vasyunin [20] investigated similar problems for BMO functions on $\mathbb{R}$; and more.

Coming back to (1.3) and the restriction (1.4), we easily see that if $Y \neq |X|$, then the lower bound for $||F||_{L^p(\mathbb{T})}$ can be improved. For instance, suppose that $p = 2$ and put $u = \operatorname{Re} F$ and $v = \operatorname{Im} F$. Since u, v satisfy

Cauchy-Riemann equations, we may write

$$\begin{aligned}||F||_{L^2(\mathbb{T})} &= \left(||u||^2_{L^2(\mathbb{T})} + ||v||^2_{L^2(\mathbb{T})}\right)^{1/2}\\ &= \left(||u||^2_{L^2(\mathbb{T})} + ||v - v(0)||^2_{L^2(\mathbb{T})} + |v(0)|^2\right)^{1/2}\\ &= \left(||u||^2_{L^2(\mathbb{T})} + ||u - u(0)||^2_{L^2(\mathbb{T})} + |v(0)|^2\right)^{1/2}\\ &= \left(2||u||^2_{L^2(\mathbb{T})} - |u(0)|^2 + |v(0)|^2\right)^{1/2} \geq (2Y^2 - X^2)^{1/2}\end{aligned}$$

and equality can hold, for instance, if we take $F(z) = z\sqrt{2Y^2 - 2X^2} + X$. What about other values of p? This question is answered in Theorem 1.1, to formulate which we will need some auxiliary notation. Let X, Y be two real numbers satisfying $|X| \leq Y$. For $1 < p \leq 2$, define

$$C_p(X,Y) = \left[\sin^{-p}\left(\frac{\pi}{2p}\right)(Y^p - |X|^p) + |X|^p\right]^{1/p},$$

while for $2 < p < \infty$, let

$$C_p(X,Y) = \frac{Y}{\cos\phi_p},$$

where ϕ_p is the unique number $\phi \in [0, \pi/(2p))$, satisfying

$$\left(\frac{|X|}{Y}\right)^p = \frac{\cos(p\phi)}{\cos^p\phi}. \tag{1.5}$$

The existence and uniqueness of ϕ_p follows from the fact that the right-hand side of (1.5) is a continuous and strictly decreasing function of ϕ, which takes value 1 at $\phi = 0$ and converges to 0 as $\phi \to \pi/(2p)$.

We are ready to state our main result.

Theorem 1.1. *Let $1 < p < \infty$. Then for any holomorphic function $F : \mathbb{D} \to \mathbb{C}$, satisfying $\operatorname{Re} F(0) = X$ and $||\operatorname{Re} F||_{L^p(\mathbb{T})} = Y$, we have*

$$||F||_{L^p(\mathbb{T})} \geq C_p(X,Y). \tag{1.6}$$

For each p, X and Y, the number $C_p(X,Y)$ cannot be replaced by a smaller number.

The proof of this statement rests on the existence of certain families of subharmonic functions on the plane. In the next section we study the case $1 < p \leq 2$ of the above theorem, and Section 3 of the paper is devoted to the case $2 < p < \infty$. In the final part of the paper we sketch some ideas which lead to the discovery of special functions used in Sections 2 and 3.

2. The case $1 < p \leq 2$

As we have mentioned above, the proof will rest on the existence of certain special subharmonic functions. Introduce $U_p : [0,\infty)^2 \to \mathbb{R}$ by

$$U_p(x,y) = \begin{cases} R^p - \sin^{-p}\left(\frac{\pi}{2p}\right)|x|^p & \text{if } \theta \leq \frac{\pi}{2} - \frac{\pi}{2p}, \\ \cot\left(\frac{\pi}{2p}\right) R^p \cos\left(p\left(\theta - \frac{\pi}{2}\right)\right) & \text{if } \theta > \frac{\pi}{2} - \frac{\pi}{2p}. \end{cases}$$

Here $x = R\cos\theta$, $y = R\sin\theta$, where $R \geq 0$ and $\theta \in [0, \pi/2]$, stand for the polar coordinates. Let us extend U_p to the whole plane $\mathbb{R}^2$ by the requirement $U_p(x,y) = U_p(-x,y) = U_p(x,-y)$ for all x, $y \in \mathbb{R}$. One easily checks that the function U_p is continuous; further properties of U_p are gathered in a lemma below.

Lemma 2.1. *The function U_p enjoys the following properties.*

(i) We have $U_p(x,y) \geq U_p(x,0)$ for all x, $y \in \mathbb{R}$.

(ii) For all x, $y \in \mathbb{R}$ we have the majorization

$$U_p(x,y) \leq R^p - \sin^{-p}\left(\frac{\pi}{2p}\right)|x|^p. \tag{2.1}$$

(iii) The function U_p is subharmonic.

Proof. (i) By the symmetry of U_p, it is enough to prove that $U_{py}(x,y) \geq 0$ for x, $y \geq 0$. This estimate is evident if $\theta \leq \pi/2 - \pi/(2p)$. If $\theta > \pi/2 - \pi/(2p)$, then, using the identities $R_y = y/R$ and $\theta_y = x/R^2$, we compute that

$$U_{py}(x,y) = p\cot\left(\frac{\pi}{2p}\right) R^{p-1}\sin\left(\theta + p\left(\frac{\pi}{2} - \theta\right)\right).$$

This is positive, since $\theta + p(\pi/2 - \theta) \in (0, \pi)$.

(ii) By the symmetry of U_p, we may restrict ourselves to nonnegative x and y. Furthermore, we may assume that $\theta \in [\pi/2 - \pi/(2p), \pi/2]$, since for the remaining θ's both sides are equal. Under these additional assumptions, the majorization can be rewritten in the equivalent form

$$G(\theta) := \left[\cot\left(\frac{\pi}{2p}\right)\cos\left(p\left(\theta - \frac{\pi}{2}\right)\right) - 1\right]\cos^{-p}\theta + \sin^{-p}\left(\frac{\pi}{2p}\right) \leq 0.$$

However, we have $G(\pi/2 - \pi/(2p)) = 0$ and, as we will show now, G is nonincreasing on $(\pi/2 - \pi/(2p), \pi/2)$. Since

$$G'(\theta) = p \cdot \frac{\cot(\pi/(2p))\sin(\theta - p(\theta - \pi/2)) - \sin\theta}{\cos^{p+1}\theta},$$

the announced monotonicity of G is equivalent to saying that the numerator is nonpositive. After the substitution $\psi = \pi/2 - \theta$, the latter can be rewritten as

$$\frac{\cos((p-1)\psi)}{\cos((p-1)\frac{\pi}{2p})} \leq \frac{\cos\psi}{\cos\frac{\pi}{2p}}, \qquad \text{for } \psi \in (0, \pi/2p).$$

But recall that we work in the case $1 < p \le 2$; therefore, it is enough to show that for any $0 \le s \le t \le \pi/2$, the function

$$\xi(\alpha) = \frac{\cos(\alpha s)}{\cos(\alpha t)}, \qquad \alpha \in [0,1],$$

is nondecreasing. A direct differentiation shows that

$$\begin{aligned}\xi'(\alpha) &= \frac{-s\sin(\alpha s)\cos(\alpha t) + t\sin(\alpha t)\cos(\alpha s)}{\cos^2(\alpha t)}\\ &\ge \frac{-t\sin(\alpha s)\cos(\alpha t) + t\sin(\alpha t)\cos(\alpha s)}{\cos^2(\alpha t)} = \frac{t\sin(\alpha(t-s))}{\cos^2(\alpha t)} \ge 0.\end{aligned}$$

This proves $G' \le 0$ on $(\pi/2 - \pi/(2p), \pi/2)$ and establishes the majorization (2.1).

(iii) It is easy to check that U_p is of class C^1 and hence, by the symmetry, it is enough to verify the subharmonicity on $(0,\infty)^2$. Clearly, U_p is harmonic in the angle $\theta \in (\pi/2 - \pi/(2p), \pi/2)$. On the other hand, if $\theta \in (0, \pi/2 - \pi/(2p))$, we compute that

$$\begin{aligned}\Delta U_p(x,y) &= \left[\frac{\partial^2}{\partial R^2} + \frac{1}{R}\frac{\partial}{\partial R} + \frac{1}{R^2}\frac{\partial^2}{\partial^2\theta}\right] U_p(x,y)\\ &= p(p-1)R^{p-2}\left[\frac{p}{p-1} - \sin^{-p}\left(\frac{\pi}{2p}\right)\cos^{p-2}\theta\right].\end{aligned}$$

Since $1 < p \le 2$, the expression in the square brackets is larger than

$$\frac{p}{p-1} - \sin^{-p}\left(\frac{\pi}{2p}\right)\cos^{p-2}\left(\frac{\pi}{2} - \frac{\pi}{2p}\right) = \frac{p}{p-1} - \sin^{-2}\left(\frac{\pi}{2p}\right),$$

which is nonnegative. Indeed, after the substitution $r = 1/p \in [1/2, 1)$, we get

$$\sin^2\left(\frac{\pi}{2p}\right) - \frac{p-1}{p} = \sin^2\left(\frac{\pi r}{2}\right) - 1 + r,$$

which is zero for $r = 1/2$ and is an increasing function of $r \in (1/2, 1)$. □

Proof of (1.6). We are ready to establish the lower bound. Let us fix a function F as in the statement and gather some information. First, since $\operatorname{Re} F$ belongs to $L^p(\mathbb{T})$, the restriction $F|_{\mathbb{T}}$ also has this property, by Riesz' theorem. But, clearly, we have $|U_p(x,y)| \le c_p R^p$ for all x, y and some c_p depending only on p, so the restriction of $U_p \circ F$ to the unit circle is integrable. Finally, observe that by the third part of the above lemma, the composition $U_p \circ F$ is a subharmonic function on the unit disc. Therefore, using the first two parts of the lemma and the mean-value property, we obtain

$$U_p(X,0) \le U_p(F(0)) \le \int_{\mathbb{T}} U_p \circ F(u)\mathrm{d}m(u) \le ||F||_p^p - \sin^p\left(\frac{\pi}{2p}\right)||\operatorname{Re} F||_p^p.$$

This is equivalent to

$$||F||_p \ge \left(\sin^{-p}\left(\frac{\pi}{2p}\right)(Y^p - X^p) + X^p\right)^{1/p},$$

which is precisely the desired lower bound. □

Sharpness. Let X be an arbitrary real number and fix $\varepsilon > 0$. Pick $\varphi_0 \in (0, \pi/2 - \pi/(2p))$ and $M < 0$ such that the angle $A = \{(x,y) : x > 0,\ y \geq M + x\tan\varphi_0\}$ contains the point $(X, 0)$. Let $\mu_{\partial A}^{(X,0)}$ denote the harmonic measure on ∂A with respect to the point $(X,0)$. Clearly, we have

$$\int_{\partial A} x \mathrm{d}\mu_{\partial A}^{(X,0)} = X.$$

We will prove that if φ_0 and M are chosen appropriately, then

$$\int_{\partial A} |x|^p \mathrm{d}\mu_{\partial A}^{(X,0)} = Y^p \tag{2.2}$$

and

$$\int_{\partial A} (|x|^2 + |y|^2)^{p/2} \mathrm{d}\mu_{\partial A}^{(X,0)} < \sin^{-p}\left(\frac{\pi}{2p}\right)(Y^p - |X|^p) + |X|^p + \varepsilon. \tag{2.3}$$

This will yield the claim: if we take F to be the conformal map sending $\mathbb{D}$ onto A and 0 onto $(X,0)$, then we will have $\operatorname{Re} F(0) = X$ and, by (2.2) and (2.3),

$$||\operatorname{Re} F||_{L^p(\mathbb{T})} = Y \quad \text{and} \quad ||F||_{L^p(\mathbb{T})} < (C_p(X,Y)^p + \varepsilon)^{1/p},$$

so the sharpness will hold due to the fact that ε is arbitrary.

By symmetry and continuity, we may and do assume that $X > 0$. Let us start with (2.2). The left-hand side does not change if we translate the angle A and the point $(X,0)$ by the vector $(0,-M)$. For the translated angle, the analysis of the harmonic measure is simpler: the function

$$H(x,y) = \frac{\cos^p \varphi_0}{\cos(p(\varphi_0 - \pi/2))} R^p \cos(p(\varphi - \pi/2))$$

is harmonic on $A + (0,-M)$ and equals $(x,y) \mapsto |x|^p$ on the boundary of this set. Hence

$$\begin{aligned}
\int_{\partial A} |x|^p \mathrm{d}\mu_A^{(X,0)} &= \int_{\partial A} |x|^p \mathrm{d}\mu_{A+(0,-M)}^{(X,-M)} \\
&= H(X,-M) \\
&= (-M)^p \cdot \frac{\cos^p \varphi_0}{\cos(p(\varphi_0 - \pi/2))} \frac{\cos(p(\psi - \pi/2))}{\sin^p \psi},
\end{aligned}$$

where ψ is the angle corresponding to the point $(X,-M)$ (that is, $\psi = \arctan(-M/X)$). So, we can rewrite the above identity in the equivalent form

$$\begin{aligned}
&\int_{\partial A} |x|^p \mathrm{d}\mu_A^{(X,0)} \\
&= \frac{\cos^p \varphi_0}{\cos(p(\varphi_0 - \pi/2))} \cdot (M^2 + X^2)^{p/2} \cos(p(\arctan(-M/X) - \pi/2)).
\end{aligned} \tag{2.4}$$

Now, suppose that φ_0 is close to $\pi/2 - \pi/(2p)$. The largest allowed value of M is $-X\tan\varphi_0$: then the point $(X,0)$ lies at the boundary of A, so $\mu_A^{(X,0)} =$

$\delta_{(X,0)}$ and hence $\int_{\partial A} |x|^p \mathrm{d}\mu_A^{(X,0)} = X^p$. On the other hand, if we let $M \to -\infty$, then $\psi \to \pi/2$ and $\int_{\partial A} |x|^p \mathrm{d}\mu_A^{(X,0)} \to \infty$. Finally, if we fix X and φ_0, then the right-hand side of (2.4) is a strictly increasing function of M. Hence there is a unique number $M = M(\varphi_0)$ such that (2.2) holds. The crucial observation is that $M(\varphi)$ is bounded and $\psi \to \pi/2 - \pi/(2p)$ as $\varphi_0 \to \pi/2 - \pi/(2p)$. Indeed, when φ_0 approaches $\pi/2 - \pi/(2p)$, then we have

$$\frac{\cos^p \varphi_0}{\cos(p(\varphi_0 - \pi/2))} \to \infty,$$

so, if the last expression in (2.4) is equal to Y^p, the term $\cos(p(\psi - \pi/2))$ must tend to 0. This implies $\psi \to \pi/2 - \pi/(2p)$ and, in consequence, $M(\varphi_0) \to -X \cot(\pi/(2p))$. To deal with (2.3), note that

$$\begin{aligned}&\int_{\partial A} (|x|^2 + |y|^2)^{p/2} \mathrm{d}\mu_{\partial A}^{(X,0)} \\ &= \int_{\partial A \cap \{|y| \leq M\}} (|x|^2 + |y|^2)^{p/2} \mathrm{d}\mu_{\partial A}^{(X,0)} + \int_{\partial A \cap \{|y| \geq M\}} (|x|^2 + |y|^2)^{p/2} \mathrm{d}\mu_{\partial A}^{(X,0)}.\end{aligned}$$

As $\varphi_0 \to \pi/2 - \pi/(2p)$, the first integral tends to X^p, since the measures $\mu_{\partial A}^{(X,0)}$ converge weakly to $\delta_{(X,0)}$. To deal with the second integral, observe that $|y| \leq |x| \tan \varphi_0$ when $(x, y) \in \partial A$ and $|y| \geq M$. Consequently,

$$\begin{aligned}&\int_{\partial A \cap \{|y| \geq M\}} (|x|^2 + |y|^2)^{p/2} \mathrm{d}\mu_{\partial A}^{(X,0)} \\ &\leq \cos^{-p} \varphi_0 \int_{\partial A \cap \{|y| \geq M\}} |x|^p \mathrm{d}\mu_{\partial A}^{(X,0)} \\ &= \cos^{-p} \varphi_0 \left[\int_{\partial A} |x|^p \mathrm{d}\mu_{\partial A}^{(X,0)} - \int_{\partial A \cap \{|y| < M\}} |x|^p \mathrm{d}\mu_{\partial A}^{(X,0)} \right] \\ &\to \sin^{-p} \left(\frac{\pi}{2p} \right) \left[Y^p - X^p \right].\end{aligned}$$

Combining all the above facts, we obtain

$$\liminf_{\varphi_0 \to \pi/2 - \pi/(2p)} \int_{\partial A} (|x|^2 + |y|^2)^{p/2} \mathrm{d}\mu_{\partial A}^{(X,0)} = C_p(X, Y),$$

which yields (2.3). This completes the proof. □

3. The case $2 \leq p < \infty$

Recall the number ϕ_p defined in (1.5) and introduce the parameters

$$c_p = \frac{\sin(p\phi_p)}{\cos^{p-1} \phi_p \sin((p-1)\phi_p)}$$

and

$$\alpha_p = -\frac{\sin \phi_p}{\sin((p-1)\phi_p)}.$$

Consider the function $U_p : [0,\infty)\times\mathbb{R}\to\mathbb{R}$, given by

$$U_p(x,y) = \begin{cases} \alpha_p R^p \cos(p\theta) & \text{if } |\theta|\leq \phi_p,\\ R^p - c_p x^p & \text{if } |\theta| > \phi_p,\end{cases}$$

where, as previously, we have used the polar coordinates. Let us extend U_p to the whole $\mathbb{R}^2$, setting $U_p(x,y) = U_p(-x,-y)$ for all x, $y\in\mathbb{R}$. As in the previous section, first we study some elementary properties of this special function.

Lemma 3.1. *(i) We have $U_p(x,y)\geq U_p(x,0)$ for all x, $y\in\mathbb{R}$.*

(ii) For all x, $y\in\mathbb{R}$ we have the majorization

$$U_p(x,y)\leq R^p - c_p|x|^p. \tag{3.1}$$

(iii) The function U_p is subharmonic.

Proof. (i) It suffices to show the inequality $\frac{\partial}{\partial y}U_p(x,y)\geq 0$ for x, $y>0$. This is evident if $\theta\geq\phi_p$, so let us assume that $\theta\in(0,\phi_p)$. A direct differentiation gives

$$\frac{\partial}{\partial y}U_p(x,y) = -p\alpha_p R^{p-1}\sin((p-1)\theta) > 0,$$

as needed.

(ii) By symmetry, we may assume that x, $y>0$. Clearly, it suffices to verify the majorization for $\theta\in(0,\phi_p)$. We rewrite the bound in the equivalent form

$$G(\theta) = \frac{\alpha_p\cos p\theta - 1}{\cos^p\theta}\leq -c_p.$$

Both sides are equal when $\theta=\phi_p$, so it is enough to prove that G is nondecreasing. We derive that $G'(\theta)$ equals

$$\frac{p}{\cos^{p+1}\theta}\big[-\alpha_p\sin((p-1)\theta)-\sin\theta\big] = \frac{p\sin\phi_p}{\cos^{p+1}\theta}\left[\frac{\sin((p-1)\theta)}{\sin((p-1)\phi_p)} - \frac{\sin\theta}{\sin\phi_p}\right].$$

Since $p-1\geq 1$ and $(p-1)\theta\leq(p-1)\phi_p\leq\pi/2$, we will be done if we prove that for any fixed $0<s<t$, the function

$$\xi(\alpha) = \frac{\sin\alpha s}{\sin\alpha t}$$

is nondecreasing on $(0,\pi/(2y))$. We compute that

$$\xi'(\alpha) = \frac{\sin\alpha s}{\alpha\sin\alpha t}\left[\alpha s\cot\alpha s - \alpha t\cot\alpha t\right]$$

and note that the function $\zeta(u) = u\cot u$ is decreasing on $(0,\pi/2)$: $\zeta'(u) = (2\sin^2 u)^{-1}(\sin 2u - 2u)\leq 0$. This implies that $\xi'\leq 0$ and the majorization follows.

(iii) The function U_p is of class C^1 on the plane, and it is harmonic on the set $\{|\theta|<\phi_p\}$. Consequently, it suffices to check that the Laplacian of U_p

is nonnegative on $\{|\theta| > \phi_p\}$. We derive that on this set, we have

$$\begin{aligned}\Delta U_p(x,y) &= p(p-1)R^{p-2}\left[\frac{p}{p-1} - c_p\cos^{p-2}\theta\right]\\ &\geq p(p-1)R^{p-2}\left[\frac{p}{p-1} - c_p\cos^{p-2}\phi_p\right]\\ &= p(p-1)R^{p-2}\left[\frac{p}{p-1} - \frac{\sin(p\phi_p)}{\cos\phi_p\sin((p-1)\phi_p)}\right].\end{aligned}$$

Note that

$$\lim_{r\to 0}\left[\frac{p}{p-1} - \frac{\sin(pr)}{\cos r\sin((p-1)r)}\right] = 0.$$

Therefore, we will be done if we show that the function

$$\xi(r) = \frac{\sin(pr)}{\cos r\sin((p-1)r)}, \qquad r\in(0,\pi/(2p)),$$

is nonincreasing. A direct differentiation shows that

$$\xi'(r) = \frac{\sin\big[2(p-1)r\big] - (p-1)\sin 2r}{2\cos^2 r\sin^2((p-1)r)}.$$

If we denote the numerator by $\zeta(r)$, we see that $\zeta(0) = 0$ and

$$\zeta'(r) = 2(p-1)\big\{\cos\big[2(p-1)r\big] - \cos 2r\big\} \leq 0 \qquad \text{for } r\in(0,\pi/(2p)),$$

since the cosine function is decreasing on $(0,(p-1)\pi/p)$. Thus, ζ is nonpositive on $(0,\pi/(2p))$ and hence ξ is nonincreasing. This proves the claim. □

Equipped with the above lemma, we turn our attention to Theorem 1.1.

Proof of (1.6). The reasoning is the same as in the case $1 < p \leq 2$: we obtain

$$U_p(x,0) \leq U_p(F(0)) \leq \int_{\mathbb{T}} U_p\circ F(u)\mathrm{d}m(u) \leq ||F||_p^p - c_p||\operatorname{Re}F||_p^p,$$

or, equivalently,

$$||F||_p^p \geq c_pY^p + \alpha_p|X|^p.$$

Now, by the definition of c_p, α_p and the identity (1.5), the latter estimate is equivalent to

$$||F||_p \geq \frac{Y}{\cos\phi_p} = C_p(X,Y),$$

which is the claim. □

Sharpness. Here the reasoning is a bit simpler than in the case $1 < p \leq 2$. Consider the angle $A = \{(x,y) : x > 0,\ |\theta| \leq \phi_p\}$ and let μ be the harmonic measure on ∂A with respect to the point $(X,0)$. The restriction of the function U_p to the set A is harmonic and $U_p(x,y) = x^p(\cos^{-p}\phi_p - c_p)$ for $(x,y)\in\partial A$. Consequently, by the mean-value property, we see that

$$\int_{\partial A} x\mathrm{d}\mu = X \quad \text{and} \quad \int_{\partial A} x^p\mathrm{d}\mu = \frac{U_p(X,0)}{\cos^{-p}\phi_p - c_p} = \frac{\alpha_pX^p}{\cos^{-p}\phi_p - c_p} = Y^p.$$

Consequently,

$$\int_{\partial A}(x^2+y^2)^{p/2}\mathrm{d}\mu=\int_{\partial A}(x^2+x^2\tan^2\phi_p)^{p/2}\mathrm{d}\mu=\frac{Y^p}{\cos^p\phi_p}.$$

Therefore, if F is the univalent mapping which sends $\mathbb{D}$ onto A and 0 onto $(X,0)$, then $\operatorname{Re}F(0)=X$, $||\operatorname{Re}F||_{L^p(\mathbb{T})}=Y$ and

$$||F||_{L^p(\mathbb{T})}=\left(\int_{\partial A}(x^2+y^2)^{p/2}\mathrm{d}\mu\right)^{1/p}=\frac{Y}{\cos\phi_p}.$$

So, the lower bound (1.6) is attained and the proof is complete. □

4. On the search of the special functions

In this section we will explain briefly some informal argumentation which leads to the discovery of the special functions U_p used above. We will focus on the case $2\leq p<\infty$, for $1<p<2$ the reasoning is similar. For the sake of clarity, let us start with the general idea behind the proof of Theorem 1.1. Given $2\leq p<\infty$ and a constant $\beta>0$, one searches for the optimal (i.e., the largest) constant $\gamma(p,\beta)$ such that

$$||F||_{L^p(\mathbb{T})}^p\geq\beta||\operatorname{Re}F||_{L^p(\mathbb{T})}^p+\gamma(p,\beta)|\operatorname{Re}F(0)|^p \tag{4.1}$$

for all holomorphic functions F on the unit disc. This clearly gives some initial insight into (1.6): having analyzed (4.1), we see that

$$||F||_{L^p(\mathbb{T})}\geq\sup\left\{(\beta Y^p+\gamma(p,\beta)|X|^p)^{1/p}:\beta>0\right\}. \tag{4.2}$$

Is this bound optimal? To answer this question, *suppose* that for each p and β, there is an extremizer: a nonzero function $F=F^{p,\beta}$ for which both sides are equal. Clearly, for any p and β such an object is not unique: the inequality (4.1) is homogeneous of order p, so if we multiply an extremizer by a constant, we again obtain an extremizer. It is evident how to proceed: we take the number β for which the supremum in (4.2) is attained, consider the extremizer of (4.1), scaled so that $\operatorname{Re}F^{p,\beta}(0)=X$, and verify that it satisfies $||\operatorname{Re}F||_p=Y$. So, we see that the problem boils down to a thorough analysis of the inequality (4.1).

Next, the reasoning presented in the papers [2], [6] and [16] links the validity of the estimate (4.1) with the existence of special functions on the plane. Namely, given p and β, we search for a largest subharmonic function $U_{p,\beta}$ on $\mathbb{R}^2$, satisfying the majorization

$$U_{p,\beta}(x,y)\leq(y^2+x^2)^{p/2}-\beta|x|^p. \tag{4.3}$$

Then, as we have already seen in the previous sections, the mean-value property yields

$$||F||_{L^p(\mathbb{T})}^p\geq\beta||\operatorname{Re}F||_{L^p(\mathbb{T})}^p+U_{p,\beta}(F(0)).$$

Hence, one is forced to take $\gamma(p,\beta)=\inf_{x,y}U_{p,\beta}(x,y)/|x|^p$.

How to find $U_{p,\beta}$? Such a function, if it exists, must satisfy the symmetry condition $U_{p,\beta}(x,y)=U_{p,\beta}(x,-y)=U_{p,\beta}(-x,y)$ for all x, $y\in\mathbb{R}$; indeed, if this did not hold, we could replace $U_{p,\beta}$ with a larger subharmonic function

$$(x,y)\mapsto \max\left\{U_{p,\beta}(x,y),U_{p,\beta}(-x,y),U_{p,\beta}(x,-y),U_{p,\beta}(-x,-y)\right\},$$

for which the majorization (4.3) is still valid. A similar argument shows that $U_{p,\beta}$ must be homogeneous of order p: otherwise $U_{p,\beta}$ would be strictly majorized by the subharmonic function $\sup_{\lambda>0}\lambda^p U_{p,\beta}(\cdot/\lambda,\cdot/\lambda)$. So, $U_{p,\beta}$ can be written in polar coordinates as

$$U_{p,\beta}(x,y)=R^p g_{p,\beta}(\theta),$$

for some function $g_{p,\beta}$ to be found. Now, we *assume* that $U_{p,\beta}$ is of class C^2; despite the fact that the function we obtain at the end does not have this regularity, it will facilitate our further considerations. A closer look at the papers [2], [6] and [16] suggests that there is a number $\kappa(p,\beta)>0$ such that either

(i) $U_{p,\beta}$ is harmonic on the set $\{(x,y):|y|<\kappa(p,\beta)|x|\}$, and $U_{p,\beta}(x,y)=(y^2+x^2)^{p/2}-\beta|x|^p$ on $\{(x,y):|y|\geq\kappa(p,\beta)|x|\}$,

or

(ii) $U_{p,\beta}$ is harmonic on the set $\{(x,y):|x|<\kappa(p,\beta)|y|\}$, and $U_{p,\beta}(x,y)=(y^2+x^2)^{p/2}-\beta|x|^p$ on $\{(x,y):|x|\geq\kappa(p,\beta)|y|\}$.

So, we have two possibilities to check. Suppose that (i) holds true and take $\phi(p,\beta)\in(0,\pi/2)$ such that $\kappa(p,\beta)=\tan\phi(p,\beta)$. A direct calculation shows that $\Delta U_{p,\beta}=0$ if and only if $g''(\theta)+p^2g(\theta)=0$, so we see that

$$g(\theta)=a_1\cos(p\theta)+a_2\sin(p\theta),\qquad \theta\in[-\phi(p,\beta),\phi(p,\beta)],$$

for some unknown constants a_1, a_2. Since $U_{p,\beta}(x,y)=U_{p,\beta}(x,-y)$, we conclude that $a_2=0$. To derive a_1, we use the fact that $U_{p,\beta}$ is of class C^1 and see what happens on the common boundary of the sets $\{(x,y):|y|>\kappa(p,\beta)|x|\}$ and $\{(x,y):|y|\leq\kappa(p,\beta)|x|\}$. By the continuity of $U_{p,\beta}$, we get that

$$a_1\cos(p\phi(p,\beta))=1-\beta\cos^p\phi(p,\beta),$$

while the comparison of the partial derivatives yields

$$-a_1\sin(p\phi(p,\beta))=\beta\cos^{p-1}\phi(p,\beta)\sin\phi(p,\beta).$$

The system of these two equations can be easily solved: we get

$$\beta=\frac{\sin(p\phi(p,\beta))}{\cos^{p-1}\phi(p,\beta)\sin((p-1)\phi(p,\beta))}$$

and

$$a_1=-\frac{\sin\phi(p,\beta)}{\sin((p-1)\phi(p,\beta))}.$$

This suggests to use the number $\phi=\phi(p,\beta)\in(0,\pi/2)$ as a “free” parameter (instead of β). We obtain the candidate for the special function $U_{p,\beta}$; it can be checked, with similar calculations as those presented in Section 3, that it enjoys all the required properties; furthermore,

$$\gamma(p,\beta) = \inf_{x,y>0} \frac{U_{p,\beta}(x,y)}{|x|^p} = a_1 = -\frac{\sin\phi(p,\beta)}{\sin((p-1)\phi(p,\beta))}.$$

It remains to check that the right-hand side of (4.2) is precisely the constant $C_p(X,Y)$; when we take $\phi(p,\beta)=\phi_p$ (defined in (1.5)), then $U_{p,\beta}$ is precisely the function U_p used above.

Finally, let us say a few words about the search for the appropriate extremizers F in (4.1). A look at the above proof immediately gives three conditions on F. First, we must have equality in the majorization (4.3), i.e.,

$$U_{p,\beta}(F(e^{i\theta})) = |F(e^{i\theta})|^p - \beta|\operatorname{Re} F(e^{i\theta})|^p$$

for almost all $\theta \in [-\pi,\pi)$. That is, F must send the unit circle into the set $\{(x,y) : |x| \geq \kappa(p,\beta)|y|\}$. The second condition is that the mean value property, applied to the subharmonic function $U_{p,\beta}\circ F$, must return equality: this suggests that F must send the open unit disc into $\{(x,y) : x>0,\ |y| < \kappa(p,\beta)|x|\}$, inside which $U_{p,\beta}$ is harmonic. Finally, we must have $F(0)=X$; this will guarantee the equality $U_{p,\beta}(F(0)) = \gamma(p,\beta)|X|^p$. Let us combine the three observations: we see that a natural choice for F is a conformal mapping of $\mathbb{D}$ onto the angle $A=\{(x,y) : x>0, |y|\leq\kappa(p,\beta)x\}$, sending $0\in\mathbb{D}$ onto $(X,0)\in A$. One can check that this is indeed the right choice.

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