BEST CONSTANTS IN THE WEAK-TYPE ESTIMATES FOR UNCENTERED MAXIMAL OPERATORS

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ABSTRACT. Let μ be a Borel measure on $\mathbb{R}$. The paper contains the proofs of the estimates

$$||\mathcal{M}_\mu f||_{L^{q,\infty}(A,\mu)} \leq c_{p,q}||f||_{L^p(\mathbb{R},\mu)}\, \mu(A)^{1/q-1/p}, \qquad 1 \leq p < \infty,\, q \in (0,p],$$

and

$$||\mathcal{M}_\mu f||_{L^{q,\infty}(A,\mu)} \leq C_{p,q}||f||_{L^{p,\infty}(\mathbb{R},\mu)}\, \mu(A)^{1/q-1/p}, \quad 1 < p < \infty, q \in (0,p].$$

Here A is a subset of $\mathbb{R}$, f is a μ-locally integrable function, $\mathcal{M}_\mu$ is the uncentered maximal operator with respect to μ and $c_{p,q}$, $C_{p,q}$ are finite constants depending only on the parameters indicated. In the case when μ is the Lebesgue measure, the optimal choices for $c_{p,q}$ and $C_{p,q}$ are determined. As an application, we present some related tight bounds for the strong maximal operator on $\mathbb{R}^n$ with respect to a general product measure.

1. Introduction

Suppose that μ is a nonnegative Borel measure on $\mathbb{R}^n$ and let $f : \mathbb{R}^n \to \mathbb{R}$ be a μ-locally integrable function. The uncentered maximal function of f with respect to μ is given by the formula

$$(\mathcal{M}_\mu f)(x) = \sup_{x\in B} \frac{1}{\mu(B)} \int_B |f| \mathrm{d}\mu,$$

where the supremum is taken over all closed balls B which contain the point x. If μ is the Lebesgue measure, then $\mathcal{M}_\mu$ is the usual uncentered maximal operator of Hardy and Littlewood. It is well-known (see e.g. Stein [6]) that if μ satisfies the doubling condition

$$\mu(B(x, 2r)) \leq C\mu(B(x, r)) \qquad \text{for some } C < \infty \text{ and all } x \in \mathbb{R}^n,\ r > 0$$

(here $B(x, r)$ denotes the closed ball of center x and radius r), then $\mathcal{M}_\mu$ maps $L^p(\mathbb{R}^n, \mu)$ into itself for $p > 1$, and $L^1(\mathbb{R}^n, \mu)$ into $L^{1,\infty}(\mathbb{R}^n, \mu)$. This is still true, without the doubling property, if and only if $n = 1$ (see [1], [2] and [5]).

The question about the precise evaluation of the strong and weak norms of $\mathcal{M}_\mu$ has gained some interest in the literature, and the objective of this paper is to establish two new results of this type. We will be particularly interested in the one-dimensional case. We have the following L^p-estimates for $\mathcal{M}_\mu$: for any μ-locally integrable f and $1 < p < \infty$ we have

$$||\mathcal{M}_\mu f||_{L^p(\mathbb{R},\mu)} \leq c_p||f||_{L^p(\mathbb{R},\mu)}, \tag{1.1}$$

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where c_p is the unique positive solution of the equation

$$(p-1)x^p - px^{p-1} - 1 = 0. \tag{1.2}$$

This statement, with μ being the Lebesgue measure, was proved by Grafakos and Montgomery-Smith in [3]; for the general case, consult Grafakos and Kinnunen [2]. In general, the constant c_p in (1.1) cannot be replaced by a smaller number, see [3]. The L^1-inequality does not hold in general with any finite constant c_1, but we have the sharp weak-type estimate

$$||\mathcal{M}_\mu f||_{L^{1,\infty}(\mathbb{R},\mu)} \leq 2||f||_{L^1(\mathbb{R},\mu)},$$

as proved in [2]. Here, as usual, for any Borel subset A of $\mathbb{R}$ and any $0 < p < \infty$, we define the weak p-th norm of f on A by the formula

$$||f||_{L^{p,\infty}(A,\mu)} = \sup_{\lambda > 0} \lambda \big[\mu(\{x \in A : |f(x)| > \lambda\})\big]^{1/p}.$$

There is a natural question about the best constants in the corresponding weak-type (p,p) estimates for $\mathcal{M}_\mu$, $1 < p < \infty$. In fact, we will study this question in a more general setting and compare the weak q-th norm of $\mathcal{M}_\mu f$ to the p-th norm of f, where $p \geq 1$ and $q \in (0,p]$. Introduce the constant

$$C_p = \frac{(p-1)(2^{p/(p-1)}-1)}{p}\Big((p-1)(2^{p/(p-1)}-2)\Big)^{-1/p}$$

when $1 < p < \infty$, and put $C_1 = 2$. We will establish the following result.

Theorem 1.1. *For any μ-locally integrable function $f : \mathbb{R} \to \mathbb{R}$, any Borel subset A of $\mathbb{R}$ and any $1 \leq p < \infty$, $q \in (0,p]$ we have*

$$||\mathcal{M}_\mu f||_{L^{q,\infty}(A,\mu)} \leq C_p ||f||_{L^p(\mathbb{R},\mu)} \mu(A)^{1/q-1/p}. \tag{1.3}$$

If μ is the Lebesgue measure, then the constant C_p is the best possible.

In particular, if $p = q$, then (1.3) yields the weak-type (p,p) estimate

$$||\mathcal{M}_\mu f||_{L^{p,\infty}(\mathbb{R},\mu)} \leq C_p ||f||_{L^p(\mathbb{R},\mu)}, \tag{1.4}$$

which, as we will see, is also sharp provided μ is the Lebesgue measure.

The next problem we will study concerns the sharp comparison of the weak norms of f and $\mathcal{M}_\mu f$. Here the constants c_p of Grafakos and Montgomery-Smith come into play; we will prove the following statement.

Theorem 1.2. *For any μ-locally integrable function $f : \mathbb{R} \to \mathbb{R}$, any Borel subset A of $\mathbb{R}$ and any $1 < p < \infty$, $q \in (0,p]$ we have*

$$||\mathcal{M}_\mu f||_{L^{q,\infty}(A,\mu)} \leq c_p ||f||_{L^{p,\infty}(\mathbb{R},\mu)} \mu(A)^{1/q-1/p}. \tag{1.5}$$

If μ is the Lebesgue measure, then the constant c_p is the best possible.

As previously, let us distinguish the choice $p = q \in (1,\infty)$. It gives the bound

$$||\mathcal{M}_\mu f||_{L^{p,\infty}(\mathbb{R},\mu)} \leq c_p ||f||_{L^{p,\infty}(\mathbb{R},\mu)}, \tag{1.6}$$

which will be proved to be sharp in the case when μ is the Lebesgue measure.

Theorems 1.1 and 1.2 will be established in the next section. In Section 3 we will apply these two theorems to obtain related results in the higher-dimensional setting: more precisely, we will show tight weak-type estimates for the so-called strong maximal operator on $\mathbb{R}^n$, $n \geq 2$.

2. Proofs of Theorems 1.1 and 1.2

We start with recalling the main lemma from [2] (see also [3] for the special case in which μ is the Lebesgue measure). This result can be regarded as the appropriate version of the weak-type estimate for $\mathcal{M}_\mu$. Here and below, we use the notation $\{f>\lambda\}$ for the set $\{x\in\mathbb{R}: f(x)>\lambda\}$.

Lemma 2.1. *If f is a nonnegative and μ-locally integrable function on $\mathbb{R}$, then for any $\lambda>0$ we have*

$$\lambda\Big(\mu\left(\{\mathcal{M}_\mu f>\lambda\}\right)+\mu\left(\{f>\lambda\}\right)\Big)\leq\int_{\{\mathcal{M}_\mu f>\lambda\}} f\,d\mu+\int_{\{f>\lambda\}} f\,d\mu. \tag{2.1}$$

In other words, for any f, λ as in the statement above, we have

$$\int_{\mathbb{R}} u\big(f(x)/\lambda, \mathcal{M}_\mu f(x)/\lambda\big)\mathrm{d}\mu(x)\leq 0, \tag{2.2}$$

where $u:[0,\infty)\times[0,\infty]\to\mathbb{R}$ is the function given by the formula

$$u(x,y)=\big(\chi_{\{x>1\}}+\chi_{\{y>1\}}\big)(1-x).$$

Introduce the parameters

$$r_p=\frac{p}{(p-1)(2^{p/(p-1)}-1)},\qquad s_p=\frac{p\,2^{1/(p-1)}}{(p-1)(2^{p/(p-1)}-1)}$$

and

$$\alpha_p=\frac{2^{p/(p-1)}-1}{2^{p/(p-1)}-2}.$$

Lemma 2.2. *For any $0\leq x\leq y$ and any $1<p<\infty$, we have*

$$\alpha_p u(x,y)\geq\chi_{\{y>1\}}-C_p^p x^p. \tag{2.3}$$

Proof. If $y\leq 1$, then the estimate becomes $0\geq -C_p^p x^p$, which is obvious. Suppose that $y>1$ and $x\leq 1$. Then (2.3) is equivalent to

$$F(x):=\alpha_p(1-x)-1+C_p^p x^p\geq 0,$$

which holds true for all $x\geq 0$. This is the consequence of the fact that F is a convex function, combined with the equalities $F(r_p)=F'(r_p)=0$. Finally, if both x and y are larger than 1, the inequality (2.3) can be rewritten in the form

$$G(x):=2\alpha_p(1-x)-1+C_p^p x^p\geq 0,$$

which follows from the convexity of G and the equalities $G(s_p)=G'(s_p)=0$. □

Proof of (1.3). We may assume that f is a nonnegative function which satisfies $||f||_{L^p(\mathbb{R},\mu)}<\infty$. Combining (2.2) and (2.3), we obtain that for $p>1$,

$$\lambda^p\mu\big(\{\mathcal{M}_\mu f>\lambda\}\big)\leq C_p^p||f||^p_{L^p(\mathbb{R},\mu)}. \tag{2.4}$$

This bound is also true for $p=1$, as we have already mentioned above. Thus, since $\mu\big(\{x\in A:\mathcal{M}_\mu f(x)>\lambda\}\big)\leq\min\big\{\mu(A),\mu(\{\mathcal{M}_\mu f>\lambda\})\big\}$, we have

$$\begin{aligned}\lambda^q\mu(\{x\in A:\mathcal{M}_\mu f(x)>\lambda\})&\leq\lambda^q\mu(\{\mathcal{M}_\mu f\geq\lambda\})^{q/p}\mu(A)^{1-q/p}\\&\leq C_p^q||f||^q_{L^p(\mathbb{R},\mu)}\mu(A)^{1-q/p},\end{aligned} \tag{2.5}$$

where the latter passage is due to (2.4). It remains to take supremum over λ in (2.5) to obtain (1.3). □

Sharpness for the Lebesgue measure. Let r_p, s_p be as above and introduce the parameter $\beta_p = 2(s_p-1)/(1-r_p)$. Consider the function

$$f = s_p\chi_{[-1,1]} + r_p\left(\chi_{[-\beta_p-1,-1)} + \chi_{(1,\beta_p+1]}\right)$$

and let $A = [-\beta_p-1, \beta_p+1]$. The identity

$$\frac{1}{\big|[-\beta_p-1,1]\big|}\int_{-\beta_p-1}^{1} f(x)\mathrm{d}x = \frac{1}{\big|[-1,\beta_p+1]\big|}\int_{-1}^{\beta_p+1} f(x)\mathrm{d}x = \frac{2s_p+\beta_p r_p}{2+\beta_p} = 1$$

and the definition of the maximal operator imply that $\mathcal{M}_{|\cdot|}f(x) \geq 1$ for $x \in A$. Therefore,

$$\frac{|\{x\in A : \mathcal{M}_{|\cdot|}f(x)\geq 1\}|}{||f||^q_{L^p(\mathbb{R},|\cdot|)}|A|^{1-q/p}} = \left(\frac{|A|}{||f||^p_{L^p(\mathbb{R},|\cdot|)}}\right)^{q/p} = \left(\frac{2(\beta_p+1)}{2\beta_p r_p^p + 2s_p^p}\right)^{q/p},$$

and the latter expression is easily checked to be equal to C_p^q. This proves the sharpness of (1.3). The same example yields the optimality of C_p in (1.4): we have

$$||\mathcal{M}_{|\cdot|}f||^p_{L^{p,\infty}(\mathbb{R},|\cdot|)} \geq |\{\mathcal{M}_{|\cdot|}f\geq 1\}| \geq |A| = C_p^p||f||^p_{L^p(\mathbb{R},|\cdot|)}. \quad \square$$

Proof of (1.5). It suffices to consider functions f which are nonnegative and satisfy $0 < ||f||_{L^{p,\infty}(\mathbb{R},\mu)} < \infty$. In addition, by homogeneity, we may and do assume that $||f||_{L^{p,\infty}(\mathbb{R},\mu)} = 1$. Rewrite (2.1) in the form

$$\lambda\mu(\{\mathcal{M}_\mu f > \lambda\}) \leq \int_{\{\mathcal{M}_\mu f>\lambda\}} f\mathrm{d}\mu + \int_{\{f>\lambda\}}\big(f-\lambda\big)\mathrm{d}\mu.$$

The well-known inequality of Hardy and Littlewood (see e.g. [4]) states that if h is a nonnegative function and A is a Borel subset of $\mathbb{R}$, then

$$\int_A h\mathrm{d}\mu \leq \int_0^{\mu(A)} h^*(t)\mathrm{d}t, \tag{2.6}$$

where $h^*(t) = \inf\big\{s>0 : \mu(\{f>s\})\leq t\big\}$ is the nonincreasing rearrangement of h. Since $||f||_{L^{p,\infty}(\mathbb{R},\mu)} = 1$, we have $\mu(\{f>\lambda\}) \leq \lambda^{-p}$ for all $\lambda > 0$ and hence $f^*(t) \leq t^{-1/p}$ for all positive t. Putting all these facts together, we obtain

$$\begin{aligned}\lambda\,\mu(\{\mathcal{M}_\mu f>\lambda\}) &\leq \int_0^{\mu(\{\mathcal{M}_\mu f>\lambda\})} t^{-1/p}\mathrm{d}t + \int_0^{\lambda^{-p}}(t^{-1/p}-\lambda)\mathrm{d}t\\ &= \frac{p}{p-1}\mu(\{\mathcal{M}_\mu f>\lambda\})^{(p-1)/p} + \frac{\lambda^{1-p}}{p-1}.\end{aligned}$$

Multiplying both sides by $(p-1)\lambda^{p-1}$ yields

$$(p-1)\lambda^p\mu(\{\mathcal{M}_\mu f>\lambda\}) \leq p\big(\lambda^p\mu(\{\mathcal{M}_\mu f>\lambda\})\big)^{(p-1)/p} + 1.$$

In view of (1.2), this implies

$$\lambda^p\mu(\{\mathcal{M}_\mu f>\lambda\}) \leq c_p^p = c_p^p||f||_{L^{p,\infty}(\mathbb{R},\mu)}. \tag{2.7}$$

Indeed, we have $c_p \geq 1$ and the function $x \mapsto (p-1)x^p - px^{p-1}$ is increasing on $[1,\infty)$). Thus we have established (1.6). Furthermore, (2.7) yields

$$\lambda^q\mu(\{x\in A : \mathcal{M}_\mu f(x)>\lambda\}) \leq c_p^q||f||^q_{L^{p,\infty}(\mathbb{R},\mu)}\mu(A)^{1-q/p},$$

which can be seen by repeating the argument leading from (2.4) to (2.5). The proof of (1.5) is complete. $\square$

Sharpness for the Lebesgue measure. Fix $p > 1$ and let $f : \mathbb{R} \to \mathbb{R}$ be given by $f(t) = |2t|^{-1/p}$. It is easy to check that $||f||_{L^{p,\infty}(\mathbb{R})} = 1$. Furthermore, for any $x > 0$ we have

$$\frac{1}{|[-c_p^{-p}x, x]|}\int_{-c_p^p x}^{x} f(t)\mathrm{d}t = (2x)^{-1/p}\frac{p(1+c_p^{1-p})}{(p-1)(1+c_p^{-p})} = c_p(2x)^{-1/p}, \tag{2.8}$$

where the latter equality follows from (1.2). Thus, by the definition of the maximal operator, we have $\mathcal{M}_{|\cdot|}f(x) \geq c_p(2x)^{-1/p}$ for $x > 0$ and, similarly, $\mathcal{M}_{|\cdot|}f(x) \geq c_p(-2x)^{-1/p}$ for negative x. Consequently, $||\mathcal{M}_{|\cdot|}f||_{L^{p,\infty}(\mathbb{R},|\cdot|)} \geq c_p$ and the equality in (1.6) is attained. Next, putting $A = \{\mathcal{M}_{|\cdot|}f \geq 1\}$, we see that $[-c_p^p/2, c_p^p/2] \subseteq A$ and hence

$$||\mathcal{M}_{|\cdot|}f||^q_{L^{q,\infty}(A,|\cdot|)} \geq |A| \geq c_p^q|A|^{1-q/p} = c_p^q|A|^{1-q/p}||f||^q_{L^{p,\infty}(\mathbb{R},|\cdot|)}.$$

This yields the desired optimality of c_p in (1.5). □

3. Estimates for the strong maximal function

This section contains applications of the previous results to the study of maximal operators in higher dimensions. Let $n \geq 1$ be a fixed integer and let μ be a product measure on $\mathbb{R}^n$: $\mu = \mu_1 \otimes \mu_2 \otimes \ldots \otimes \mu_n$ for some Borel measures $\mu_1, \mu_2, \ldots, \mu_n$ on $\mathbb{R}$. The strong maximal operator M_μ is an operator which acts on μ-locally integrable functions f by the formula

$$M_\mu f(x) = \sup_{x\in D}\frac{1}{\mu(D)}\int_D |f|\mathrm{d}\mu,$$

where the supremum is taken over all closed rectangles D, with sides parallel to the axes, satisfying $x \in D$. Observe that for $n = 1$ the operators M_μ and $\mathcal{M}_\mu$ coincide.

We will prove the following fact.

Theorem 3.1. *Let μ and M_μ be as above.*

(i) *If $n \geq 2$, then in general M_μ* does not *map $L^1(\mathbb{R}^n, \mu)$ into $L^{1,\infty}(\mathbb{R}^n, \mu)$.*

(ii) *If $1 < p < \infty$, then for any $f : \mathbb{R}^n \to \mathbb{R}$ we have*

$$||M_\mu f||_{L^{p,\infty}(\mathbb{R}^n,\mu)} \leq C_p c_p^{n-1}||f||_{L^p(\mathbb{R}^n\mu)}. \tag{3.1}$$

If μ is the Lebesgue measure on $\mathbb{R}^n$, then the constant has the optimal order $O((p-1)^{1-n})$ as $p \to 1$.

(iii) *If $1 < p < \infty$, then for any $f : \mathbb{R}^n \to \mathbb{R}$ we have*

$$||M_\mu f||_{L^{p,\infty}(\mathbb{R}^n,\mu)} \leq c_p^n||f||_{L^{p,\infty}(\mathbb{R}^n,\mu)}. \tag{3.2}$$

If μ is the Lebesgue measure on $\mathbb{R}^n$, then the constant is the best possible.

Remark 3.2. *By the argument from the previous section,* (3.1) *and* (3.2) *imply the estimates*

$$||M_\mu f||_{L^{q,\infty}(A,\mu)} \leq C_p c_p^{n-1}||f||_{L^p(\mathbb{R}^n\mu)}\mu(A)^{1/q-1/p}$$

and

$$||M_\mu f||_{L^{q,\infty}(A,\mu)} \leq c_p^n||f||_{L^{p,\infty}(\mathbb{R}^n,\mu)}\mu(A)^{1/q-1/p} \tag{3.3}$$

for all μ-locally integrable functions $f : \mathbb{R}^n \to \mathbb{R}$, all Borel subsets A of $\mathbb{R}^n$ and all $1 < p < \infty$, $0 < q \leq p$. We will prove below that (3.3) *is sharp provided μ is the Lebesgue measure.*

Proof of Theorem 3.1. (i) This will be shown in the proof of (ii) below.

(ii) The key observation is that

$$M_\mu \leq \mathcal{M}^{(1)}_{\mu_1} \circ \mathcal{M}^{(2)}_{\mu_2} \circ \ldots \circ \mathcal{M}^{(n)}_{\mu_n}, \tag{3.4}$$

where $\mathcal{M}^{(k)}_{\mu_k}$ denotes the maximal operator $\mathcal{M}_{\mu_k}$ applied to the k-th coordinate. Let f be a nonnegative function on $\mathbb{R}^n$ satisfying $||f||_{L^p(\mathbb{R}^n,\mu)} < \infty$. Using (1.4) with respect to $\mathcal{M}_{\mu_1}$ and then (1.1) with respect to $\mathcal{M}_{\mu_2}$, $\mathcal{M}_{\mu_3}$, ..., $\mathcal{M}_{\mu_n}$, we obtain

$$\begin{aligned}
&\lambda^p \mu\big(\{\mathcal{M}^{(1)}_{\mu_1} \circ \mathcal{M}^{(2)}_{\mu_2} \circ \ldots \circ \mathcal{M}^{(n)}_{\mu_n} f > \lambda\}\big) \\
&= \int_{\mathbb{R}^{n-1}} \lambda^p \mu_1 \left(\{x_1 : \mathcal{M}^{(1)}_{\mu_1} \circ \ldots \circ \mathcal{M}^{(n)}_{\mu_n} f(x_1, x_2, \ldots, x_n) > \lambda\}\right) \mathrm{d}\mu_2(x_2) \ldots \mathrm{d}\mu_n(x_n) \\
&\leq C_p^p \int_{\mathbb{R}^{n-1}} \left[\int_{\mathbb{R}} \big[\mathcal{M}^{(2)}_{\mu_2} \circ \ldots \circ \mathcal{M}^{(n)}_{\mu_n} f(x_1, x_2, \ldots, x_n)\big]^p \mathrm{d}\mu_1(x_1)\right] \mathrm{d}\mu_2(x_2) \ldots \mathrm{d}\mu_n(x_n) \\
&= C_p^p \int_{\mathbb{R}^n} \big[\mathcal{M}^{(2)}_{\mu_2} \circ \ldots \circ \mathcal{M}^{(n)}_{\mu_n} f(x_1, x_2, \ldots, x_n)\big]^p \mathrm{d}\mu_1(x_1)\mathrm{d}\mu_2(x_2) \ldots \mathrm{d}\mu_n(x_n) \\
&= C_p^p \int_{\mathbb{R}^{n-1}} \left[\int_{\mathbb{R}} \big[\mathcal{M}^{(2)}_{\mu_2} \circ \ldots \circ \mathcal{M}^{(n)}_{\mu_n} f(x)\big]^p \mathrm{d}\mu_2(x_2)\right] \mathrm{d}\mu_1(x_1)\mathrm{d}\mu_3(x_3) \ldots \mathrm{d}\mu_n(x_n) \\
&\leq C_p^p c_p^p \int_{\mathbb{R}^n} \big[\mathcal{M}^{(3)}_{\mu_3} \circ \ldots \circ \mathcal{M}^{(n)}_{\mu_n} f(x_1, x_2, \ldots, x_n)\big]^p \mathrm{d}\mu_1(x_1)\mathrm{d}\mu_2(x_2) \ldots \mathrm{d}\mu_n(x_n) \\
&\leq \ldots \\
&\leq C_p^p c_p^{(n-1)p} ||f||^p_{L^p(\mathbb{R}^n,\mu)}.
\end{aligned}$$

This yields (3.1). It is not difficult to check that $1 \leq C_p \leq 2$ and $\frac{p}{p-1} \leq c_p \leq \frac{2p}{p-1}$ for $1 < p < \infty$, so the constant $C_p c_p^{n-1}$ is of order $O((p-1)^{1-n})$ when $p \to 1$. To see that this order is optimal when μ is the Lebesgue measure, take $p \in (1, 2)$, $n \geq 2$ and put $f = \chi_{[-1,1]^n}$. Then, for any $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$, we have

$$M_\mu f(x) \geq \prod_{k=1}^n \min\left(\frac{2}{|x_k| + 1}, 1\right),$$

which can be verified by considering the smallest rectangle which contains x and the cube $[-1, 1]^n$. Thus, for any $\lambda \in (0, 1)$ we may write

$$\begin{aligned}
|\{M_\mu f > \lambda\}| &\geq 2^n \left|\left\{x \in [1, \infty)^n : \prod_{k=1}^n \frac{2}{x_k + 1} > \lambda\right\}\right| \\
&= 2^n \int_1^{a_1} \int_1^{a_2} \ldots \int_1^{a_n} \mathrm{d}x_n \mathrm{d}x_{n-1} \ldots \mathrm{d}x_1,
\end{aligned} \tag{3.5}$$

where $a_1 = 2/\lambda - 1$ and

$$a_k = \frac{2^k}{\lambda(x_1 + 1) \ldots (x_{k-1} + 1)} - 1, \qquad k = 2, 3, \ldots, n.$$

Denote the right-hand side of (3.5) by γ_n. Deriving the inner integral with respect to x_n gives the identity

$$\gamma_n = 2^n \int_1^{a_1} \int_1^{a_2} \ldots \int_1^{a_{n-1}} \frac{2^n}{\lambda(x_1 + 1) \ldots (x_{n-1} + 1)} \mathrm{d}x_{n-1} \ldots \mathrm{d}x_1 - 4\gamma_{n-1},$$

valid for $n \geq 2$. By induction, we easily verify that

$$\int_1^{a_k} \dots \int_1^{a_{n-1}} \frac{1}{(x_k+1)\dots(x_{n-1}+1)} \mathrm{d}x_{n-1}\dots \mathrm{d}x_k = \frac{1}{(n-k)!}\left(\log \frac{a_k+1}{2}\right)^{n-k}$$

and hence

$$\frac{\gamma_n}{4^n} = \frac{\left(\log \lambda^{-1}\right)^{n-1}}{\lambda(n-1)!} - \frac{\gamma_{n-1}}{4^{n-1}}. \tag{3.6}$$

This, in turn, implies that for $n \geq 3$,

$$\frac{\gamma_n}{4^n} = \frac{\left(\log \lambda^{-1}\right)^{n-1}}{\lambda(n-1)!} - \frac{\left(\log \lambda^{-1}\right)^{n-2}}{\lambda(n-2)!} + \frac{\gamma_{n-2}}{4^{n-2}} > \frac{\left(\log \lambda^{-1}\right)^{n-1}}{\lambda(n-1)!} - \frac{\left(\log \lambda^{-1}\right)^{n-2}}{\lambda(n-2)!}. \tag{3.7}$$

This is also true for $n = 2$: we have $\gamma_1 = 4(\lambda^{-1}-1)$ and hence, by (3.6),

$$\frac{\gamma_2}{4} = \frac{\log \lambda^{-1}}{\lambda} - \frac{1}{\lambda} + 1.$$

Consequently, we have $\lim_{\lambda\to 0} \lambda|\{M_\mu f > \lambda\}| = \infty$ and (i) is proved. Next, if we plug $\lambda = \exp(-(n-1)/(p-1))$ into (3.7), we obtain that

$$\begin{aligned}\frac{||M_\mu f||^p_{L^{p,\infty}(\mathbb{R}^n,|\cdot|)}}{||f||^p_{L^p(\mathbb{R}^n,|\cdot|)}} &\geq \frac{\lambda^p|\{M_\mu f > \lambda\}|}{2^n} \\ &> 2^n e^{1-n}\frac{(n-1)^{n-1}}{(n-1)!}\frac{2-p}{(p-1)^{n-1}} \\ &\geq \frac{\kappa_n}{(p-1)^{(n-1)p}},\end{aligned}$$

for some constant κ_n depending only on n. This gives the optimality of the order.

(iii) Introduce the operators $T_k = \mathcal{M}^{(k)}_{\mu_k} \circ \mathcal{M}^{(k+1)}_{\mu_{k+1}} \circ \dots \circ \mathcal{M}^{(n)}_{\mu_n}$, $k = 1, 2, \dots, n$, and let $T_{n+1} =$Id. We will prove that

$$||T_k f||_{L^{p,\infty}(\mathbb{R},\mu)} \leq c_p ||T_{k+1} f||_{L^{p,\infty}(\mathbb{R}^n \mu)} \tag{3.8}$$

for any f and any $k \in \{1, 2, \dots, n\}$; this will immediately yield (3.2). To do this, fix $\lambda > 0$ and let $A_\lambda = \{T_k f > \lambda\}$, $B_\lambda = \{T_{k+1} f > \lambda\}$. Let $\mu^{(k)}$ denote the product measure $\mu_1 \otimes \mu_2 \otimes \dots \otimes \mu_{k-1} \otimes \mu_{k+1} \otimes \dots \otimes \mu_n$ on $\mathbb{R}^{n-1}$. By (2.1), applied to $\mathcal{M}^{(k)}_{\mu_k}$, the measure μ_k and the function $t \mapsto T_{k+1} f(x_1, \dots, x_{k-1}, t, x_{k+1}, \dots, x_n)$, $t \in \mathbb{R}$,

$$\begin{aligned}&\lambda\mu_k\big(\{x_k \in \mathbb{R} : T_k f(x_1, x_2, \dots, x_n) > \lambda\}\big) \\ &\leq \int_{\{x_k\in\mathbb{R}:T_k f(x)>\lambda\}} T_{k+1} f(x)\mathrm{d}\mu_k(x_k) + \int_{\{x_k\in\mathbb{R}:T_{k+1} f(x)>\lambda\}} \big(T_{k+1} f(x) - \lambda\big)\mathrm{d}\mu_k(x_k).\end{aligned}$$

Integrating this over $\mathbb{R}^{n-1}$ with respect to $\mathrm{d}\mu^{(k)}(x_1, x_2, \dots, x_{k-1}, x_{k+1}, \dots, x_n)$ and multiplying both sides by λ^{p-1}, we obtain

$$\lambda^p \mu(A_\lambda) \leq \lambda^{p-1}\left[\int_{A_\lambda} T_{k+1} f(x)\mathrm{d}\mu(x) + \int_{B_\lambda} \big(T_{k+1} f(x) - \lambda\big)\mathrm{d}\mu(x)\right].$$

Let $(T_{k+1} f)^*$ be the nonincreasing rearrangement of $T_{k+1} f$ (the definition is analogous to that of the one-dimensional setting). We have

$$\mu(B_\lambda) = \mu(\{T_{k+1} f > \lambda\}) \leq \lambda^{-p} ||T_{k+1} f||^p_{L^{p,\infty}(\mathbb{R}^n,\mu)}, \tag{3.9}$$

so $(T_{k+1}f)^*(t) \leq t^{-1/p}||T_{k+1}f||_{L^{p,\infty}(\mathbb{R}^n,\mu)}$ for any $t>0$. Therefore, using the version of the inequality (2.6) in $\mathbb{R}^n$, we obtain

$$\lambda^p\mu(A_\lambda) \leq \lambda^{p-1}\left[\int_0^{\mu(A_\lambda)} t^{-1/p}||T_{k+1}f||_{L^{p,\infty}(\mathbb{R}^n,\mu)}\mathrm{d}t + \int_0^{\mu(B_\lambda)} \left(t^{-1/p}||T_{k+1}f||_{L^{p,\infty}(\mathbb{R}^n,\mu)}-\lambda\right)\mathrm{d}t\right].$$

If we apply (3.9) and compute the integrals above, we obtain an inequality which can be rewritten in the equivalent form

$$(p-1)\frac{\lambda^p\mu(A_\lambda)}{||T_{k+1}f||^p_{L^{p,\infty}(\mathbb{R}^n,\mu)}} \leq p\left(\frac{\lambda^p\mu(A_\lambda)}{||T_{k+1}f||^p_{L^{p,\infty}(\mathbb{R}^n,\mu)}}\right)^{1-1/p}+1.$$

By virtue of (1.2), this yields $\lambda^p\mu(A_\lambda) \leq c_p||T_{k+1}f||_{L^{p,\infty}(\mathbb{R}^n,\mu)}$ and (3.8) follows.

We turn to the sharpness. Let $\mu = |\cdot|$ be the Lebesgue measure on $\mathbb{R}^n$, fix $p'>p$ and consider the function

$$f(x_1,x_2,\ldots,x_n) = \prod_{k=1}^n |2x_k|^{-1/p'}\chi_{[-1,1]^n}(x).$$

It belongs to $L^p(\mathbb{R}^n,|\cdot|)$, so in particular $||f||_{L^{p,\infty}(\mathbb{R}^n,|\cdot|)}<\infty$. By (2.8), applied to each coordinate (here we use the product structure of f), we have $M_{|\cdot|}f \geq c^n_{p'}f$ on $\mathbb{R}^n$. Therefore, $||M_{|\cdot|}f||_{L^{p,\infty}(\mathbb{R}^n,|\cdot|)} \geq c^n_{p'}||f||_{L^{p,\infty}(\mathbb{R}^n,|\cdot|)}$ and it remains to let $p'\to p$ to see that c_p^n is optimal in (3.2). Finally, to prove the sharpness of (3.3), let f be as above. Fix $\kappa>1$ and choose $\lambda>0$ such that $\lambda^p|\{f>\lambda\}|\cdot\kappa > ||f||^p_{L^{p,\infty}(\mathbb{R}^n,\mu)}$. If we put $A=\{f>\lambda\}$, then $M_{|\cdot|}f > c^n_{p'}\lambda$ on A, so

$$\frac{||M_{|\cdot|}f||_{L^{q,\infty}(A,|\cdot|)}}{||f||_{L^{p,\infty}(\mathbb{R}^n,|\cdot|)}} \geq \frac{c^n_{p'}\lambda|A|^{1/q}}{\kappa^{1/p}\lambda|A|^{1/p}} = \frac{c^n_{p'}}{\kappa}|A|^{1/q-1/p}.$$

Since $\kappa>1$ and $p'>p$ were arbitrary, the constant c_p^n is the best in (3.3). □

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