DOOB-TYPE ESTIMATES FOR DIFFERENTIALLY SUBORDINATED MARTINGALES

ADAM OSĘKOWSKI

Abstract. Let X, Y be local martingales such that Y is differentially subordinate to X. The paper contains the proof of the sharp estimates

$$||\sup_{t\geq 0} Y_t||_p \leq \frac{p}{p-1}||X||_p, \qquad 1 < p < 2,$$
$$||\sup_{t\geq 0} Y_t||_p \leq p||X||_p, \qquad 2 \leq p < \infty.$$

As an application, we establish sharp inequalities for stochastic integrals with respect to Brownian motion in $\mathbb{R}^d$, derive some maximal bounds for harmonic functions and Riesz system, and present tight estimates for a stopped three-dimensional Bessel process.

1. Introduction

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space, equipped with a filtration $(\mathcal{F}_t)_{t\geq 0}$, a nondecreasing right-continuous family of sub-σ-fields of $\mathcal{F}$. As usual, we assume that $\mathcal{F}_0$ contains all the events of probability 0. Let $X = (X_t)_{t\geq 0}$ be an adapted real-valued local martingale with right-continuous trajectories that have limits from the left. Let $Y = H \cdot X$ be the Itô integral of H with respect to X,

$$Y_t = H_0 X_0 + \int_{(0,t]} H_s dX_s, \qquad t \geq 0,$$

where H is a predictable process with values in $[-1, 1]$. Denote by $X^*$and $|X|^*$ the one- and two-sided maximal functions of X, given by $\sup_{t\geq 0} X_t$ and $\sup_{t\geq 0} |X_t|^*$, respectively. Furthermore, we shall write $||X||_p = \sup_{t\geq 0} ||X_t||_p$ for the p-th moment of X, $1 \leq p \leq \infty$.

The purpose of this paper is to determine optimal constants in some basic maximal inequalities involving X and Y. The literature on this subject is very extensive and it is impossible to review it here, so we shall only focus on recalling results and methods which are closely related to those appearing in this paper. In [8], Burkholder invented a powerful technique of proving estimates for martingale transforms and stochastic integrals, and used it to show the following.

Theorem 1.1. *If X and Y are as above, then we have*

$$||Y||_1 \leq \gamma|| \, |X|^*||_1, \tag{1.1}$$

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where $\gamma = 2.536\ldots$ is the unique solution of the equation

$$\gamma - 3 = -\exp\big(\frac{1-\gamma}{2}\big).$$

The constant is the best possible.

Then it was proved by author in [11], that if X is nonnegative, then the constant decreases to $2 + (3e)^{-1} = 2.1226\ldots$. The paper [12] contains the proof of the related estimate in which the first moment of Y is replaced by the first moment of its one-sided maximal function. Precisely, it is shown there that

$$||Y^*||_1 \leq \eta|| \, |X|^*||_1, \tag{1.2}$$

where $\eta = 2.0856\ldots$ is the unique positive solution to the equation $2\log\left(\frac{8}{3} - \eta\right) = 1 - \eta$. In addition, if X is assumed to be nonnegative, the best constant in (1.2) equals $14/9 = 1.555\ldots$. This result has been considerably strengthened in [13], where it was proved that

$$|| \, |Y|^*||_1 \leq 3.4351\ldots|| \, |X|^*||_1,$$

with the constant arising from an intrinsic complicated system of differential equations. For $X \geq 0$, the optimal constant reduces to 3 ([14]).

In the present paper we shall study related estimates which compare the p-th moments of X and Y^*. In fact, our approach works for a wider class of processes: we show the maximal estimates under the assumption of *differential subordination*. Suppose that X, Y are adapted $\mathbb{R}^d$-valued local martingales with the usual assumptions on the paths and let $[X,X]$, $[Y,Y]$ denote the quadratic covariance processes of X and Y (when $d = 1$, see e.g. Dellacherie and Meyer [9] for details; in the vector case, we let $[X,X] = \sum_{k=1}^d [X^k, X^k]$, where X^k stands for the k-th coordinate of X). Following Bañuelos and Wang [1] and Wang [20], we say that Y is differentially subordinate to X if the process $([X,X]_t - [Y,Y]_t)_{t\geq 0}$ is nondecreasing and nonnegative as a function of t. For example, as in the beginning of this section, if X is real and $Y = H \cdot X$, for some predictable H taking values in $[-1,1]$, then Y is differentially subordinate to X. This is a consequence of the identity

$$[X,X]_t - [Y,Y]_t = \int_0^t (1 - |H_s|^2)d[X,X]_s, \qquad t \geq 0.$$

Some further examples will be presented in Section 5 below.

Our main result can be stated as follows.

Theorem 1.2. *Let X, Y be real local martingales such that Y is differentially subordinate to X. Then for any $1 < p < \infty$ we have*

$$||Y^*||_p \leq C_p||X||_p, \tag{1.3}$$

where $C_p = p/(p-1)$ for $1 < p < 2$ and $C_p = p$ for $p \geq 2$. The inequality is sharp, even if X is assumed to be a stopped Brownian motion and Y is the Itô integral with respect to X of some predictable process taking values in $\{-1, 1\}$.

This is a very surprising result, both for $1 < p < 2$ and $p \geq 2$. In the first case, the constant is the same as in the one-sided version of Doob's maximal inequality

$$||X^*||_p \leq \frac{p}{p-1}||X||_p, \qquad 1 < p < \infty, \tag{1.4}$$

which is known to be sharp (see e.g. Peskir [15]). Thus, (1.3) can be regarded as a considerably stronger version of this classical estimate. For $p \geq 2$, our bound is an immediate consequence of (1.4) and a celebrated inequality of Burkholder

$$||Y||_p \leq (p^* - 1)||X||_p, \qquad 1 < p < \infty,$$

where $p^* = \max\{p, p/(p-1)\}$ (see [4] and [7]). What is quite unexpected, $C_p = p$ is the best possible in (1.3).

A few words about the proof and the organization of the paper. As already observed above, we need to establish (1.3) only for $1 < p < 2$. We shall exploit Burkholder's method from [8], which translates the problem of showing (1.3) into that of finding a certain special function which has some convex-type properties. However, in comparison with the reasoning appearing in [8] and [11]–[14], our approach will require some additional effort. Namely, the special function we manage to construct in the next section does not allow to handle local martingales with jumps. Nonetheless, it has all the necessary properties to establish (1.3) in the case when X is a stopped Brownian motion and $Y = H \cdot X$ for some predictable H taking values in $\{-1, 1\}$. Using embedding, this enables us to show the discrete-time version of (1.3). Having this done, the results from [8] provide us with another, abstract special function. This object has all the properties we need and hence we are able to establish (1.3) in full generality. This three-step proof of (1.3) is carried out in Section 3. Section 4 contains the construction of examples showing the optimality of C_p. The final part of the paper is devoted to some higher-dimensional generalizations and applications of Theorem 1.2. These include maximal estimates for harmonic functions on Euclidean domains, a tight inequality for a Riesz system in $\mathbb{R}^{n+1}$ and some sharp bounds for infimum of three-dimensional Bessel process.

2. The special function and its properties

Throughout this section, p is a fixed number from the interval $(1, 2)$. Let

$$\alpha_p = \frac{p^p}{p-1}, \qquad \beta_p = \frac{p^p}{(p-1)(2p-1)^{p-1}}$$

and consider the subsets D_1, D_2 of $[0, \infty) \times (-\infty, 1]$, given by

$$D_1 = \{(x, y) : py \leq (p-1)x + 1\},$$
$$D_2 = \{(x, y) : py > (p-1)x + 1\}.$$

Introduce the function $u_p : [0, \infty) \times (-\infty, 1] \to \mathbb{R}$ by the following formula. If $(x, y) \in D_1$, set

$$u_p(x, y) = 1 - \alpha_p \left(x - y + \frac{1}{p} \right)^{p-1} \left((-p^2 + 3p - 1)x + (p-1)^2 \left(y - \frac{1}{p} \right) \right),$$

while for $(x, y) \in D_2$, define

$$u_p(x, y) = 1 - \beta_p \left(x + y - \frac{1}{p} \right)^{p-1} \left((p^2 - p + 1)x - (p^2 - 1) \left(y - \frac{1}{p} \right) \right).$$

In the two lemmas below, we study the properties of u_p which will be needed in our further considerations. For any set A, let A^o stand for its interior.

Lemma 2.1. *(i) The function u_p is of class C^1 on $(0, \infty) \times (-\infty, 1)$.*

(ii) We have $\lim_{x \downarrow 0} u_{px}(x, y) \leq 0$ for any $y < 1$.

(iii) For any $x \geq 0$, the function $u_p(x, \cdot)$ is convex on $(-\infty, 1)$.

(iv) For any $x > 0$ we have

$$\lim_{y\uparrow 1}\big[pu_p(x,y) - xu_{px}(x,y) - u_{py}(x,y)\big] \leq 0.$$

(v) For any $(x,y) \in [0,\infty) \times (-\infty, 1]$ we have

$$u_p(x,y) \geq 1 - \left(\frac{p}{p-1}\right)^p x^p. \tag{2.1}$$

(vi) For any $y < 1$, the function $u_{px}(\cdot, y)$ is convex on $(0,\infty)$.

Proof. (i) We easily check that u_p is continuous: the limit of u_p at a point $(x,y) \in \partial D_1 \cap \partial D_2$ equals $1 - \frac{2p-1}{p-1}x^p$. Of course, u_p is of class C^1 on $D_1^o \cup D_2^o$. In addition, a little calculation shows that

$$u_{px}(x,y) = \begin{cases} -p\alpha_p(x-y+\frac{1}{p})^{p-2}\big[(-p^2+3p-1)x + (p-2)(py-1)\big] & \text{on } D_1^o, \\ -p\beta_p(x+y-\frac{1}{p})^{p-2}\big[(p^2-p+1)x + (2-p)(py-1)\big] & \text{on } D_2^o \end{cases}$$

and

$$u_{py}(x,y) = \begin{cases} (p-1)\alpha_p(x-y+\frac{1}{p})^{p-2}\big[p(2-p)x + (p-1)(py-1)\big] & \text{on } D_1^o, \\ (p-1)\beta_p(x+y-\frac{1}{p})^{p-2}\big[p(2-p)x + (p+1)(py-1)\big] & \text{on } D_2^o. \end{cases}$$

Since the partial derivatives match at $\partial D_1 \cap \partial D_2$, the function u_p has the postulated smoothness.

(ii) We have

$$\lim_{x\downarrow 0} u_{px}(0,y) = \begin{cases} -p^2(2-p)\alpha_p(-y+\frac{1}{p})^{p-1} & \text{if } y \leq \frac{1}{p}, \\ -p^2(2-p)\alpha_p(y-\frac{1}{p})^{p-1} & \text{if } y > \frac{1}{p}, \end{cases}$$

which is nonpositive.

(iii) We compute that if (x,y) belongs to D_1^o, then

$$u_{pyy}(x,y) = (p-1)\alpha_p\left(x - y + \frac{1}{p}\right)^{p-3}\big[(p^3-3p^2+3p)x + (p-1)^2(1-py)\big].$$

Now, if $y < 1/p$, then both summands in the square bracket are nonnegative. If $y \geq 1/p$, then $x > (py-1)/(p-1)$ by the definition of D_1, so the expression in the square brackets exceeds

$$\frac{py-1}{p-1}\big(p^3 - 3p^2 + 3p - (p-1)^3\big) = \frac{py-1}{p-1} \geq 0.$$

Thus, $u_{pyy} \geq 0$ on D_1^o. Similarly, if $(x,y) \in D_2^o$, then

$$u_{pyy}(x,y) = (p-1)\beta_p\left(x + y - \frac{1}{p}\right)^{p-3}\big[p(-p^2+5p-3)x + (p-1)(p+1)(py-1)\big]$$

and the expression in the square brackets is nonnegative. It suffices to use (i) to get the convexity of $u_p(x,\cdot)$.

(iv) We consider separately two cases: $x \geq 1$ and $x \leq 1$. If the first possibility occurs, we rewrite the estimate using the formulas for u_{px} and u_{py} on D_1^o. After some lengthy but straightforward computations we get the equivalent bound

$$1 \leq (px - p + 1)^{p-2}[p(2-p)x + (p-1)^2].$$

Both sides are equal for $x=1$ and the right-hand side is a nondecreasing function of $x\in[1,\infty)$: its derivative equals

$$p^2(p-1)(2-p)(px-p+1)^{p-3}(x-1)\geq 0,$$

so the desired inequality holds true. If $x\leq 1$, we use the formulas for the partial derivatives of u_p on D_2^o and transform the estimate into

$$1\leq (2p-1)^{1-p}\,(px+p-1)^{p-2}\left[p(2-p)x+p^2-1\right].$$

We have equality for $x=1$ and the derivative of the right-hand side is given by

$$(2p-1)^{1-p}p^2(p-1)(2-p)(px-p+1)^{p-3}(x-1)\leq 0,$$

so the bound holds for $x\in[0,1)$ as well.

(v) Fix $x\geq 0$. Directly from the above formula for u_{py}, we infer that the left-hand side of (2.1), as a function of y, is decreasing on $(-\infty,\frac{p-2}{p-1}x+\frac{1}{p}]$ and increasing on $[\frac{p-2}{p-1}x+\frac{1}{p},1]$. It remains to note that both sides of (2.1) are equal when $y=\frac{p-2}{p-1}x+\frac{1}{p}$.

(vi) If $(x,y)\in D_1^o$, then

$$\begin{aligned}
&u_{pxxx}(x,y)\\
&=(p-1)(2-p)\alpha_p\left(x-y+\frac{1}{p}\right)^{p-4}\left[p(-p^2+3p-1)x+(p^2-2p-2)\,(py-1)\right]\\
&\geq (p-1)(2-p)\alpha_p\left(x-y+\frac{1}{p}\right)^{p-4}\left[p(-p^2+3p-1)x+(p^2-2p-2)(p-1)x\right]\\
&=p(p-1)(2-p)^2\alpha_p x\left(x-y+\frac{1}{p}\right)^{p-4}\geq 0,
\end{aligned}$$

where in the second passage we have used the definition of D_1. Similarly, when $(x,y)\in D_2^o$, then

$$\begin{aligned}
&u_{pxxx}(x,y)\\
&=(p-1)(2-p)\beta_p\left(x+y-\frac{1}{p}\right)^{p-4}\left[p(p^2-p+1)x+(-p^2+6p-2)\,(py-1)\right]
\end{aligned}$$

is nonnegative, since so are both summands appearing in the square bracket. It remains to check that u_{pxx} is continuous on $(0,\infty)\times(-\infty,1)$; this can be done as in (i) above. □

Lemma 2.2. *The function u_p is concave along the lines of slope ± 1 contained in $[0,\infty)\times(-\infty,1]$.*

Proof. In virtue of the part (i) from the previous lemma, it suffices to check that $u_{pxx}\pm 2u_{pxy}+u_{pyy}\leq 0$ on $D_1^o\cup D_2^o$. Suppose first that (x,y) belongs to the interior of D_1. Then u_p is locally linear along the line segment of slope 1 passing through (x,y), so $u_{pxx}(x,y)+2u_{pxy}(x,y)+u_{pyy}(x,y)=0$. Moreover, a little calculation shows that

$$\begin{aligned}
u_{pxx}(x,y)&-2u_{pxy}(x,y)+u_{pyy}(x,y)\\
&=-4u_{pxy}(x,y)\\
&=4p(p-1)(2-p)\alpha_p\left(x-y+\frac{1}{p}\right)^{p-3}\big[-(p-1)x+py-1\big],
\end{aligned}$$

which is nonpositive, by the definition of D_1. Similarly, if (x, y) lies in the interior of D_2, then it is evident that $u_{pxx}(x, y) - 2u_{pxy}(x, y) + u_{pyy}(x, y) = 0$; in addition,

$$\begin{aligned} u_{pxx}(x, y) &+ 2u_{pxy}(x, y) + u_{pyy}(x, y) \\ &= 4u_{pxy}(x, y) \\ &= 4p(p-1)(2-p)\beta_p \left(x + y - \frac{1}{p}\right)^{p-3} \big[(p-1)x - py + 1\big] \leq 0, \end{aligned}$$

which completes the proof. □

Let U_p, $V_p : \mathbb{R} \times \mathbb{R} \times (0, \infty) \to \mathbb{R}$ be given by

$$U_p(x, y, z) = (y \vee z)^p u_p\left(\frac{|x|}{y \vee z}, \frac{y}{y \vee z}\right),$$
$$V_p(x, y, z) = (y \vee z)^p - \left(\frac{p}{p-1}\right)^p |x|^p.$$

In the next section we will study the properties of the process $(U_p(X_t, Y_t, Y_t^*))_{t\geq 0}$ (for X, Y as in Theorem 1.2), using Itô's formula. Since U_p is not of class C^2, we need to modify it slightly to ensure the necessary smoothness. Suppose that $\psi : \mathbb{R}^3 \to [0, \infty)$ is a radial C^∞ function, supported on the unit ball of $\mathbb{R}^3$, satisfying $\int_{\mathbb{R}^3} \psi = 1$ and such that $\psi(x, y, z)$ decreases as $|(x, y, z)|$ increases. Let $D = \{(x, y, z) : x \in \mathbb{R},\ z \geq \max\{y, 0\}\}$. For a given $\delta > 0$ and $(x, y, z) \in D$, define

$$U_p^\delta(x, y, z) = \int_{[-1,1]^3} U_p(x + \delta r, y - \delta + \delta s, z + \delta + \delta t)\psi(r, s, t)\mathrm{d}r\mathrm{d}s\mathrm{d}t$$

and

$$V_p^\delta(x, y, z) = \int_{[-1,1]^3} V_p(x + \delta r, y - \delta + \delta s, z + \delta + \delta t)\psi(r, s, t)\mathrm{d}r\mathrm{d}s\mathrm{d}t.$$

Note that the use of the asymmetric expressions $y - \delta + \delta s$, $z + \delta - \delta t$ under the integrals guarantees the following: if $(x, y, z) \in D$, then $y - \delta + \delta s \leq z + \delta - \delta t$ (at least when (r, s, t) belongs to the support of ψ) and hence

$$U_p^\delta(x, y, z) = \int_{[-1,1]^3} (z+\delta+\delta t)^p u_p\left(\frac{|x + \delta r|}{z + \delta + \delta t}, \frac{y - \delta + \delta s}{z + \delta + \delta t}\right)\psi(r, s, t)\mathrm{d}r\mathrm{d}s\mathrm{d}t. \tag{2.2}$$

This equation will enable us to transfer the key properties of u_p to U_p^δ.

Lemma 2.3. *(i) The function U_p^δ is of class C^∞.*

(ii) We have the majorization $U_p^\delta \geq V_p^\delta$.

(iii) $U_{pxx}^\delta \pm 2U_{pxy}^\delta + U_{pyy}^\delta \leq 0$.

(iv) Fix $\varepsilon > 0$ and $N > 1$. If δ is sufficiently small, then $U_{pz}^\delta(x, y, y) \leq \varepsilon$ for $|x| \leq N$ and $y \in [N^{-1}, N]$.

(v) For any $(x, y, z) \in D$ such that $x \geq 0$ and $y < z$ we have

$$U_{pyy}^\delta(x, y, z) \geq 0$$

and

$$xU_{pxx}^\delta(x, y, z) - U_{px}^\delta(x, y, z) \geq 0. \tag{2.3}$$

Proof. The property (i) is evident and (ii) follows from $U_p \geq V_p$, which, in turn, is a consequence of (2.1). The condition (iii) is due to Lemma 2.2. To check the fourth statement, note that for $x \in \mathbb{R}$ and $0 < y < z$,

$$U_{pz}(x, y, z) = z^{p-1}\big[pu_p(x/z, y/z) - (x/z)u_{px}(x/z, y/z) - (y/z)u_{py}(x/z, y/z)\big],$$

so

$$U^\delta_{pz}(x, y, y) = \int_{[-1,1]^3} (y + \delta - \delta t)^{p-1} A(x, y, r, s, t, \delta)\psi(r, s, t)\mathrm{d}r\mathrm{d}s\mathrm{d}t,$$

where

$$\begin{aligned} A(x, y, r, s, t, \delta) = {} & pu_p\left(\frac{x + \delta r}{y + \delta - \delta t}, \frac{y - \delta + \delta s}{y + \delta - \delta t}\right) \\ & - \frac{x + \delta r}{y + \delta - \delta t} u_{px}\left(\frac{x + \delta r}{y + \delta - \delta t}, \frac{y - \delta + \delta s}{y + \delta - \delta t}\right) \\ & - \frac{y - \delta + \delta s}{y + \delta - \delta t} u_{py}\left(\frac{x + \delta r}{y + \delta - \delta t}, \frac{y - \delta + \delta s}{y + \delta - \delta t}\right). \end{aligned}$$

Now we apply Lemma 2.1 (iv) and use the fact that the partial derivatives of u_p are locally Lipschitz. Consequently, if δ is sufficiently small, then for x, y as in the statement, the expression $(y + \delta - \delta t)^{p-1}A(x, y, r, s, t, \delta)$ does not exceed ε.

(v) By part (iii) of Lemma 2.1, the function $U_p(x, \cdot, z)$ is convex for any $x \geq 0$ and $z > 0$. This immediately yields the first estimate. To get the second one, observe that both sides are equal for $x = 0$; hence it suffices to prove that $U^\delta_{pxxx}(x, y, z) \geq 0$ for (x, y, z) as in the statement. Consider the function $s \mapsto u_{px}(|s|, y) \operatorname{sgn} s$, $s \in \mathbb{R}$ (for $s = 0$, set the value to be 0). Using Lemma 2.1, we easily see that this function is odd and, when restricted to $(0, \infty)$, it is convex and negative. Now we easily check the midpoint convexity of $U^\delta_{px}(\cdot, y, z)$ on $(0, \infty)$, using (2.2) and the radial monotonicity of ψ. □

3. Proof of (1.3) for $1 < p < 2$

3.1. **Proof for stochastic integrals with respect to Brownian motion.** First we shall establish the inequality (1.3) in the special case when $X = B^\eta$ is a stopped standard one-dimensional Brownian motion, starting from some $x_0 \in \mathbb{R}$, and Y is an Itô integral, with respect to X, of some predictable process H taking values in $\{-1, 1\}$. Let us start with some reductions. Of course, we may assume that $x_0 \neq 0$. Next, it suffices to consider only those X, which are bounded in L^p (otherwise the claim is trivial). Then so is Y^*, in virtue of Doob's and Burkholder's inequalities. Finally, we may assume that Y satisfies $Y_0 = |x_0|$ almost surely. Indeed, if it is not the case, we replace it with $\overline{Y} = (Y_t + |X_0| - Y_0)_{t\geq 0}$ (or, in other words, change H_0 to $\operatorname{sgn} B_0$). Then $\overline{Y}$ starts from $|X_0| = |x_0| > 0$, is a stochastic integral as required and, for any $t \geq 0$,

$$\overline{Y}^*_t \geq Y^*_t \geq Y_0 \geq -\overline{Y}_0 \geq -\overline{Y}^*_t,$$

so $||Y^*||_p \leq ||\overline{Y}^*||_p$.

Now, suppose that $\varepsilon > 0$ and $N > 1/|x_0|$ are given and fixed. Let U^δ_p be the function constructed in the previous section, corresponding to the δ for which Lemma 2.3 (iv) is satisfied. Introduce the stopping time

$$\tau = \inf\{t \geq 0 : |X_t| \geq N \text{ or } |Y_t| \geq N\}$$

and consider the process $(Z_t)_{t\geq 0} = ((X_{\tau\wedge t}, Y_{\tau\wedge t}, Y^*_{\tau\wedge t}))_{t\geq 0}$. By Itô's formula, for any $t \geq 0$,

$$U_p^\delta(Z_t) = U_p^\delta(Z_0) + I_1 + I_2 + I_3/2, \tag{3.1}$$

where

$$I_1 = \int_{0+}^{\tau\wedge t} U^\delta_{px}(Z_s)\mathrm{d}X_s + \int_{0+}^{\tau\wedge t} U^\delta_{py}(Z_s)\mathrm{d}Y_s,$$
$$I_2 = \int_{0+}^{\tau\wedge t} U^\delta_{pz}(Z_s)\mathrm{d}Y^*_s,$$
$$I_3 = \int_{0+}^{\tau\wedge t} U^\delta_{pxx}(Z_s)\mathrm{d}[X,X]_s + 2\int_{0+}^{\tau\wedge t} U^\delta_{pxy}(Z_s)\mathrm{d}[X,Y]_s + \int_{0+}^{\tau\wedge t} U^\delta_{pyy}(Z_s)\mathrm{d}[Y,Y]_s.$$

Let us now analyze each of the terms I_1, I_2, I_3 separately. The stochastic integrals in I_1 are martingales, so $\mathbb{E}I_1 = 0$. To deal with I_2, note that the support of $\mathrm{d}Y^*_s$ is contained in the set $\{(x,y,z) : N^{-1} \leq |x_0| \leq y = z \leq N\}$, on which U^δ_{pz} is bounded from above by ε: here we use the part (iv) of Lemma 2.3. Consequently, $I_2 \leq \varepsilon(Y^*_{\tau\wedge t} - Y_0) \leq \varepsilon Y^*_{\tau\wedge t}$. Finally, since X is a Brownian motion and $H \in \{-1, 1\}$,

$$I_3 = \int_{0+}^{\tau\wedge t} \left[U^\delta_{pxx}(Z_s) + 2U^\delta_{pxy}(Z_s)H_s + U^\delta_{pyy}(Z_s)\right]\mathrm{d}s \leq 0,$$

in virtue of Lemma 2.3 (iii). Therefore, integrating both sides of (3.1) and applying Lemma 2.3 (ii), we get

$$\mathbb{E}V_p^\delta(Z_t) \leq \mathbb{E}U_p^\delta(Z_0) + \varepsilon\mathbb{E}Y^*_{\tau\wedge t}.$$

Letting $\delta \to 0$ yields, by Lebesgue's dominated convergence theorem (X and Y are bounded in L^p),

$$\mathbb{E}V_p(Z_t) \leq \mathbb{E}U_p(Z_0) + \varepsilon\mathbb{E}Y^*_{\tau\wedge t} \leq \varepsilon\mathbb{E}Y^*_{\tau\wedge t},$$

where the latter passage follows from the estimate $u_p(1,1) \leq 0$. Now let $\varepsilon \to 0$ and $N \to \infty$ to obtain $\mathbb{E}V_p(X_t, Y_t, Y^*_t) \leq 0$, or

$$\mathbb{E}(Y^*_t)^p \leq \left(\frac{p}{p-1}\right)^p \mathbb{E}|X_t|^p \leq \left(\frac{p}{p-1}\right)^p ||X||_p^p.$$

It remains to let $t \to \infty$ and the proof is complete.

3.2. **Proof for martingale transforms.** Now we shall prove the estimate (1.3) in the discrete-time setting. Let us provide the necessary definitions. Assume that $(\mathcal{F}_n)_{n\geq 0}$ is a discrete filtration and let $f = (f_n)_{n\geq 0}$ be a martingale, with the difference sequence $df = (df_n)_{n\geq 0}$ given by $df_0 = f_0$ and $df_n = f_n - f_{n-1}$ for $n \geq 1$. Let g be the transform of f by a predictable real sequence $v = (v_n)_{n\geq 0}$ bounded in absolute value by 1. That is, assume that we have $dg_n = v_n df_n$ for $n = 0, 1, 2, \ldots$. We shall establish the bound

$$||g^*||_p \leq \frac{p}{p-1}||f||_p, \tag{3.2}$$

where $||f||_p$ and g^* are defined in analogy to the continuous-time case. As in the previous subsection, we may restrict ourselves to $f \in L^p$ and to g satisfying $g_0 \geq 0$ with probability 1. We shall exploit the following version of the decomposition lemma of Burkholder (see Lemma A.1 from [7]). The proof is identical as in the original setting and we do not present it here.

Lemma 3.1. *For any f, g as above, there exist martingales $F^j = (F_n^j)_{n\geq 0}$ and Borel measurable functions $\phi_j : [-1,1] \to \{-1,1\}$ such that, for $j \geq 1$ and $n \geq 0$,*

$$f_n = F_{2n+1}^j, \quad g_n = \sum_{j=1}^{\infty} 2^{-j}\phi_j(v_0)G_{2n+1}^j.$$

Here G^j is the transform of F^j by $V = ((-1)^k)_{k\geq 0}$ and satisfies $G_0^j \geq 0$ almost surely.

Now, for any fixed j, there is a Brownian motion X and a stochastic integral Y as in the previous subsection such that the martingale pair (F^j, G^j) can be embedded into (X, Y). That is, there is a nondecreasing sequence $(\tau_n)_{n\geq 0}$ of stopping times (depending on j) such that $((F_n^j, G_n^j))_{n\geq 0}$ has the same law as $((X_{\tau_n}, Y_{\tau_n}))_{n\geq 0}$. Consequently, for any n,

$$||(G_n^j)^*||_p \leq ||Y_{\tau_n}^*||_p \leq \frac{p}{p-1}||X_{\tau_n}||_p = \frac{p}{p-1}||F_n^j||_p.$$

Since $G_0^j \geq 0$ almost surely, we have $(G_n^j)^* \geq 0$ for each n and thus

$$\begin{aligned} ||g_n^*||_p &\leq \left|\left|\sum_{j=1}^{\infty} 2^{-j}\phi_j(v_0)(G_{2n+1}^j)^*\right|\right|_p \\ &\leq \sum_{j=1}^{\infty} 2^{-j}\left|\left|(G_{2n+1}^j)^*\right|\right|_p \leq \frac{p}{p-1}\sum_{j=1}^{\infty} 2^{-j}\left|\left|F_{2n+1}^j\right|\right|_p = \frac{p}{p-1}||f_n||_p. \end{aligned}$$

It suffices to let $n \to \infty$ to obtain (3.2).

3.3. **Proof in the general case.** The inequality for martingale transforms gives rise to another special function, which can be regarded as an enhancement of U_p. For any x, $y \in \mathbb{R}$, let $\mathcal{M}(x, y)$ denote the class of all pairs (f, g) of simple discrete-time martingales, given on the Lebesgue's interval $([0,1], \mathcal{B}([0,1]), |\cdot|)$, such that $(f_0, g_0) \equiv (x, y)$ and $dg_n = v_n df_n$, $n = 1, 2, \ldots$, for some deterministic sequence $(v_n)_{n\geq 1}$ taking values in $[-1,1]$. Let γ be an arbitrary constant larger than $p/(p-1)$. For x, $y \in \mathbb{R}$ and $z \geq 0$, define

$$W_p(x, y, z) = \sup\left\{||g^* \vee z||_p^p - \gamma^p||f||_p^p\right\},$$

where the supremum is taken over all (f, g) from the class $\mathcal{M}(x, y)$.

Lemma 3.2. *The function W_p has the following properties.*

(i) $W_p(x, y, z)$ is finite for any $(x, y, z) \in \mathbb{R}^2 \times \mathbb{R}_+$.

(ii) $W_p(x, y, z) \geq V_p^\gamma(x, y, z) = |y \vee z|^p - \gamma^p|x|^p$ for $(x, y, z) \in \mathbb{R}^2 \times \mathbb{R}_+$.

(iii) $W_p(x, y, z_1) \leq W_p(x, y, z_2)$ for x, $y \in \mathbb{R}$ and $0 \leq z_1 \leq z_2$.

(iv) For any $x \in \mathbb{R}$ and $z \geq 0$, the function $W_p(x, \cdot, z)$ is convex on $(-\infty, z]$.

(v) For any $z > 0$, the function $W_p(\cdot, \cdot, z)$ is concave along any line segment of slope in $[-1,1]$ (contained in $\mathbb{R} \times (-\infty, z]$).

Proof. (i) For any $(f, g) \in \mathcal{M}(x, y)$, we have $||g^* \vee z||_p^p \leq ||g^*||_p^p + z^p$. Furthermore, by Jensen's inequality, there is an absolute constant c, depending only on γ and p, such that

$$||f||_p^p \geq \left(\frac{p}{(p-1)\gamma}\right)^p ||y - x + f||_p^p - c|x - y|^p.$$

Therefore, we obtain

$$||g^* \vee z||_p^p - \gamma^p ||f||_p^p \leq z^p + c\gamma^p |x-y|^p + \left[||g^*||_p^p - \left(\frac{p}{p-1}\right)^p ||y-x+f||_p^p \right]$$

and the expression in the square brackets is nonpositive, due to (3.2) (since g is a transform of $y-x+f$). It remains to take supremum over (f,g) to get the finiteness of W_p.

(ii) This follows immediately from the fact that the constant pair (x,y) belongs to $\mathcal{M}(x,y)$.

(iii) This is evident from the very definition, since for any g we have $g^* \vee z_1 \leq g^* \vee z_2$ almost surely.

(iv) Take any y_1, $y_2 \leq z$, $\lambda \in (0,1)$ and put $y = \lambda y_1 + (1-\lambda) y_2$. For any $(f,g) \in \mathcal{M}(x,y)$, we have that $(f, y_1 - y + g) \in \mathcal{M}(x, y_1)$ and $(f, y_2 - y + g) \in \mathcal{M}(x, y_2)$. Furthermore, for any n,

$$\begin{aligned}
\mathbb{E}(g_n^* \vee z)^p &\leq \mathbb{E}\big[\big((\lambda(y_1 - y + g)_n^* + (1-\lambda)(y_2 - y + g)_n^*\big) \vee z\big]^p \\
&\leq \mathbb{E}\big[\lambda\big((y_1 - y + g)_n^* \vee z\big) + (1-\lambda)\big((y_2 - y + g)_n^* \vee z\big)\big]^p \\
&\leq \lambda \mathbb{E}\big((y_1 - y + g)_n^* \vee z\big)^p + (1-\lambda)\mathbb{E}\big((y_2 - y + g)_n^* \vee z\big)^p,
\end{aligned}$$

which implies

$$\mathbb{E}(g_n^* \vee z)^p - \gamma^p \mathbb{E}|f_n|^p \leq \lambda W_p(x, y_1, z) + (1-\lambda) W_p(x, y_2, z).$$

It suffices to take supremum over f, g and n to get the convexity.

(v) This can be done by modifying appropriately the so called "splicing argument" due to Burkholder. See e.g. page 11 in [7] for details. □

Now we shall follow the pattern from Subsection §3.1 and approximate W_p by a smooth function. For ψ as above and $\delta > 0$, introduce W_p^δ, $(V_p^\gamma)^\delta$ on $D = \{(x,y,z) : x \in \mathbb{R},\ z \geq \max\{y, 0\}\}$ by the convolutions

$$W_p^\delta(x,y,z) = \int_{[-1,1]^3} W_p(x + \delta r, y - \delta + \delta s, z + \delta + \delta t)\psi(r,s,t)\mathrm{d}r\mathrm{d}s\mathrm{d}t$$

and

$$(V_p^\gamma)^\delta(x,y,z) = \int_{[-1,1]^3} V_p^\gamma(x + \delta r, y - \delta + \delta s, z + \delta + \delta t)\psi(r,s,t)\mathrm{d}r\mathrm{d}s\mathrm{d}t$$

(recall that $V_p^\gamma(x,y,z) = |y \vee z|^p - \gamma^p |x|^p$ for $(x,y,z) \in \mathbb{R}^2 \times \mathbb{R}_+$). It is evident that W_p^δ satisfies appropriate versions of the properties (i), (ii) and (iii) of Lemma 2.3. Even more, the partial derivatives of W_p^δ can be extended to continuous functions on the whole D and

$$W_{pxx}^\delta + 2sW_{pxy}^\delta + s^2 W_{pyy}^\delta \leq 0 \qquad \text{for } |s| \leq 1, \tag{3.3}$$

since in the part (v) of the above lemma we allow the slope to be an arbitrary number from $[-1,1]$. The main advantage of W_p^δ over U_p^δ is that the new function satisfies the following stronger version of Lemma 2.3 (iv):

$$W_{pz}^\delta(x,y,z) \leq 0 \qquad \text{for } x \in \mathbb{R},\ y \leq z. \tag{3.4}$$

The validity of this bound is an immediate consequence of Lemma 3.2 (iii). This stronger property will enable us to handle the jump part of the pair (X,Y). Namely, we shall prove the following statement.

Lemma 3.3. *For any* $(x, y, z) \in D$ *and any* h, $k \in \mathbb{R}$ *such that* $|k| \leq |h|$ *we have*

$$W_p^\delta(x+h, y+k, z) \geq W_p^\delta(x, y, z) + W_{px}^\delta(x, y, z)h + W_{py}^\delta(x, y, z)k. \tag{3.5}$$

Proof. We may assume that $h \neq 0$. When $y + k \leq z$, then the assertion follows immediately from (3.3); thus, it suffices to deal with $y+k > z$ and, again by (3.3), we may assume that $y = z$. Fix $\kappa > 0$. Let X be a standard Brownian motion, starting from 0, let $Y = kX/h$ and consider the stopping time $\tau = \inf\{t : X_t \in \{-\kappa h, h\}\}$. Observe that the process

$$([x + X, x + X]_t - [y + Y, y + Y]_t)_{t\geq 0} = (x^2 - y^2 + [X, X]_t - [Y, Y]_t)_{t\geq 0}$$

is nondecreasing as a function of t. In consequence, the argumentation presented in §3.1 leads to the inequality

$$\mathbb{E}W_p^\delta(x + X_{\tau\wedge t}, y + Y_{\tau\wedge t}, Y_{\tau\wedge t}^* \vee z) \leq W_p^\delta(x, y, z), \qquad t \geq 0.$$

Therefore, letting $t \to \infty$ and using Lebesgue's dominated convergence theorem yields

$$\begin{aligned}
&W_p^\delta(x, y, z)\\
&\geq \mathbb{E}W_p^\delta(x + X_\tau, y + Y_\tau, Y_\tau^* \vee z)\\
&= \mathbb{E}W_p^\delta(x + X_\tau, y + Y_\tau, Y_\tau^* \vee z)1_{\{X_\tau = h\}}\\
&\quad + \mathbb{E}W_p^\delta(x + X_\tau, y + Y_\tau, Y_\tau^* \vee z)1_{\{X_\tau = -\kappa h\}}\\
&\geq \frac{\kappa}{1+\kappa} W_p^\delta(x + h, y + k, (y + k) \vee z) + \frac{1}{1+\kappa} W_p^\delta(x - \kappa h, y - \kappa k, (y - \kappa k) \vee z),
\end{aligned}$$

where in the latter passage we have used (3.4). It suffices to subtract $W_p^\delta(x-\kappa h, y-\kappa k, (y - \kappa k) \vee z)$ from both sides, divide throughout by κ and let $\kappa \to 0$ to obtain the claim. □

Lemma 3.4. *For any* $(x, y, z) \in D$ *there is* $c = c(x, y, z) \leq 0$ *such that if* h, $k \in \mathbb{R}$, *then*

$$W_{pxx}^\delta(x, y, z)h^2 + 2W_{pxy}^\delta(x, y, z)hk + W_{pyy}^\delta(x, y, z)k^2 \leq c(x, y, z)(h^2 - k^2). \tag{3.6}$$

Proof. By Lemma 3.2 (iv) and (v), we have $W_{pyy}^\delta \geq 0$, $W_{pxx}^\delta \leq 0$ and

$$W_{pxx}^\delta(x, y, z) \pm 2W_{pxy}^\delta(x, y, z) + W_{pyy}^\delta(x, y, z) \leq 0. \tag{3.7}$$

We shall prove that the claim holds with $c = (W_{pxx}^\delta - W_{pyy}^\delta)/2$. Indeed, this function has the required sign and the bound (3.6) can be rewritten in the form

$$(W_{pxx}^\delta(x, y, z) + W_{pyy}^\delta(x, y, z))(h^2 + k^2) + 4W_{pxy}^\delta(x, y, z)hk \leq 0,$$

which follows immediately from (3.7) and the trivial estimate $2|hk| \leq h^2 + k^2$. □

We shall also need the following decomposition lemma of Wang [20]. Recall that for any semimartingale X there exists a unique continuous local martingale part X^c of X satisfying

$$[X, X]_t = |X_0|^2 + [X^c, X^c]_t + \sum_{0<s\leq t} |\triangle X_s|^2$$

for all $t \geq 0$. Here $\triangle X_s = X_s - X_{s-}$ denotes the jump of X at time s. Furthermore, we have that $[X^c, X^c] = [X, X]^c$, the pathwise continuous part of $[X, X]$.

Lemma 3.5. *If X and Y are semimartingales, then Y is differentially subordinate to X if and only if Y^c is differentially subordinate to X^c, $|Y_0| \leq |X_0|$ and $|\Delta Y_s| \leq |\Delta X_s|$ for all s.*

Now we are ready to establish (1.3) in the general case. As above, we may assume that X is bounded in L^p and Y satisfies $Y_0 \geq 0$ almost surely. For any $N > 1$, let $\tau = \inf\{t \geq 0 : |X_t| + |Y_t| \geq N\}$ and consider the process $Z = ((X_{\tau\wedge t}, Y_{\tau\wedge t}, Y^*_{\tau\wedge t}))_{t\geq 0}$. An application of Itô's formula gives

$$W_p^\delta(Z_t) = W_p^\delta(Z_0) + I_1 + I_2 + I_3/2 + I_4, \tag{3.8}$$

where

$$\begin{aligned} I_1 &= \int_{0+}^{\tau\wedge t} W_{px}^\delta(Z_{s-})\mathrm{d}X_s + \int_{0+}^{\tau\wedge t} W_{py}^\delta(Z_{s-})\mathrm{d}Y_s, \\ I_2 &= \int_{0+}^{\tau\wedge t} W_{pz}^\delta(Z_{s-})\mathrm{d}Y_s^*, \\ I_3 &= \int_{0+}^{\tau\wedge t} W_{pxx}^\delta(Z_{s-})\mathrm{d}[X,X]_s^c \\ &\quad + 2\int_{0+}^{\tau\wedge t} W_{pxy}^\delta(Z_{s-})\mathrm{d}[X,Y]_s^c + \int_{0+}^{\tau\wedge t} W_{pyy}^\delta(Z_{s-})\mathrm{d}[Y,Y]_s^c \end{aligned}$$

and

$$I_4 = \sum_{0<s\leq t} \left[W_p^\delta(Z_s) - W_p^\delta(Z_{s-}) - W_{px}^\delta(Z_{s-})\Delta X_s - W_{py}^\delta(Z_{s-})\Delta Y_s\right].$$

We have that I_1 is a local martingale and hence we may assume that $\mathbb{E}I_1 = 0$ (if this is not the case, we take the localizing sequence $(\tau_n)_{n\geq 0}$, repeat the reasoning with Z replaced by Z^{τ_n} and take $n \to \infty$ at the end). Using (3.4) and arguing as previously, we see that $I_2 \leq 0$. To deal with I_3, fix $0 \leq s_0 < s_1 \leq t$. For any $j \geq 0$, let $(\eta_i^j)_{1\leq i\leq i_j}$ be a sequence of nondecreasing finite stopping times with $\eta_0^j = s_0$, $\eta_{i_j}^j = s_1$ such that $\lim_{j\to\infty} \max_{1\leq i\leq i_j-1} |\eta_{i+1}^j - \eta_i^j| = 0$. Keeping j fixed, we apply, for each $i = 0, 1, 2, \ldots, i_j$, the estimate (3.6) to $x = X_{s_0-}$, $y = Y_{s_0-}$, $z = Y^*_{s_0-}$ and $h = h_i^j = X^c_{\eta_{i+1}^j} - X^c_{\eta_i^j}$, $k = k_i^j = Y^c_{\eta_{i+1}^j} - Y^c_{\eta_i^j}$. Summing the obtained $i_j + 1$ inequalities and letting $j \to \infty$ yields

$$\begin{aligned} W_{pxx}^\delta(Z_{s_0-})[X^c,X^c]_{s_0}^{s_1} + 2W_{pxy}^\delta(Z_{s_0-})[X^c,Y^c]_{s_0}^{s_1} + W_{pyy}^\delta(Z_{s_0-})[Y^c,Y^c]_{s_0}^{s_1} \\ \leq c(Z_{s_0-})\big([X^c,X^c]_{s_0}^{s_1} - [Y^c,Y^c]_{s_0}^{s_1}\big) \leq 0, \end{aligned}$$

where we have used the notation $[S,T]_{s_0}^{s_1} = [S,T]_{s_1} - [S,T]_{s_0}$ and the last inequality is due to the differential subordination of Y^c to X^c (see Lemma 3.5). This implies $I_3 \leq 0$, by approximation of I_3 by discrete sums. Finally, $I_4 \leq 0$ follows from (3.5) and Lemma 3.5. Thus, taking expectation in (3.8), we get

$$\mathbb{E}(V_p^\gamma)^\delta(Z_t) \leq \mathbb{E}W_p^\delta(Z_t) \leq \mathbb{E}W_p^\delta(Z_0).$$

Letting $\delta \to 0$ and using Lebesgue's dominated convergence theorem gives

$$\mathbb{E}V_p^\gamma(Z_t) \leq \mathbb{E}W_p(Z_0) \leq 0,$$

where the latter estimate is due to the pointwise bound $W_p(x,y,y) \leq 0$ for $x \geq y$, a direct consequence of the definition of W_p. It suffices to let N, $t \to \infty$ to get

$$||Y^*||_p \leq \gamma||X||_p.$$

Since $\gamma > p/(p-1)$ was arbitrary, the result follows.

4. Sharpness

4.1. **The case** $1 < p < 2$**.** Let B be a standard, one-dimensional Brownian motion starting from 1. It is well known that the inequality

$$||B^*_\tau||_p \leq \frac{p}{p-1}||B_\tau||_p, \qquad \tau \in L^{p/2},$$

is sharp; for the sake of completeness, let us outline this fact. Fix $\gamma < p/(p-1)$ and introduce the stopping time $\tau = \inf\{t \geq 0 : B^*_t = \gamma B_t\}$. Obviously, τ is finite almost surely and $||B^*_\tau||_p = \gamma||B_\tau||_p$, so we will be done if we show that both sides are finite (or equivalently, $\tau \in L^{p/2}$, due to Burkholder-Davis-Gundy inequality). To do this, consider the function $K : \mathbb{R}\times[0,\infty) \to \mathbb{R}$ given by $K(x,y) = py^{p-1}\left(y - \frac{p}{p-1}|x|\right)$. This function is of class C^∞ on $(0,\infty)\times(0,\infty)$ and satisfies $K_{xx}(x,y) = 0$, $K_y(y,y) = 0$ for x, $y > 0$. Consequently, applying Itô's formula yields

$$-\frac{p}{p-1} = K(1,1) = \mathbb{E}K(B_{\tau\wedge t}, B^*_{\tau\wedge t}). \tag{4.1}$$

On the other hand, by the definition of τ,

$$K(B_{\tau\wedge t}, B^*_{\tau\wedge t}) \leq p(B^*_{\tau\wedge t})^{p-1}\left(B^*_{\tau\wedge t} - \frac{p}{(p-1)\gamma}B^*_{\tau\wedge t}\right) = \frac{p}{\gamma}\left(\gamma - \frac{p}{p-1}\right)(B^*_{\tau\wedge t})^p,$$

so $B^*_\tau \in L^p$ and we are done. Let us mention here that letting $t \to \infty$ in (4.1) yields

$$||B^*_\tau||_p^p = \left(\frac{p}{\gamma} - p + 1\right)^{-1}$$

so in particular $||\tau||_{p/2} \to \infty$ as $\gamma \to p/(p-1)$, in virtue of Burkholder-Davis-Gundy inequality.

4.2. **The case** $p \geq 2$**.** Here the example is a bit more complicated. As previously, let B be a standard Brownian motion starting from 1 and consider the process $D = \left(1 - \int_{0+}^t \operatorname{sgn} B_s \mathrm{d}B_s\right)_{t\geq 0}$, which, by Itô-Tanaka formula, is another Brownian motion starting from 1. In order to define the martingales X and Y, we need to stop B and D appropriately. Introduce $\eta = \inf\{t \geq 0 : B_t \in \{0,2\}\}$. Next, for a fixed $\gamma > p-1$, set $\tau = \eta$ on $\{B_\eta = 2\}$ and $\tau = \inf\{t \geq \eta : D_t = \gamma|B_t|\}$ on $\{B_\tau = 0\}$. Finally, define $X = B^\tau$ and $Y = D^\tau$. By Itô-Tanaka formula, we have that $|B| + D = 2 + L$, where L stands for the local time of B at 0. In particular, this implies

$$|B| + D = D^* \qquad \text{on the set } \{B_\eta = 0\}, \tag{4.2}$$

since the measure $\mathrm{d}L^\tau$ is concentrated on $\{B = 0\}$. Therefore, on the set $\{B_\eta = 0\}$ we have

$$\tau = \inf\{t \geq \eta : D_t = \gamma|B_t|\} = \inf\left\{\tau \geq \eta : D^*_t = \frac{\gamma+1}{\gamma}D_t\right\}.$$

We have $\frac{\gamma+1}{\gamma} < \frac{p}{p-1}$ and $\lim_{\gamma\to p-1}\frac{\gamma+1}{\gamma} = \frac{p}{p-1}$. In consequence, using the strong Markov property and arguing as in the case $1 < p < 2$, we show that $\tau \in L^{p/2}$ and

$||\tau||_{p/2} \to \infty$ as $\gamma \to p-1$. In addition, on $\{B_\eta = 0\}$ we may write

$$Y^* = D^*_\tau = \frac{\gamma+1}{\gamma} D_\tau = (\gamma+1)|B_\tau| = (\gamma+1)|X_\tau|$$

(see (4.2)), so for any $\varepsilon > 0$,

$$\begin{aligned} ||Y^*||_p \geq ||Y^* 1_{\{B_\eta=0\}}||_p &= (\gamma+1)||X_\tau 1_{\{B_\eta=0\}}||_p \\ &\geq (\gamma+1)||X_\tau||_p - (\gamma+1)||X_\tau 1_{\{B_\eta=2\}}||_p \\ &\geq (\gamma+1)||X||_p - 2(\gamma+1) \\ &\geq (\gamma+1-\varepsilon)||X||_p \end{aligned}$$

provided γ is sufficiently close to $p-1$. This shows that the constant p is indeed the best possible in (1.3).

5. Further extensions and applications

5.1. **Vector-valued case.** We will prove the following generalization of Theorem 1.2 to the higher dimensional setting, under the additional assumption of the continuity of paths.

Theorem 5.1. *Suppose that X is an $\mathbb{R}^d$-valued continuous-path martingale and that Y is a real, continuous-path martingale which is differentially subordinate to X. Then for $1 < p < \infty$ we have*

$$||Y^*||_p \leq C_p ||X||_p \tag{5.1}$$

and the inequality is sharp.

Essentially, the proof goes along the same lines as in §3.1 and §3.3, so we shall only present the necessary modifications and leave the details to the reader. We need to establish (5.1) only for $1 < p < 2$; for the remaining values of p the inequality follows from the bounds of Doob and Burkholder. The proof for $p \in (1,2)$ rests on the following vector version of the function U^δ_p. For $1 < p < 2$, introduce $\mathcal{U}^\delta_p : \mathbb{R}^d \times \mathbb{R} \times [0,\infty) \to \mathbb{R}$ by

$$\mathcal{U}^\delta_p(x,y,z) = U^\delta_p(|x|,y,z).$$

We have the following counterpart of Lemma 3.4.

Lemma 5.1. *For any $x \in \mathbb{R}^d \setminus \{0\}$, $z \geq \max\{y,0\}$ and $h \in \mathbb{R}^d$, $k \in \mathbb{R}$ we have*

$$\begin{aligned} \sum_{i,j=1}^d \mathcal{U}^\delta_{px_ix_j}(x,y,z)h_ih_j + \sum_{i=1}^d \mathcal{U}^\delta_{px_iy}(x,y,z)h_ik + \mathcal{U}^\delta_{pyy}(x,y,z)k^2 \\ \leq w(|x|,y,z)(|h|^2-k^2), \end{aligned} \tag{5.2}$$

where $w = (U^\delta_{pxx} - U^\delta_{pyy})/2 \leq 0$.

Proof. Let $\langle x_1, x_2\rangle$ denote the scalar product of x_1, $x_2 \in \mathbb{R}^d$. We have

$$2U^\delta_{pxy}(|x|,y,z)\left\langle \frac{x}{|x|}, h\right\rangle k \leq 2\left|U^\delta_{pxy}(|x|,y,z)hk\right|$$

and, by Lemma 2.3 (v),

$$\left(U^\delta_{pxx}(|x|,y,z) - \frac{U^\delta_{px}(|x|,y,z)}{|x|}\right)\left(\left\langle \frac{x}{|x|}, h\right\rangle^2 - |h|^2\right) \leq 0.$$

If we add these two inequalities, we get the estimate which is equivalent to

$$\sum_{i,j=1}^{d}\mathcal{U}^{\delta}_{px_ix_j}(x,y,z)h_ih_j+2\sum_{i=1}^{d}\mathcal{U}^{\delta}_{px_iy}(x,y,z)h_ik+\mathcal{U}^{\delta}_{pyy}(x,y,z)k^2$$
$$\leq U^{\delta}_{pxx}(|x|,y,z)|h|^2+2\big|U^{\delta}_{pxy}(|x|,y,z)\big||h||k|+U^{\delta}_{pyy}(x,y,z)k^2.$$

The right-hand side does not exceed $w(|x|,y,z)(|h|^2-k^2)$. This is done exactly in the same manner as in Lemma 3.4, with the aid of part (iii) of Lemma 2.3. To check the sign of w, write $2w=(U^{\delta}_{pxx}+U^{\delta}_{pyy})-2U^{\delta}_{pyy}$ and note that $\Delta U^{\delta}_p\leq 0$ by Lemma 2.3 (iii) and $U^{\delta}_{pyy}\geq 0$ by Lemma 2.3 (v). □

Proof of Theorem 5.1. Fix $\varepsilon>0$, $N>1$ and pick $\delta>0$ such that U^{δ}_p satisfies the property (iv) from Lemma 2.3. The function $\mathcal{U}^{\delta}_p$ has a singularity at each point of the form $(0,y,z)$, so we need an additional argument to ensure that the process $|X|$ stays away from 0. Increasing the dimension d if necessary, we may and do assume that X^1, the first coordinate of X, vanishes for all t with probability 1. Introduce the stopping time

$$\tau=\tau^N=\inf\{t:|X_t|+|Y_t|\geq N\}$$

and the process

$$Z=\big((\varepsilon,X^2_{\tau\wedge t},X^3_{\tau\wedge t},\ldots,X^d_{\tau\wedge t}),Y_{\tau\wedge t},Y^*_{\tau\wedge t})\big)_{t\geq 0}.$$

Now we are allowed to apply Itô's formula to get

$$\mathcal{U}^{\delta}_p(Z_t)=\mathcal{U}^{\delta}_p(Z_0)+I_1+I_2+I_3/2,\tag{5.3}$$

where I_1, I_2 are as in §3.1 and

$$I_3=\sum_{i,j=1}^{d}\int_0^{\tau\wedge t}\mathcal{U}^{\delta}_{px_ix_j}(Z_s)\mathrm{d}[X^i,X^j]_s$$
$$+2\sum_{i=1}^{d}\int_0^{\tau\wedge t}\mathcal{U}^{\delta}_{px_iy}(Z_s)\mathrm{d}[X^i,Y]_s+\int_0^{\tau\wedge t}\mathcal{U}^{\delta}_{pyy}(Z_s)\mathrm{d}[Y,Y]_s.$$

Repeating the reasoning from that subsection we get $\mathcal{U}_p(Z_0)\leq 0$, $\mathbb{E}I_1=0$ and $I_2\leq\varepsilon Y^*_{\tau\wedge t}$. Using the approximation arguments from §3.3, we see that Lemma 5.2 gives $I_3\leq 0$. It remains to let $\delta\to 0$, $\varepsilon\to 0$, $N\to\infty$ and apply the majorization $V_p\leq U_p$ to obtain

$$\mathbb{E}|Y^*_t|^p\leq\left(\frac{p}{p-1}\right)^p\mathbb{E}|X_t|^p\leq\left(\frac{p}{p-1}\right)^p||X||_p^p.$$

Letting $t\to\infty$ yields the claim. □

5.2. **Improved bounds for stochastic integrals.** Assume that B is a Brownian motion in $\mathbb{R}^n$, starting from an arbitrary point, and let H, K be two predictable processes taking values in the class of matrices of dimensions $d\times n$ and $1\times n$, respectively. Define X, Y by stochastic integrals

$$X_t=X_0+\int_{0+}^t H_s\cdot\mathrm{d}B_s\qquad Y_t=Y_0+\int_{0+}^t K_s\cdot\mathrm{d}B_s,$$

for any $t \geq 0$. For any matrix $A = (a_{ij})_{1\leq i\leq d,\, 1\leq j\leq n}$, its Hilbert-Schmidt norm is given by

$$||A||_{HS} = \left(\sum_{i=1}^{d}\sum_{j=1}^{n} a_{ij}^2\right)^{1/2}.$$

Theorem 5.2. *With the above notation, if $|Y_0| \leq |X_0|$ and $|K_s| \leq ||H_s||_{HS}$ for any $s > 0$, then*

$$||Y^*||_p \leq C_p ||X||_p, \qquad 1 < p < \infty,$$

and the inequality is sharp.

Proof. We have that X, Y are continuous-path martingales taking values in $\mathbb{R}^d$ and $\mathbb{R}$, respectively, and Y is differentially subordinate to X. Indeed, for any $t \geq 0$,

$$[X,X]_t - [Y,Y]_t = X_0^2 - Y_0^2 + \int_{0+}^{t} \left(||H_s||_{HS}^2 - |K_s|^2\right)\mathrm{d}s.$$

Therefore the assertion follows from Theorem 5.1. □

5.3. **Bounds for harmonic functions on Euclidean domains.** Suppose that $n \geq 1$ is a fixed integer, let D be a connected open subset of $\mathbb{R}^n$ and fix $\xi \in D$. Let D_0 be a bounded subdomain of D, satisfying $\xi \in D_0 \subset D_0 \cup \partial D_0 \subset D$. Denote by $\mu_{D_0}^{\xi}$ the harmonic measure on ∂D_0 with respect to ξ. Consider two harmonic functions u, v on D, taking values in $\mathbb{R}^d$ (in other words, each coordinate of u and v is a real harmonic function). Following [6], we say that v is differentially subordinate to u if

$$|\nabla v(x)| \leq |\nabla u(x)| \quad \text{for } x \in D, \tag{5.4}$$

where the gradient of a vector-valued function $u = (u^1, u^2, \ldots, u^d)$ is given by

$$|\nabla u|^2 = \sum_{j=1}^{d} |\nabla u^j|^2 = \sum_{j=1}^{d}\sum_{k=1}^{n}\left|\frac{\partial u^j}{\partial x_k}\right|^2.$$

Define the p-th norm of u by

$$||u||_p = \left[\int_{\partial D_0} |u(x)|^p \mu_{D_0}^{\xi}(\mathrm{d}x)\right]^{1/p}.$$

Suppose that B is a Brownian motion in $\mathbb{R}^n$, starting from ξ, and let $\tau = \inf\{t \geq 0 : B_t \notin D_0\}$ be the exit time of B from D_0. We define v^*, the Brownian maximal function of v, by the formula $v^* = v^*(\xi) = \sup_{s\in[0,\tau]} v(B_s)$.

Theorem 5.3. *Assume that u is $\mathbb{R}^d$-valued harmonic function and v is a real-valued harmonic function which is differentially subordinate to u. If, in addition, we have $|v(\xi)| \leq |u(\xi)|$, then*

$$||v^*||_p \leq C_p ||u||_p.$$

Proof. The processes $X = (u(B_{\tau\wedge t}))_{t\geq 0}$, $Y = (v(B_{\tau\wedge t}))_{t\geq 0}$ are martingales, which have the representation

$$X_t = u(\xi) + \int_{0+}^{\tau\wedge t} \nabla u(B_s)\cdot \mathrm{d}B_s, \qquad Y_t = v(\xi) + \int_{0+}^{\tau\wedge t} \nabla v(B_s)\cdot \mathrm{d}B_s$$

for $t \geq 0$. It suffices to apply Theorem 5.2, the assumptions of which are satisfied due to the properties of u and v. □

A particularly interesting example corresponds to the case when D is the upper half-space $(0,\infty)\times\mathbb{R}^{n-1}$. We shall prove the following statement (see Burkholder, Gundy and Silverstein [2] and Burkholder [3] for related results and further connections to nontangential maximal functions).

Theorem 5.4. *Let u, v be real valued harmonic functions on D satisfying* (5.4). *Then*

$$\sup_{t\geq 0}\left[\int_{\mathbb{R}^{n-1}}\mathbb{E}|v^*(t,x)-v(t,x)|^p\,dx\right]^{1/p}\leq C_p\sup_{t\geq 0}\left[\int_{\mathbb{R}^{n-1}}|u(t,x)|^p\,dx\right]^{1/p}.$$

Proof. We may and do assume that the right-hand side is finite. Fix $t>0$, $x\in\mathbb{R}^{n-1}$ and apply the previous theorem to u, $v-v(t,x)$, $\xi=(t,x)$ and the rectangle $D_0=(\varepsilon,M)\times(-N,N)$, where ε, M, N are chosen so that $\xi\in D_0$. Letting M, $N\to\infty$ yields

$$\mathbb{E}|\sup_{s\leq\tau}v(B_s)-v(t,x)|^p\leq C_p^p\int_{\mathbb{R}^{n-1}}|u(\varepsilon,r)|^pP_{t,x,\varepsilon}(r)\mathrm{d}r,$$

where $\tau=\inf\{s>0:B_s^1=\varepsilon\}$ and $P_{t,x,\varepsilon}$ stands for the Poisson kernel

$$P_{t,x,\varepsilon}(r)=\mathbb{P}(B_\tau\in\mathrm{d}r)=\frac{\Gamma\left(n/2\right)}{\pi^{n/2}}\frac{t-\varepsilon}{(|x-r|^2+(t-\varepsilon)^2)^{n/2}}.$$

It suffices to integrate both sides over x with respect to the Lebesgue measure on $\mathbb{R}^{n-1}$ and use the equality $\int_{\mathbb{R}^{n-1}}P_{t,x,\varepsilon}\mathrm{d}x=1$. The proof is complete. ☐

5.4. **An application to Riesz system of harmonic functions.** Put $D=(0,\infty)\times\mathbb{R}^n$ and let u be a real harmonic function on D. Let us introduce $u_k=\partial u/\partial x_k$ and $u_{jk}=\partial^2u/\partial x_j\partial x_k$ for j, $k\in\{0,\,1,\,2,\,\ldots,\,n\}$. Then u_0, u_1, …, u_n are harmonic and satisfy the generalized Cauchy-Riemann equations

$$\sum_{j=0}^n u_{jj}=0\qquad\text{and}\qquad u_{jk}=u_{kj}.\tag{5.5}$$

This system of harmonic functions was studied in depth by Stein and Weiss [19]. They showed it is a natural object on which the classical theory of Hardy spaces can be extended.

To apply our results, observe that the function u_0 is differentially subordinate to $n^{1/2}w=n^{1/2}(u_1,u_2,\ldots,u_n)$. Indeed, using (5.5), we obtain

$$\begin{aligned}|\nabla u_0|^2&=|u_{00}|^2+\sum_{k=1}^n|u_{0k}|^2\\&=\left|\sum_{k=1}^n u_{kk}\right|^2+\sum_{k=1}^n|u_{k0}|^2\\&\leq(n-1)\sum_{k=1}^n|u_{kk}|^2+\left(\sum_{k=1}^n|u_{kk}|^2+\sum_{k=1}^n|u_{k0}|^2\right)\\&\leq(n-1)|\nabla w|^2+|\nabla w|^2\\&=n|\nabla w|^2.\end{aligned}$$

Note that the constant n is optimal here: consider the gradient of $nx_0^2 - x_1^2 - \ldots - x_n^2$. Consequently, if $\xi \in D$ satisfies $|u_0(\xi)| \leq n^{1/2}|w(\xi)|$, then for any $1 < p < \infty$ we have

$$||u_0^*||_p \leq C_p \left|\left| \left(\sum_{j=1}^n |u_j|^2 \right)^{1/2} \right|\right|_p .$$

See Essén [10] and Stein [18] for related results and further discussion.

5.5. **A sharp bound for three-dimensional Bessel process.** Let us turn to our final application. Assume that B is a Brownian motion in $\mathbb{R}^3$ starting from 0 and let $\rho = (|B_t|)_{t \geq 0}$ be the three-dimensional Bessel process. Theorem 1.2 allows us to obtain some sharp estimates for the stopped infimum of ρ. To be precise, we shall prove the following fact.

Theorem 5.5. *Let ρ be as above and let β be an adapted one-dimensional Brownian motion. Then for any $1 < p < \infty$ and any stopping time $\tau \in L^{p/2}$,*

$$|| \inf_{t \geq \tau} \rho||_p \leq C_p ||\beta_\tau||_p$$

and

$$||\beta_\tau||_p \leq C_p || \inf_{t \geq \tau} \rho||_p.$$

Both inequalities are sharp.

Proof. By Pitman's theorem (see e.g. Pitman [16] or Revuz and Yor [17]), there is an adapted standard one-dimensional Brownian motion D such that $(\rho, J) = (2D^* - D, D^*)$. Then D is differentially subordinate to β and β is differentially subordinate to D, since both processes have the same square bracket. Thus, the desired inequalities follow at once from (1.3). To get the optimality of the constant C_p, use the reasoning from Section 4 to construct the appropriate Brownian motion β and the stopping time τ. □

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Department of Mathematics, Informatics and Mechanics, University of Warsaw, Banacha 2, 02-097 Warsaw, Poland
E-mail address: ados@mimuw.edu.pl