# ON CARLSON-LEVIN INEQUALITIES

ADAM OSĘKOWSKI

Abstract. We present a new proof of Carlson-Levin inequality and some of its extensions, based on a dynamic programing-type approach.

## 1. Introduction.

In 1934, F. Carlson proved in [5] that for any sequence $(a_n)_{n\geq 1}$ of nonnegative numbers, we have the inequality

$$\left(\sum_{n=1}^{\infty} a_n\right)^4 \leq \pi^2 \sum_{n=1}^{\infty} a_n^2 \sum_{n=1}^{\infty} n^2 a_n^2. \tag{1}$$

Furthermore, for any measurable function $f : [0,\infty) \to [0,\infty)$, we have

$$\left(\int_0^{\infty} f(u)\mathrm{d}u\right)^4 \leq \pi^2 \int_0^{\infty} f^2(u)\mathrm{d}u \int_0^{\infty} u^2 f^2(u)\mathrm{d}u. \tag{2}$$

Both inequalities are sharp: the constant $\pi^2$ cannot be replaced by a smaller number. These two estimates have been extended in many directions and found many interesting applications, including harmonic analysis (cf. [4, 10]), interpolation theory (see [8, 12, 14]), and optimal sampling ([1]). See also the papers [2], [6], [7], [9] and [11] for related results in this direction.

By now, several different proofs of (1) and (2) have been invented. Carlson realized that the estimates cannot be established by a direct use of Hölder's inequality, and his original proof exploits the theory of analytic functions. Probably the most universal argument is due to Hardy [9], who showed that (1) and (2) do follow from Hölder's inequality if combined with a clever splitting procedure. This splitting argument was applied and further extended by many authors. In particular, Levin [13] exploited it to identify the best constant $C_{p,q,\lambda,\mu}$ in the estimate

$$\int_0^{\infty} f(u)\mathrm{d}u \leq C_{p,q,\lambda,\mu} \left(\int_0^{\infty} u^{p-1-\lambda} f^p(u)\mathrm{d}u\right)^s \left(\int_0^{\infty} u^{q-1+\mu} f^q(u)\mathrm{d}u\right)^t, \tag{3}$$

where $p$, $q$ are assumed to be larger than 1, the numbers $\lambda$, $\mu$ are positive parameters and $s = \mu/(p\mu + q\lambda)$, $t = \lambda/(p\mu + q\lambda)$. The formula for the best constant is

$$C_{p,q,\lambda,\mu} = \frac{(\lambda+\mu)^{s+t-1}}{(ps)^s (qt)^t} \left[\mathcal{B}\left(\frac{s}{1-s-t}, \frac{t}{1-s-t}\right)\right]^{1-s-t},$$

where $\mathcal{B}(\alpha,\beta) = \int_0^1 u^{\alpha-1}(1-u)^{\beta-1}\mathrm{d}u$ is the usual Beta function. One easily verifies that, due to homogeneity reasons, for fixed $p$, $q$, $\lambda$ and $\mu$, the above choice of $s$ and

2000 *Mathematics Subject Classification.* Primary: 58E35. Secondary: 26A46.
*Key words and phrases.* Carlson inequality, best constants.

$t$ is the only one which produces a non-trivial estimate (for other $s$, $t$, one is forced to take the constant to be infinite).

The purpose of this note is to present an alternative proof of the inequality (3), with the use of a dynamic-programming-type approach. Actually, as we will indicate, the method can be applied to a much wider class of estimates and we will manage to obtain a certain extension of (3). Roughly speaking, our technique enables to deduce an inequality of the above type from the existence of a certain special function of two variables, enjoying an appropriate monotonicity condition. We also refer the interested reader to [3] for the description of related methods.

## 2. A method.

We start with a general setup. Fix a Borel function $V : [0,1] \times [0,\infty) \to [0,\infty)$, which is bounded on bounded sets. Suppose that for a fixed $y \geq 0$, we want to find an effective lower bound for the quantity

$$\inf \int_0^1 V(u, f(u))\, \mathrm{d}u, \tag{4}$$

where the infimum is taken over all nonnegative and continuous functions $f$ on $[0,1]$ satisfying $\int_0^1 f(u)\mathrm{d}u = y$. To study this problem, we extend it to a slightly more general setting: for any $x \in [0,1]$ and any $y \geq 0$, let

$$B(x,y) = \inf \int_0^x V(u, f(u))\mathrm{d}u,$$

where the infimum is taken over all nonnegative and continuous functions $f$ on $[0,x]$ satisfying $\frac{1}{x}\int_0^x f(u)\mathrm{d}u = y$. Here and below, we use the convention $\frac{1}{x}\int_0^x f(u)\mathrm{d}u = f(0)$ when $x = 0$. Directly from the definition, we see that $B$ enjoys the following:

1° We have $B(0,y) = 0$ for all $y \geq 0$.

2° For any nonnegative and continuous $f$ on $[0,1]$, the function

$$\xi_f : x \mapsto B\left(x, \frac{1}{x}\int_0^x f(u)\mathrm{d}u\right) + \int_x^1 V(u, f(u))\mathrm{d}u$$

is nonincreasing on $[0,1]$.

Indeed, the first condition is evident. To prove the second, pick $0 \leq w < x \leq 1$ and fix an arbitrary continuous function $f : [0,x] \to [0,\infty)$. For any $\varepsilon > 0$ there is a continuous function $\tilde{f} : [0,w] \to [0,\infty)$ such that $\frac{1}{w}\int_0^w \tilde{f}(u)\mathrm{d}u = \frac{1}{w}\int_0^w f(u)\mathrm{d}u$ and

$$B\left(w, \frac{1}{w}\int_0^w f(u)\mathrm{d}u\right) + \varepsilon > \int_0^w V(u, \tilde{f}(u))\mathrm{d}u.$$

Actually, modifying $\tilde{f}$ slightly if necessary, we may also assume that $\tilde{f}(w) = f(w)$, without violating the preceding properties. Extend $\tilde{f}$ to $[0,x]$ by setting $\tilde{f} = f$ on $(w,x]$. Then $\tilde{f}$ is continuous on $[0,x]$ and we have $\int_0^x \tilde{f}(u)\mathrm{d}u = \int_0^x f(u)\mathrm{d}u$, so

$$\begin{aligned} B\left(x, \frac{1}{x}\int_0^x f(u)\mathrm{d}u\right) &\leq \int_0^x V(u, \tilde{f}(u))\mathrm{d}u \\ &= \int_0^w V(u, \tilde{f}(u))\mathrm{d}u + \int_w^x V(u, \tilde{f}(u))\mathrm{d}u) \\ &< B\left(w, \frac{1}{w}\int_0^w f(u)\mathrm{d}u\right) + \varepsilon + \int_w^x V(u, f(u))\mathrm{d}u. \end{aligned}$$

Since $\varepsilon$ was arbitrary, the property 2° follows.

A simple but very important observation is that if we manage to find *some* function $B$ which satisfies the conditions 1° and 2°, then

$$\int_0^1 V(u, f(u))\mathrm{d}u \geq B\left(1, \int_0^1 f(u)\mathrm{d}u\right) \tag{5}$$

for any continuous function $f : [0,1] \to [0,\infty)$. This follows immediately from the fact that the left-hand side is equal to $\xi_f(0)$, while the right-hand side is precisely $\xi_f(1)$. Thus, we see that the problem of proving an effective lower bound for the infimum (4) boils down to the problem of constructing an appropriate special function.

So, suppose that we have fixed $V$ and we search for the corresponding function $B$. It is clear that the main difficulty lies in handling the second condition. If $B$ is sufficiently regular, then we can use the following simple observation.

**Lemma 1.** *Suppose that $B$ is continuous on $[0,1] \times [0,\infty)$ and differentiable in the interior of this set. Assume further that $B$ satisfies the inequality*

$$B_x(x,y) + \left(\frac{d}{x} - \frac{y}{x}\right) B_y(x,y) \leq V(x,d) \tag{6}$$

*for any $x \in (0,1)$, $y > 0$ and $d \geq 0$. Then 2° holds true.*

*Proof.* Fix a continuous function $f : [0,1] \to [0,\infty)$ and two numbers $w$, $x$ satisfying $0 \leq w < x \leq 1$. A direct differentiation shows that

$$\begin{aligned} &B\left(x, \frac{1}{x}\int_0^x f\right) - B\left(w, \frac{1}{w}\int_0^w f\right) \\ &= \int_w^x \left\{ B_x\left(u, \frac{1}{u}\int_0^u f\right) + \left(\frac{f(u)}{u} - \frac{1}{u^2}\int_0^u f\right) B_y\left(u, \frac{1}{u}\int_0^u f\right)\right\} \mathrm{d}u \\ &\leq \int_w^x V(u, f(u))\mathrm{d}u, \end{aligned}$$

which follows from (6), by taking $x = u$, $y = \frac{1}{u}\int_0^u f$ and $d = f(u)$. We obtained an inequality equivalent to $\xi_f(w) \geq \xi_f(x)$; this completes the proof. □

In the next section we will see how the above approach can be successfully used in the study of (3). As we shall see, the method is very effective, at the price of involving some lengthy and technical computations at some points.

## 3. Analysis of Levin's inequality.

For the sake of clarity, we have decided to split the analysis into seven separate steps.

*Step 1. Reductions.* Let us start with several useful observations which will turn the desired inequality (3) into a more convenient form. First, it suffices to show the claim for $p \neq q$; then the case $p = q$ follows from an easy limiting argument. Actually, we may assume that $p < q$. Indeed, if $p > q$, then we substitute $w = 1/u$ under the three integrals in (3): we obtain (3) again, applied to the function $u \mapsto f(1/u)/u^2$, with the parameters $p' = q$, $q' = p$, $\lambda' = p - q + \mu$ and

$\mu' = p - q + \lambda$ (enjoying $p' < q'$, $\lambda'$, $\mu' > 0$, as desired). Secondly, it suffices to study the "localized" version of (3):

$$\int_0^1 f(u)\mathrm{d}u \leq C_{p,q,\lambda,\mu} \left( \int_0^1 u^{p-1-\lambda} f^p(u)\mathrm{d}u \right)^s \left( \int_0^1 u^{q-1+\mu} f^q(u)\mathrm{d}u \right)^t, \tag{7}$$

in which the integration is taken over the interval $[0,1]$. To see how (7) implies (3), simply apply the substitution $w = ru$ and let $r \to \infty$. Next, note that by standard approximation, in the analysis of (7) we may restrict ourselves to continuous functions $f : [0,1] \to [0,\infty)$. Our final remark introduces a somewhat different reformulation of (7). Namely, for each $c \geq 0$, we will search for the best lower bound for the quantity

$$\int_0^1 u^{p-1-\lambda} f^p(u) + cu^{q-1+\mu} f^q(u)\mathrm{d}u \tag{8}$$

depending only on $c$ and $\int_0^1 f(u)\mathrm{d}u$; then an appropriate optimization with respect to $c$ will allow us to deduce (7). This reduction is close to Hardy's argument, as it replaces analysis of the right-hand side of (7) by a much more convenient linear expression (8). Clearly, this expression can be studied with the use of the method introduced in the preceding section: for $c \geq 0$, we let $V = V^c : [0,1] \times [0,\infty) \to [0,\infty)$ be given by $V^c(x,y) = x^{p-1-\lambda}y^p + cx^{q-1+\mu}y^q$. So, the question is: what is the corresponding function $B : [0,1] \times [0,\infty) \to \mathbb{R}$?

In the next two steps, we will present a reasoning which leads to a candidate for this special function. Then, in Steps 4 and 5, we will verify rigorously that this object indeed enjoys all the required conditions.

*Step 2. Structural properties of $B = B^c$.* Write the abstract definition:

$$B^c(x,y) = \inf \int_0^x u^{p-1-\lambda} f^p(u) + cu^{q-1+\mu} f^q(u)\mathrm{d}u,$$

where the infimum is taken over the class of all continuous $f : [0,x] \to [0,\infty)$ with $\frac{1}{x}\int_0^x f(u)\mathrm{d}u = y$. For any such $f$ and any $r > 0$, the function $rf$ is continuous and has average $ry$ over $[0,x]$. Hence

$$\begin{aligned}
&\int_0^x u^{p-1-\lambda} f^p(u) + cu^{q-1+\mu} f^q(u)\mathrm{d}u \\
&\quad = r^{-p} \int_0^x u^{p-1-\lambda}(rf)^p(u) + cr^{p-q}u^{q-1+\mu}(rf)^q(u)\mathrm{d}u \geq r^{-p}B^{cr^{p-q}}(x,ry),
\end{aligned}$$

and taking the infimum over all $f$ as above, we obtain $B^c(x,y) \geq r^{-p}B^{cr^{p-q}}(x,ry)$. Applying this estimate with $y$, $r$, $c$ replaced by $ry$, $r^{-1}$ and $cr^{p-q}$, respectively, shows that actually both sides are equal. Next, for any $r > 0$ and any continuous $f : [0,x] \to [0,\infty)$ with $\frac{1}{x}\int_0^x f(u)\mathrm{d}u = y$, the function $u \mapsto f(ru)$ is continuous on $[0,x/r]$ and has average $y$. Consequently,

$$\begin{aligned}
&\int_0^x u^{p-1-\lambda} f^p(u) + cu^{q-1+\mu} f^q(u)\mathrm{d}u \\
&= r^{p-\lambda} \int_0^{x/r} u^{p-1-\lambda} f^p(ru) + cr^{q-p+\lambda+\mu}u^{q-1+\mu} f^q(ru)\mathrm{d}u \\
&\geq r^{p-\lambda}B^{cr^{q-p+\lambda+\mu}}(x/r,y)
\end{aligned}$$

and since $f$ was arbitrary, we get $B^c(x,y) \geq r^{p-\lambda} B^{cr^{q-p+\lambda+\mu}}(x/r, y)$. As previously, the reverse inequality is also true, which can be seen by applying the bound with $x$, $c$ and $r$ replaced by $x/r$, $cr^{q-p+\lambda+\mu}$ and $r^{-1}$, respectively.

Using the above "structural" properties of $B^c$, we get that

$$B^c(x,y) = x^{p-\lambda} B^{cx^{q-p+\lambda+\mu}}(1,y) = x^{p-\lambda} y^p B^{cx^{q-p+\lambda+\mu} y^{q-p}}(1,1).$$

Therefore, in our search for the special function (which a priori need not be equal to $B^c$: there might exist other special objects leading to the sharp bound), we may restrict ourselves to the class of functions of the above form: $B(x,y) = x^{p-\lambda} y^p \varphi(cx^{q-p+\lambda+\mu} y^{q-p})$, for some function $\varphi$ to be found.

*Step 3. On an informal search of $\varphi$.* Suppose for a while that the desired function $B$ is continuous on $[0,1] \times [0,\infty)$ and differentiable in the interior of this set. By the reasoning from Section 2, this function will satisfy 2° if the condition (6) holds. The inequality

$$B_x(x,y) + \left(\frac{d}{x} - \frac{y}{x}\right) B_y(x,y) \leq V(x,d)$$

is, after the substitution $r = cx^{q-p+\lambda+\mu} y^{q-p}$ and $D = d/y$, equivalent to

$$-\lambda\varphi(r) + (\lambda+\mu) r\varphi'(r) + (p\varphi(r) + (q-p) r\varphi'(r)) D - D^p - rD^q \leq 0. \tag{9}$$

For a fixed $r$, the left-hand side, considered as a function of $D$, attains its maximum for $D(r)$ satisfying

$$p\varphi(r) + (q-p) r\varphi'(r) = pD^{p-1}(r) + qrD^{q-1}(r). \tag{10}$$

Now, recall that we search for $\varphi$ leading to a sharp result. So, it seems very natural to *conjecture* that for any $r$ and the corresponding $D(r)$, the inequality (9) is actually an equality. This leads us to the system of differential equations, for unknown functions $D$ and $\varphi$. After some easy computations, we rewrite the system as

$$\begin{aligned} -\lambda\varphi(r) + (\lambda+\mu) r\varphi'(r) &= (1-p) D^p(r) + (1-q) rD^q(r), \\ p\varphi(r) + (q-p) r\varphi'(r) &= pD^{p-1}(r) + qrD^{q-1}(r). \end{aligned}$$

To solve the above system, we express $\varphi$ in terms of $D$:

$$\begin{aligned} \varphi(r) = &\frac{\lambda+\mu}{p\mu + q\lambda} \left(pD^{p-1}(r) + qrD^{q-1}(r)\right) \\ &+ \frac{q-p}{p\mu+q\lambda} \left((p-1) D^p(r) + (q-1) rD^q(r)\right). \end{aligned} \tag{11}$$

If we differentiate this equation, we obtain a formula for $\varphi'(r)$; plugging it and the above identity for $\varphi$ into the first equation of the system, we get the following single differential equation for the function $D(r)$ (we skip the argument $r$, for convenience):

$$\begin{aligned} &p\lambda D^{p-1} - q\mu rD^{q-1} - p(p-1) D^p - q(q-1) rD^q \\ &= rD' \big[p(p-1) D^{p-2} + q(q-1) rD^{q-2}\big] \big[\lambda + \mu + (q-p) D\big]. \end{aligned}$$

To solve this equation, substitute $D(r) = r^{1/(p-q)} E(r)$ to get

$$\begin{aligned} &\frac{\lambda(q-1) + \mu(p-1)}{q-p} (pE^{p-1} + qE^{q-1}) \\ &= rE'(p(p-1)E^{p-2} + q(q-1)E^{q-2})(\lambda + \mu + (q-p) r^{1/(p-q)} E). \end{aligned} \tag{12}$$

Multiply both sides by $(\lambda(q-1)+\mu(p-1))^{-1}(pE^{p-1}+qE^{q-1})^{1/(s+t-1)}r^{1/(p-q)-1}$ to obtain

$$\left[r^{1/(q-p)}F(E)\right]' + \frac{q-p}{\lambda+\mu}EF'(E) = 0,$$

where $F(u) = (pu^{p-1}+qu^{q-1})^{-(s+t)/(1-s-t)}$. This leads us to the following candidate for $E$:

$$r^{1/(q-p)}F(E) + \frac{q-p}{\lambda+\mu}\int_0^E uF'(u)\mathrm{d}u = 0. \tag{13}$$

Now, we will verify rigorously that $E$ given by the above equation is well-defined and the resulting function $\varphi$ has all the required properties.

*Step 4. Formal definition of $E$ and $\varphi$.* We start with some simple properties of $F$. Clearly, this function is strictly decreasing on $[0,\infty)$. Next, a simple calculation shows

$$\begin{aligned}\int_0^\infty uF'(u)\mathrm{d}u &= -\int_0^\infty F(u)\mathrm{d}u\\ &= -p^{-\frac{s+t}{1-s-t}}\int_0^\infty u^{-\frac{(p-1)(s+t)}{1-s-t}}\left(1+\frac{q}{p}u^{q-p}\right)^{-\frac{s+t}{1-s-t}}\mathrm{d}u\\ &= -\frac{p^{-s/(1-s-t)}q^{-t/(1-s-t)}}{q-p}\mathcal{B}\left(\frac{s}{1-s-t},\frac{t}{1-s-t}\right)\end{aligned} \tag{14}$$

(the last passage follows from using the substitution $w=(1+qu^{q-p}/p)^{-1}$ under the integral). Equipped with the above properties of $F$, we turn to the definition of $E$.

**Lemma 2.** *Suppose that $p<q$. For any $r>0$, there is a unique positive number $E=E(r)$ satisfying (13). Furthermore, the resulting function $E:(0,\infty)\to(0,\infty)$ is of class $C^1$ and enjoys the asymptotic behavior*

$$\begin{aligned}&\lim_{r\to\infty} qE(r)^{q-1}r^{(1-s-t)/(p-q)(s+t)}\\ &\quad= \left[\frac{p^{-s/(1-s-t)}q^{-t/(1-s-t)}}{\lambda+\mu}\mathcal{B}\left(\frac{s}{1-s-t},\frac{t}{1-s-t}\right)\right]^{-(1-s-t)/(s+t)},\end{aligned} \tag{15}$$

*and*

$$\lim_{r\to 0} E(r)r^{1/(p-q)} = \frac{\lambda}{p-1}. \tag{16}$$

*Proof.* Fix $r>0$ and denote the left-hand side of (13) by $G(E)$. To prove the existence and uniqueness of $E(r)$, we derive that

$$G'(E) = F'(E)\left[r^{1/(q-p)} + \frac{q-p}{\lambda+\mu}E\right] < 0,$$

and note that $\lim_{E\to 0}G(E)=\infty$, $\lim_{E\to\infty}G(E)<0$. The regularity of the function $E$ follows from standard theorems on implicit functions. It remains to show (15) and (16). Let us start with the first equality. By (14), the expression $\int_E^\infty uF'(u)\mathrm{d}u$ is bounded when $E$ ranges from 0 to $\infty$. This implies that $F(E(r))$ must converge to 0 as $r\to\infty$, since otherwise the left-hand side of (13) would explode at infinity. This in turn implies $\lim_{r\to\infty}E(r)=\infty$ and hence, since $q>p$,

$$\lim_{r\to\infty}(qE(r)^{q-1})^{(s+t)/(1-s-t)}F(E(r)) = 1.$$

Combining this with (13) and (14), we get (15). Finally, let us turn our attention to (16). Note that $\lim_{r\to 0} E(r) = 0$. Indeed, suppose that there is a sequence $(r_n)_{n\geq 1}$ converging to 0 such that $E(r_n)$ is bounded away from zero. Then the integrals $\int_0^{E(r_n)} uF'(u)\mathrm{d}u$ are also bounded away from zero, while the term $r_n^{1/(q-p)}F(E(r_n))$ tends to 0. This contradicts (13) and hence $E(r) \to 0$ as $r \to 0$. Since $F(u) \approx (pu^{p-1})^{-(s+t)/(1-s-t)}$ when $u$ is close to 0 (in the sense that the ratio of the two expressions converges to 1 as $u$ goes to 0), we see that

$$F(E(r)) \approx (pE(r)^{p-1})^{-(s+t)/(1-s-t)}$$

and

$$\int_0^{E(r)} uF'(u)\mathrm{d}u \approx \int_0^{E(r)} u[(pu^{p-1})^{-(s+t)/(1-s-t)}]'\mathrm{d}u$$

(where $\approx$ has a similar meaning to that above). Now divide both sides of (13) by $E(r)^{-(p-1)(s+t)/(1-s-t)+1}$, let $r \to 0$ and use the above two approximations. Calculating a little bit, we get (16). $\square$

Having defined $E$, we set $D(r) = r^{1/(p-q)}E(r)$ as above, and see that the formula (11) gives a well-defined function $\varphi$ on $(0,\infty)$. It is easy to check that in terms of $E$, this function can be expressed as

$$\begin{aligned}\varphi(r) = {} & \frac{\lambda+\mu}{p\mu+q\lambda} r^{(p-1)/(p-q)}\left[pE(r)^{p-1} + qE(r)^{q-1}\right] \\ & + \frac{q-p}{p\mu+q\lambda} r^{p/(p-q)}\left[(p-1)E(r)^p + (q-1)E(r)^q\right].\end{aligned}$$

Using (15), we see that $E(r)$ is of order $O(r^{(1-s-t)/(q-1)(q-p)(s+t)})$ as $r\to\infty$, and hence, for large $r$, the term involving $E^{q-1}(r)$ in the definition of $\varphi$ dominates over the remaining three terms. Hence

(17)

$$\begin{aligned}&\lim_{r\to\infty}\frac{\varphi(r)}{r^{\lambda/(\lambda+\mu)}} \\ &= \frac{\lambda+\mu}{p\mu+q\lambda}\lim_{r\to\infty} r^{(p-1)/(p-q)-\lambda/(\lambda+\mu)}\cdot qE(r)^{q-1} \\ &= \frac{\lambda+\mu}{p\mu+q\lambda}\left[\frac{p^{-s/(1-s-t)}q^{-t/(1-s-t)}}{\lambda+\mu}\mathcal{B}\left(\frac{s}{1-s-t},\frac{t}{1-s-t}\right)\right]^{-(1-s-t)/(s+t)}.\end{aligned}$$

Furthermore, (16) implies that $\varphi$ extends to a continuous function on the whole $[0,\infty)$, by setting $\varphi(0) = \left(\frac{\lambda}{p-1}\right)^{p-1}$.

*Step 5. Coming back to B.* So, for a given $c \geq 0$, define $B : [0,1]\times[0,\infty) \to [0,\infty)$ by

$$B(x,y) = x^{p-\lambda}y^p\varphi(cx^{q-p+\lambda+\mu}y^{q-p}). \tag{18}$$

This function enjoys 1° and 2°: the first property follows from the very definition, and the second condition was actually the base for the whole construction of $\varphi$. Hence, by the method developed in the previous section, we obtain

(19)

$$\int_0^1 u^{p-1-\lambda}f^p(u) + cu^{q-1+\mu}f^q(u)\mathrm{d}u \geq \left(\int_0^1 f(u)\mathrm{d}u\right)^p\varphi\left(c\left(\int_0^1 f(u)\mathrm{d}u\right)^{q-p}\right)$$

for any continuous $f:[0,1]\to[0,\infty)$. Before we proceed, let us mention that this bound does generalize Carlson-Levin inequality. It can be shown that the estimate is sharp for any choice of the parameter $c$; however, since we will not need this (in what follows, we focus on obtaining (3)), we will not go further into this direction.

*Step 6. Deduction of* (7). Divide both sides of (19) by $c^{\lambda/(\lambda+\mu)}$ to get

$$c^{-\lambda/(\lambda+\mu)}\int_0^1 u^{p-1-\lambda}f^p(u)\mathrm{d}u+c^{\mu/(\lambda+\mu)}\int_0^1 u^{q-1+\mu}f^q(u)\mathrm{d}u \\ \geq\left(\int_0^1 f(u)\mathrm{d}u\right)^{p+(q-p)\lambda/(\lambda+\mu)}\frac{\varphi\left(c\left(\int_0^1 f(u)\mathrm{d}u\right)^{q-p}\right)}{\left(c\left(\int_0^1 f(u)\mathrm{d}u\right)^{q-p}\right)^{\lambda/(\lambda+\mu)}}.$$

But the function $r\mapsto r^{-\lambda/(\lambda+\mu)}\varphi(r)$ is decreasing on $(0,\infty)$. Indeed, we have the identity

$$(\lambda+\mu)r^{1+\lambda/(\lambda+\mu)}\left[r^{-\lambda/(\lambda+\mu)}\varphi(r)\right]'=-\lambda\varphi(r)+(\lambda+\mu)r\varphi'(r),$$

and the right-hand side, by (9) and (10), is equal to $(1-p)D^p(r)+(1-q)rD^q(r)<0$. Hence the preceding inequality implies

$$c^{-\lambda/(\lambda+\mu)}\int_0^1 u^{p-1-\lambda}f^p(u)\mathrm{d}u+c^{\mu/(\lambda+\mu)}\int_0^1 u^{q-1+\mu}f^q(u)\mathrm{d}u \\ \geq\gamma\left(\int_0^1 f(u)\mathrm{d}u\right)^{p+(q-p)\lambda/(\lambda+\mu)},$$

where $\gamma$ is the limit from (17). Finally, optimize the left-hand side over $c$: the choice

$$c=\frac{\lambda\int_0^1 u^{p-1-\lambda}f^p(u)\mathrm{d}u}{\mu\int_0^1 u^{q-1+\mu}f^q(u)\mathrm{d}u}$$

transforms the above bound into (7), as one verifies after some lengthy calculations.

*Step 7. Sharpness.* Finally, let us address the optimality of the constant $C_{p,q,\lambda,\mu}$. It turns out that our approach also gives a hint how to search for the extremal functions (i.e., those for which both sides of (3) are equal, or almost equal). We will be brief. The idea is very simple: let us first try to find functions which are extremal for the above $B$: that is, those for which

$$\xi_f(x)=B\left(x,\frac{1}{x}\int_0^x f(u)\mathrm{d}u\right)+\int_x^1 V^c(u,f(u))\mathrm{d}u$$

is constant on $[0,1]$. If we differentiate over $x$ and repeat the previous calculations, we see that this condition holds if for any $x\in(0,1)$ we have $\frac{d}{y}=D(s)$, where $d=f(x)$, $y=\frac{1}{x}\int_0^x f(u)\mathrm{d}u$ and $s=cx^{q-p+\lambda+\mu}y^{q-p}$; that is,

$$\frac{f(x)}{\frac{1}{x}\int_0^x f(u)\mathrm{d}u}=D\left(cx^{q-p+\lambda+\mu}\left(\frac{1}{x}\int_0^x f(u)\mathrm{d}u\right)^{q-p}\right),\qquad\text{for } x\in(0,1). \tag{20}$$

In what follows, we will study the case $c=1$ only: this will lead us to the right candidates (actually, any choice of $c$ produces extremal functions). Since $D(r)=$

$r^{1/(p-q)}E(r)$, the equality (20) is equivalent to

$$E\left(x^{q-p+\lambda+\mu}\left(\frac{1}{x}\int_0^x f(u)\mathrm{d}u\right)^{q-p}\right)=f(x)x^{-(q-p+\lambda+\mu)/(p-q)}. \tag{21}$$

Now, if we put $r(x)=x^{\lambda+\mu}\left(\int_0^x f(u)\mathrm{d}u\right)^{q-p}$, then

$$r'(x)=\frac{r(x)}{x}\left[\lambda+\mu+(q-p)D(r(x))\right]$$

and hence the differentiation of both sides of (21) yields

$$\begin{aligned}&E'(r(x))r(x)\left[\lambda+\mu+(q-p)D(r(x))\right]\\&\quad=x\left[\frac{q-p+\lambda+\mu}{q-p}x^{(q-p+\lambda+\mu)/(q-p)-1}f(x)+x^{(q-p+\lambda+\mu)/(q-p)}f'(x)\right].\end{aligned}$$

Now plug this into (12) and, in the obtained equality, substitute the right-hand side of (21) for $E(r(x))$. After some calculations, we get

$$\left(pf^{p-1}(x)x^{p-1-\lambda}+qf^{q-1}(x)x^{q-1+\mu}\right)'=0.$$

This means that $f$ (or rather the whole family of the solutions) is given implicitly by

$$pf^{p-1}(x)x^{p-1-\lambda}+qf^{q-1}(x)x^{q-1+\mu}=\kappa, \tag{22}$$

where $x\in(0,1)$ and $\kappa$ is a fixed positive parameter. Now take $\kappa=1$ and extend the above equation to all $x\in(0,\infty)$. Clearly, this gives a well-defined $C^\infty$ function $f:(0,\infty)\to(0,\infty)$, satisfying

$$\lim_{x\to 0}pf^{p-1}(x)x^{p-1-\lambda}=1,\qquad \lim_{x\to\infty}qf^{q-1}(x)x^{q-1+\mu}=1. \tag{23}$$

In particular, $f\in L^1(0,\infty)$. Now, we know that the function $\xi_f$ is constant on $[0,1]$, but we can say more. Namely, if we extend $B$ to the whole $[0,\infty)\times[0,\infty)$ by the use of (18), then the above calculations imply that the function

$$x\mapsto B\left(x,\frac{1}{x}\int_0^x f(u)\mathrm{d}u\right)+\int_x^\infty V^1(u,f(u))\mathrm{d}u$$

is constant on the whole halfline $(0,\infty)$. Comparing the limits of this function at zero and infinity, and exploiting (23), we get the equality

$$\int_0^\infty V^1(u,f(u))\mathrm{d}u=\lim_{x\to\infty}B\left(x,\frac{1}{x}\int_0^x f(u)\mathrm{d}u\right),$$

or

$$\int_0^\infty x^{p-1-\lambda}f^p(x)+x^{q-1+\mu}f^q(x)\mathrm{d}x=\gamma\left(\int_0^\infty f(u)\mathrm{d}u\right)^{p+\lambda(q-p)/(\lambda+\mu)}, \tag{24}$$

where $\gamma$ is the limit from (17). Next, multiply both sides of (22) (recall that $\kappa=1$) by $x^{(\mu(p-1)+\lambda(q-1))/(q-p)}$ to get $F(f(x)x^{1+(\mu+\lambda)/(q-p)})=x^{-(\lambda+\mu)/(q-p)}$, where $F$ is the function introduced at the end of Step 3. So, if we put $u=f(x)x^{1+(\mu+\lambda)/(q-p)}$, we see that

$$-\int_0^\infty uF'(u)\mathrm{d}u=\frac{\lambda+\mu}{q-p}\int_0^\infty f(x)\mathrm{d}x.$$

By (14), we obtain the explicit value of $\int_0^\infty f(x)\mathrm{d}x$. However, multiplying both sides of (22) by $f(x)$ and integrating over $(0,\infty)$, we get

$$\int_0^\infty px^{p-1-\lambda}f^p(x)+qx^{q-1+\mu}f^q(x)\mathrm{d}x=\int_0^\infty f(x)\mathrm{d}x.$$

Combining this with (24), we can show that

$$\int_0^\infty x^{p-1-\lambda}f^p(x)\mathrm{d}x=\frac{1}{p^{s/(1-s-t)}q^{t/(1-s-t)}}\cdot\frac{s}{\lambda+\mu}\mathcal{B}\left(\frac{s}{1-s-t},\frac{t}{1-s-t}\right),$$

$$\int_0^\infty x^{q-1+\mu}f^q(x)\mathrm{d}x=\frac{1}{p^{s/(1-s-t)}q^{t/(1-s-t)}}\cdot\frac{t}{\lambda+\mu}\mathcal{B}\left(\frac{s}{1-s-t},\frac{t}{1-s-t}\right)$$

and

$$\frac{\int_0^\infty f(x)\mathrm{d}x}{\left(\int_0^\infty x^{p-1-\lambda}f^p(x)\mathrm{d}x\right)^s\left(\int_0^\infty x^{q-1+\mu}f^q(x)\mathrm{d}x\right)^t}=C_{p,q,\lambda,\mu},$$

which shows the desired sharpness.

## Acknowledgments.

The author would like to thank an anonymous referee for the careful reading of the first version of the paper and several helpful suggestions. The research was partially supported by the National Science Centre Poland grant no. DEC-2014/14/E/ST1/00532.

## References

[1] J. Bergh, *An Optimal Reconstruction of Sampled Signals*, J. Math. Anal. Appl **115** (1986), 574–577.
[2] R. Bellman, *An integral inequality*, Duke Math. J. **9** (1943), 547–550.
[3] R. Bellman, *Adaptive control processes: a guided tour*, Princeton, N. J., Princeton University Press.
[4] A. Beurling, *Sur les integrales de Fourier absolument convergentes et leur application à une transformation fonctionelle*, C. R. Neuvième Congrès Math. Scandinaves 1938, Helsinki (1939), 345–366.
[5] F. Carlson, *Une inégalité*, Ark. Mat. Astr. Fysik **25B** (1934), No. 1., 1–5.
[6] W. B. Caton, *A class of inequalities*, Duke Math. J. **6** (1940), 442–461.
[7] R. M. Gabriel, *An extension of an inequality due to Carlson*, J. London. Math. Soc. **12** (1937), 130–132.
[8] J. Gustavsson and J. Peetre, *Interpolation of Orlicz Spaces*, Studia Math. **60** (1977), 33–59.
[9] G. H. Hardy, *A Note on Two Inequalities*, J. London Math. Soc. **11** (1936), 167–170.
[10] A. Kamaly, *Fritz Carlsons Inequality and its Application*, Math. Scand. **86** (2000), 100–108.
[11] B. Kjellberg, *On some inequalities*, C. R. Dixième Congrès Math. Scandinaves 1946, Copenhagen, 333–340.
[12] N. Ya. Kruglyak, L. Maligranda and L.-E. Persson, *A Carlson Type Inequality with Blocks and Interpolation*, Studia Math. **104** (1993), 161–180.
[13] V. I. Levin, *Sharp Constants in Carlson Type Inequalities*, Dokl. Akad. Nauk, SSSR **59** (1948), 635–638.
[14] J. Peetre, *Sur le nombre de paramétres dans la définition de certains espaces d'interpolation*, Ricerche Mat. **12** (1963), 248–261.

Department of Mathematics, Informatics and Mechanics, University of Warsaw, Banacha 2, 02-097 Warsaw, Poland
*E-mail address*: ados@mimuw.edu.pl